Context-Free Languages and Pushdown Automata

Context-Free Grammars

- Here is an example of a context-free grammar G_1:

$$\begin{aligned} A &\longrightarrow \mathtt{0}A\mathtt{1} \\ A &\longrightarrow B \\ B &\longrightarrow \# \end{aligned}$$

- Each line is a substitution rule (or production).
- A, B are variables.
- $\mathtt{0}, \mathtt{1}, \#$ are terminals.
- The left-hand side of the first rule (A) is the start variable.

Grammars and Languages

$$
\begin{aligned}
A &\longrightarrow 0A1 \\
A &\longrightarrow B \\
B &\longrightarrow \#
\end{aligned}
$$

- A grammar describes a language.
- A grammar generates a string of its language as follows.
  1. Write down the start variable.
  2. Find a written variable and a rule whose left-hand side is that variable.
  3. Replace the written variable with the right-hand side of the rule.
  4. Repeat steps 2 and 3 until no variable remains.
- For example, consider the following derivation of the string $00\#11$ generated by G_1:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

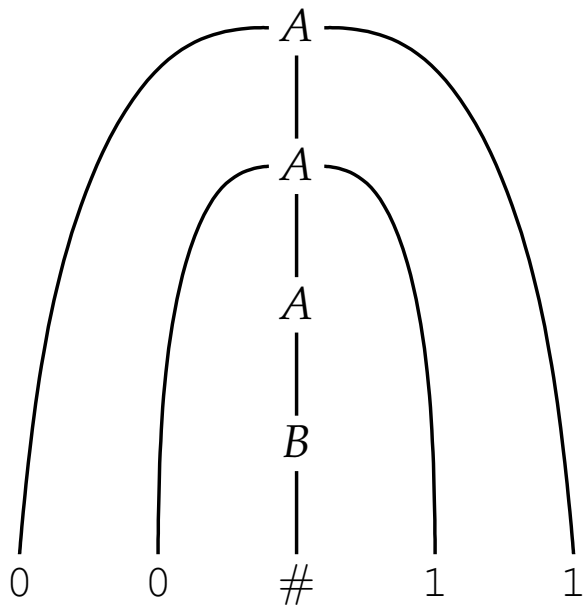
- Any language that can be generated by some context-free grammar is called a context-free language.

Grammars and Languages

- With respect to the following derivation of the string 00#11 generated by G_1:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$$

we also use a parse tree to denote a string generated by a grammar:



Context-Free Grammars – Formal Definition

Definition

A context-free grammar is a 4-tuple (V, Σ, R, S) where

- V is a finite set of variables (also called non-terminals);
- Σ is a finite set of terminals where $V \cap \Sigma = \emptyset$;
- R is a fintie set of production rules. Each rule consists of a variable and a string of variables and terminals; and
- $S \in V$ is the start variable.

- Let u, v, w are strings of variables and terminals, and $A \longrightarrow w$ a rule. We say uAv yields uwv (written $uAv \Rightarrow uwv$).
- u derives v (written $u \overset{*}{\Longrightarrow} v$) if $u = v$ or there is a sequence $u_1, u_2, \ldots, u_k$ $(k \geq 0)$ that $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow v$.
- The language of the grammar is $\{w \in \Sigma^* : S \overset{*}{\Longrightarrow} w\}$.

Context-Free Languages – Examples

Example (Balanced Parentheses)

Consider $G_3 = (\{S\}, \{(,)\}, R, S)$ where R is

$$S \longrightarrow (S) \mid SS \mid \epsilon.$$

- $A \longrightarrow w_1 \mid w_2 \mid \cdots \mid w_k$ stands for

$$\begin{aligned} A &\longrightarrow w_1 \\ A &\longrightarrow w_2 \\ &\vdots \\ A &\longrightarrow w_k \end{aligned}$$

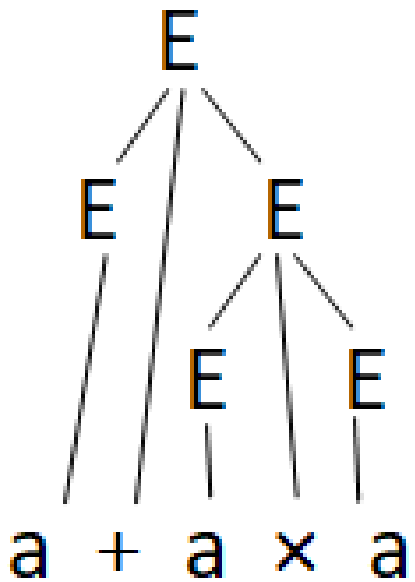
- Examples of the strings generated by G_3: $\epsilon, (), (())(), \ldots$.

Parse Trees vs. Derivation Sequences

Consider the following grammar: $E \to E + E \mid E \times E \mid (E) \mid a$

The following two derivation sequences have the same parse tree.

- $E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E \times E \Rightarrow a + E \times a \Rightarrow a + a \times a$
- $E \Rightarrow E + E \Rightarrow E + E \times E \Rightarrow a + E \times E \Rightarrow a + a \times E \Rightarrow a + a \times a$



Context-Free Languages – Examples

- From a DFA M, we can construct a context-free grammar G_M such that the language of G is $L(M)$.
- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Define $G_M = (V, \Sigma, P, S)$ where
  - $V = \{R_i : q_i \in Q\}$ and $S = \{R_0\}$; and
  - $P = \{R_i \longrightarrow \mathtt{a}R_j : \delta(q_i, \mathtt{a}) = q_j\} \cup \{R_i \longrightarrow \epsilon : q_i \in F\}$.
- Recall M_3 and construct $G_{M_3} = (\{R_1, R_2\}, \{\mathtt{0}, \mathtt{1}\}, P, \{R_1\})$ with

$$\begin{aligned} R_1 &\longrightarrow \mathtt{0}R_1 \mid \mathtt{1}R_2 \mid \epsilon \\ R_2 &\longrightarrow \mathtt{0}R_1 \mid \mathtt{1}R_2. \end{aligned}$$

- The above is a right-linear grammar for which the right-hand-side contains at most one variable at the end of the rule.

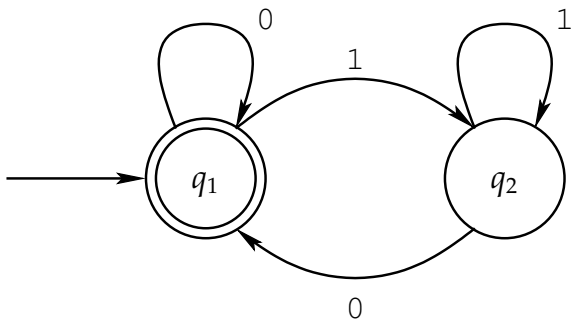


Figure: M_3

Subclasses of Context-Free Grammars

- Right-Linear Grammar

$$\begin{aligned} R_1 &\longrightarrow 0R_1 \mid 1R_2 \mid \epsilon \\ R_2 &\longrightarrow 0R_1 \mid 1R_2 \end{aligned}$$

- Left-Linear Grammar

$$\begin{aligned} R_1 &\longrightarrow R_1 0 \mid R_2 1 \mid \epsilon \\ R_2 &\longrightarrow R_1 0 \mid R_2 1 \end{aligned}$$

- Linear Grammar

$$R_1 \longrightarrow 0R_1 1 \mid \epsilon$$

Note: Left- and Right-Linear Grammars only generate regular languages, while Linear Grammar could generate non-regular languages such as $\{0^n 1^n \mid n \geq 0\}$.

- How about rules contain both RL and LL rules? (Can you use such to generate $\{0^n 1^n \mid n \geq 0\}$?)

$$\begin{aligned} R_1 &\longrightarrow R_2 1 \mid \epsilon \\ R_2 &\longrightarrow 0R_1 \end{aligned}$$

Context-Free vs. Context-Sensitive Grammars

1. Context-Free Rules: $A \longrightarrow \beta, \quad \beta \in (V \cup \Sigma)^*$
2. Context-Sensitive Rules: $\alpha A \gamma \longrightarrow \alpha \beta \gamma, \quad \alpha, \gamma \in (V \cup \Sigma)^*, \beta \in (V \cup \Sigma)^+$

- **Similarity**: both replace A by β.
- **Difference**: in (2), replacing A by β could only take place if A is surrounded by (in the context of) α and γ.
- Context-sensitive grammars are more powerful than context-free grammars.

The Chomsky Hierarchy

Grammars	Rules	Languages	Automata
Type 3 / Right-linear	$A \to aB, A \to \epsilon$	Regular	DFA/NFA
Type 2 / CFG	$A \to \alpha$	CFL	PDA
Type 1 / CSG	$\alpha A \gamma \to \alpha \beta \gamma, \lvert\beta\rvert > 0$	CSL	LBA
Type 0 / Unrestricted	$\alpha A \gamma \to \beta$	r.e.	Turing Machine

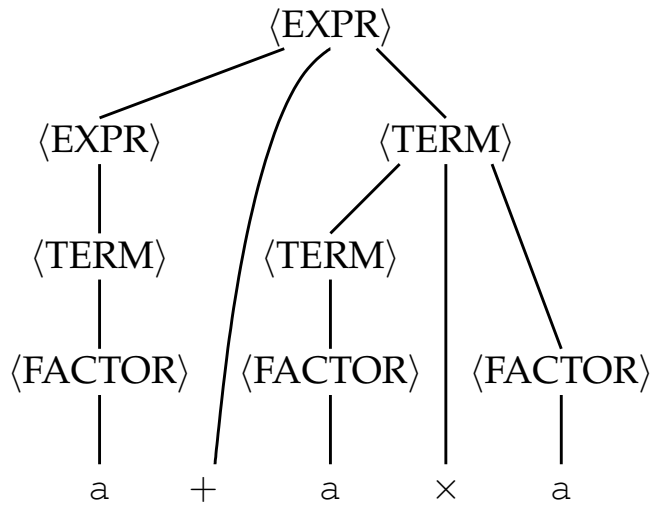
Context-Free Languages – Examples

Example (Fragment of C Grammar)

Consider $G_4 = (V, \Sigma, R, \langle \text{EXPR} \rangle)$ where

- $V = \{\langle \text{EXPR} \rangle, \langle \text{TERM} \rangle, \langle \text{FACTOR} \rangle\}$, $\Sigma = \{\mathtt{a}, +, \times, (,)\}$; and
- R is

$$\begin{aligned} \langle \text{EXPR} \rangle &\longrightarrow \langle \text{EXPR} \rangle + \langle \text{TERM} \rangle \mid \langle \text{TERM} \rangle \\ \langle \text{TERM} \rangle &\longrightarrow \langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle \mid \langle \text{FACTOR} \rangle \\ \langle \text{FACTOR} \rangle &\longrightarrow (\langle \text{EXPR} \rangle) \mid \mathtt{a} \end{aligned}$$



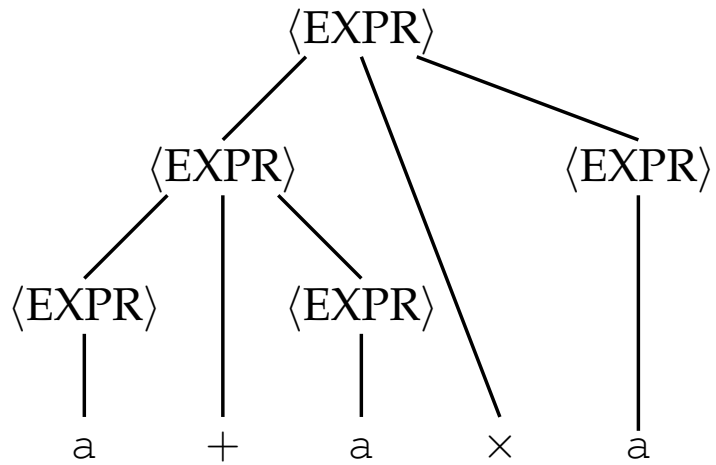
Ambiguity

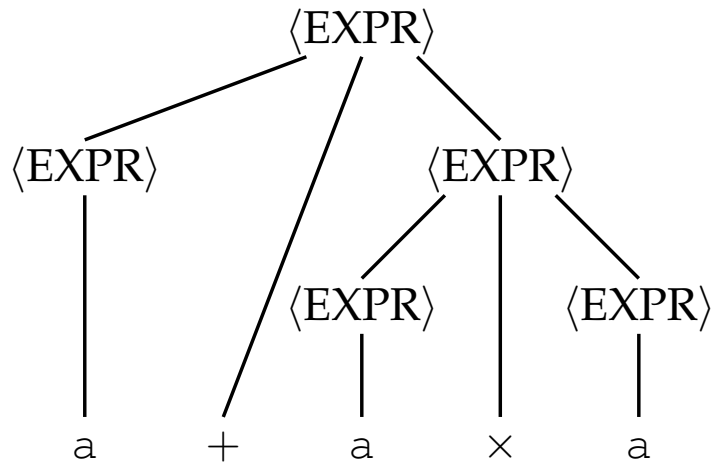
Example (Fragment of C Grammar)

Consider G_5:

$$\langle \text{EXPR} \rangle \longrightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle \mid \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \mid (\langle \text{EXPR} \rangle) \mid \texttt{a}$$

- We have two parse trees for $\texttt{a} + \texttt{a} \times \texttt{a}$.





- If a grammar generates (w.r.t. parse trees) the same in different ways, the string is derived ambiguously in that grammar.
- If a grammar generates some string ambiguously, it is ambiguous.

Ambiguity

- A derivation is a leftmost derivation if the leftmost variable is the one replaced at every step.
- Two leftmost derivations of $\mathtt{a}+\mathtt{a}\times\mathtt{a}$:

$$\begin{aligned}
\langle \text{EXPR} \rangle &\Rightarrow \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \Rightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \Rightarrow \\
&\quad \mathtt{a} + \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \Rightarrow \mathtt{a} + \mathtt{a} \times \langle \text{EXPR} \rangle \Rightarrow \mathtt{a} + \mathtt{a} \times \mathtt{a} \\
\langle \text{EXPR} \rangle &\Rightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle \Rightarrow \mathtt{a} + \langle \text{EXPR} \rangle \Rightarrow \\
&\quad \mathtt{a} + \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \Rightarrow \mathtt{a} + \mathtt{a} \times \langle \text{EXPR} \rangle \Rightarrow \mathtt{a} + \mathtt{a} \times \mathtt{a}
\end{aligned}$$

Theorem

A string is derived ambiguously in a grammar if it has two or more different leftmost derivations.

- If a language can only be generated by ambiguous grammars, we call it is inherently ambiguous.
  - $\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k : i = j \text{ or } j = k\}$ is inherently ambiguous.

Chomsky Normal Form (CNF)

Definition

A context-free grammar is in Chomsky normal form if every rule is of the form

$$\begin{aligned} S &\longrightarrow \epsilon \\ A &\longrightarrow BC \\ A &\longrightarrow \mathtt{a} \end{aligned}$$

where $\mathtt{a}$ is a terminal, S is the start variable, A is a variable, and B, C are non-start variables.

- RHS is (1) ϵ (only from S), (2) exactly two non-start variables, (3) exactly one terminal.

Theorem

Any context-free language is generated by a context-free grammar in Chomsky normal form.

Chomsky Normal Form

Proof.

Given a context-free grammar for a context-free language, we will convert the grammar into Chomsky normal form.

1. (**start variable**) Add a new start variable S_0 and a rule $S_0 \longrightarrow S$.
2. (ϵ**-rules**) For each ϵ-rule $A \longrightarrow \epsilon (A \neq S_0)$, remove it. Then for each occurrence of A on the right-hand side of a rule, add a new rule with that occurrence deleted.
   - $R \longrightarrow uAvAw$ becomes $R \longrightarrow uAvAw \mid uvAw \mid uAvw \mid uvw$.
3. (**unit rules**) For each unit rule $A \longrightarrow B$, remove it. Add the rule $A \longrightarrow u$ for each $B \longrightarrow u$.
4. For each rule $A \longrightarrow u_1 u_2 \cdots u_k (k \geq 3)$ and u_i is a variable or terminal, replace it by $A \longrightarrow u_1 A_1$, $A_1 \longrightarrow u_2 A_2$, $\ldots$, $A_{k-2} \longrightarrow u_{k-1} u_k$.
5. For each rule $A \longrightarrow u_1 u_2$ with u_1 a terminal, replace it by $A \longrightarrow U_1 u_2$, $U_1 \longrightarrow u_1$. Similarly when u_2 is a terminal. □

Chomsky Normal Form – Example

- Consider G_6 on the left. We add a new start variable on the right.

$$\begin{array}{rcl} S & \longrightarrow & ASA \mid \mathtt{a}B \\ A & \longrightarrow & B \mid S \\ B & \longrightarrow & \mathtt{b} \mid \epsilon \end{array} \qquad \begin{array}{rcl} \underline{S_0} & \longrightarrow & S \\ S & \longrightarrow & ASA \mid \mathtt{a}B \\ A & \longrightarrow & B \mid S \\ B & \longrightarrow & \mathtt{b} \mid \epsilon \end{array}$$

- Remove $B \longrightarrow \epsilon$ (left) and then $A \longrightarrow \epsilon$ (right):

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \underline{\mathtt{a}} \\ A & \longrightarrow & B \mid S \mid \underline{\epsilon} \\ B & \longrightarrow & \mathtt{b} \end{array} \qquad \begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid \underline{SA} \mid \underline{AS} \mid \underline{S} \\ A & \longrightarrow & B \mid S \\ B & \longrightarrow & \mathtt{b} \end{array}$$

- Remove $S \longrightarrow S$ (left) and then $S_0 \longrightarrow S$ (right):

$$\begin{array}{rcl} S_0 & \longrightarrow & S \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & B \mid S \\ B & \longrightarrow & \mathtt{b} \end{array} \qquad \begin{array}{rcl} S_0 & \longrightarrow & \underline{ASA} \mid \underline{\mathtt{a}B} \mid \underline{\mathtt{a}} \mid \underline{SA} \mid \underline{AS} \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & B \mid S \\ B & \longrightarrow & \mathtt{b} \end{array}$$

Chomsky Normal Form – Example

- Remove $A \longrightarrow B$ (left) and then $A \longrightarrow S$ (right):

$$\begin{array}{rcl} S_0 & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & S \mid \underline{\mathtt{b}} \\ B & \longrightarrow & \mathtt{b} \end{array} \qquad \begin{array}{rcl} S_0 & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ S & \longrightarrow & ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & \underline{\mathtt{b}} \mid \underline{ASA} \mid \underline{\mathtt{a}B} \mid \underline{\mathtt{a}} \mid \underline{SA} \mid \underline{AS} \\ B & \longrightarrow & \mathtt{b} \end{array}$$

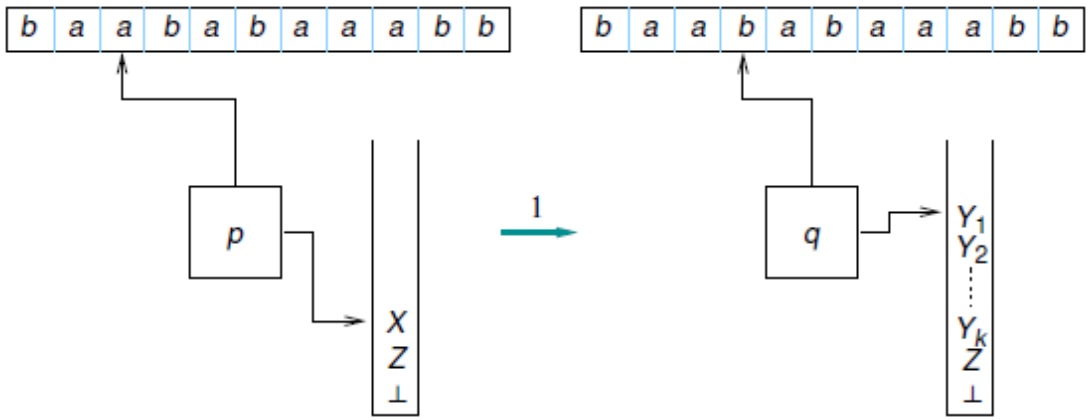
- Remove $S_0 \longrightarrow ASA$, $S \longrightarrow ASA$, and $A \longrightarrow ASA$:

$$\begin{array}{rcl} S_0 & \longrightarrow & \underline{AA_1} \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ S & \longrightarrow & \underline{AA_1} \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & \mathtt{b} \mid \underline{AA_1} \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ B & \longrightarrow & \mathtt{b} \\ \underline{A_1} & \longrightarrow & SA \end{array}$$

- Add $U \longrightarrow a$:

$$\begin{array}{rcl} S_0 & \longrightarrow & AA_1 \mid \underline{UB} \mid \mathtt{a} \mid SA \mid AS \\ S & \longrightarrow & AA_1 \mid \underline{UB} \mid \mathtt{a} \mid SA \mid AS \\ A & \longrightarrow & \mathtt{b} \mid AA_1 \mid \underline{UB} \mid \mathtt{a} \mid SA \mid AS \\ B & \longrightarrow & \mathtt{b} \\ A_1 & \longrightarrow & SA \\ \underline{U} & \longrightarrow & \mathtt{a} \end{array}$$

Schematic of Pushdown Automata



Each step of the PDA looks like:

- Read current symbol and advance head;
- Read and pop top-of-stack symbol;
- Push in a string of symbols on the stack;
- Change state.

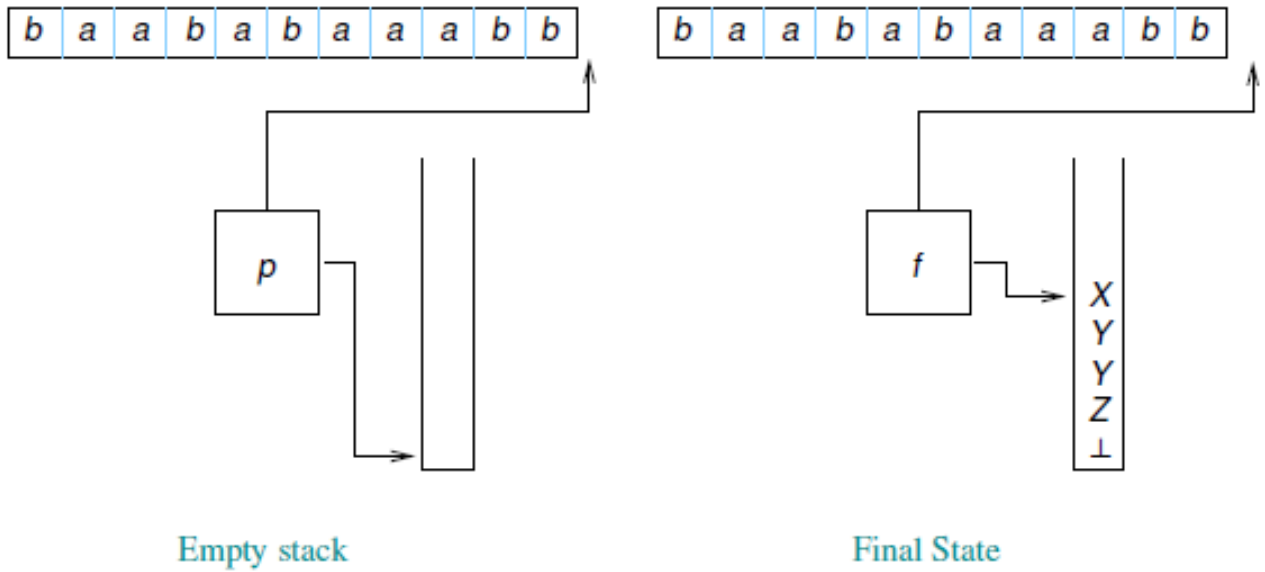
Each transition is of the form

$$(p, a, X) \to (q, Y_1 Y_2 ... Y_k)$$

Three Mechanisms of Acceptance

Accept if input is consumed and

1. Stack is empty (*Acceptance by Empty Stack*),
2. PDA is in a final state (*Acceptance by Final State*),
3. PDA is in a final state and stack is empty (*Acceptance by Final State and Empty Stack*).



It turns out that the three notions of acceptance are equivalent.

Pushdown Automata

- Consider $L = \{0^n 1^n : n \geq 0\}$.
- We have the following table:

Language	Automata
Regular	Finite
Context-free	Pushdown

- A pushdown automaton is a finite automaton with a stack.
  - A stack is a last-in-first-out storage.
  - Stack symbols can be pushed and poped from the stack.
- Computation depends on the content of the stack.
- It is not hard to see L is recognized by a pushdown automaton.

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Pushdown Automata – Formal Definition

Definition

A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Q is the set of states;
- Σ is the input alphabet;
- Γ is the stack alphabet;
- $\delta : Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma_\epsilon)$ is the transition function;
- $q_0 \in Q$ is the start state; and
- $F \subseteq Q$ is the accept states.

- Recall $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$ and $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$.
- We consider nondeterministic pushdown automata in the definition. It convers deterministic pushdown automata.
- Deterministic pushdown automata are strictly less powerful.
- For convenience, we often extend δ to $Q \times \Sigma_\epsilon \times \Gamma_\epsilon \to \mathcal{P}(Q \times \Gamma^*)$, i.e., allowing $a \in \Gamma_\epsilon$ in the stack to be replaced by $x \in \Gamma^*$.

Computation of Pushdown Automata

- A pushdown automaton $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts input w if w can be written as $w = w_1 w_2 \cdots w_m$ with $w_i \in \Sigma_\epsilon$ and there are sequences of states $r_0, r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots, s_m \in \Gamma^*$ (representing contents of the stack) such that

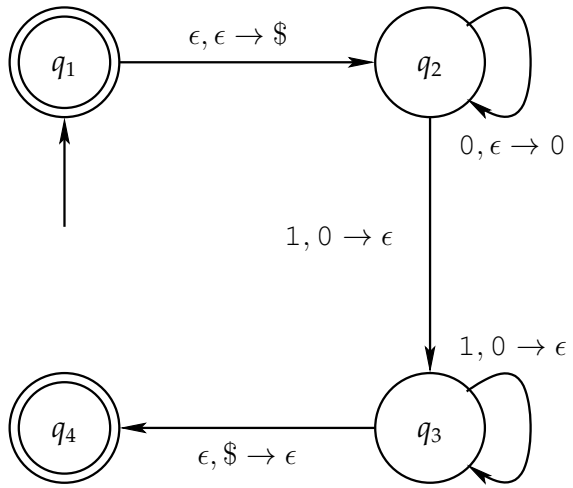
$$(r_0, s_0) \overset{w_1, -}{\to} (r_1, s_1) \cdots (r_i, at) \overset{w_{1+1},\, a \to b}{\to} (r_{1+1}, bt) \cdots \overset{w_m, -}{\to} (r_m, s_m)$$

  where
  - $r_0 = q_0$ and $s_0 = \epsilon$;
  - For $0 \le i < m$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, $s_i = at$, and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$ and $t \in \Gamma^*$.
    - ★ On reading w_{i+1}, M moves from r_i with stack at to r_{i+1} with stack bt.
    - ★ Write $c, a \to b (c \in \Sigma_\epsilon$ and $a, b \in \Gamma_\epsilon)$ to denote that the machine is reading c from the input and replacing the top of stack a with b.
  - $r_m \in F$.
- The language recognized by M is denoted by $L(M)$.
  - That is, $L(M) = \{w : M \text{ accepts } w\}$.

Pushdown Automata – Example

- Let $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, F)$ where
  - $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0, 1\}$, $\Gamma = \{0, \$\}$, $F = \{q_1, q_4\}$; and
  - δ is the following table:

input	0			1			ϵ		
stack	0	$	ϵ	0	$	ϵ	0	$	ϵ
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \epsilon)\}$					
q_3				$\{(q_3, \epsilon)\}$				$\{(q_4, \epsilon)\}$	
q_4									



- $L(M_1) = \{0^n 1^n : n \geq 0\}$

Pushdown Automata – Example

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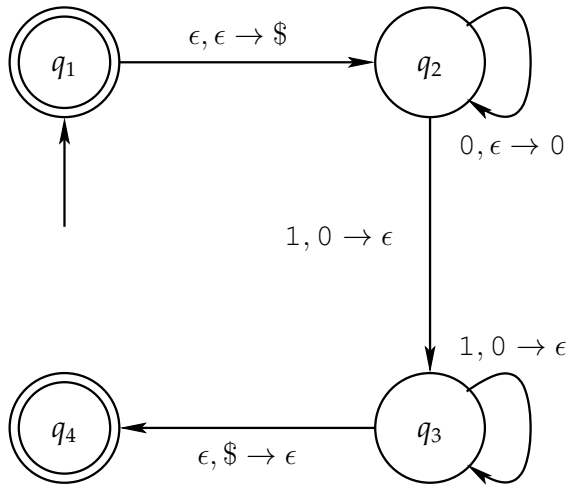
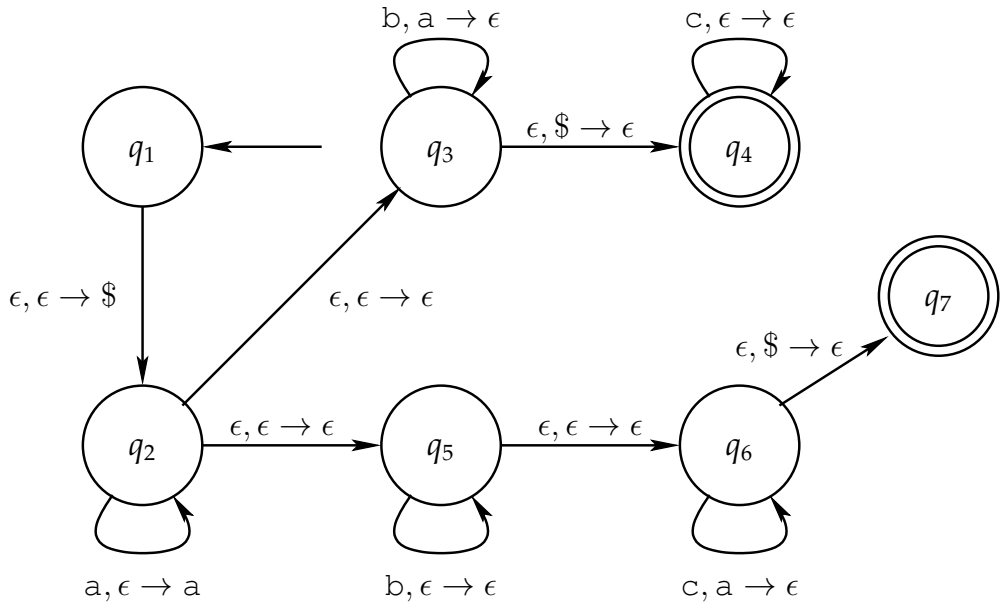
input	0			1			ϵ		
stack	0	$	ϵ	0	$	ϵ	0	$	ϵ
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \epsilon)\}$					
q_3				$\{(q_3, \epsilon)\}$				$\{(q_4, \epsilon)\}$	
q_4									



- $L(M_1) = \{0^n 1^n : n \geq 0\}$

Pushdown Automata – Example

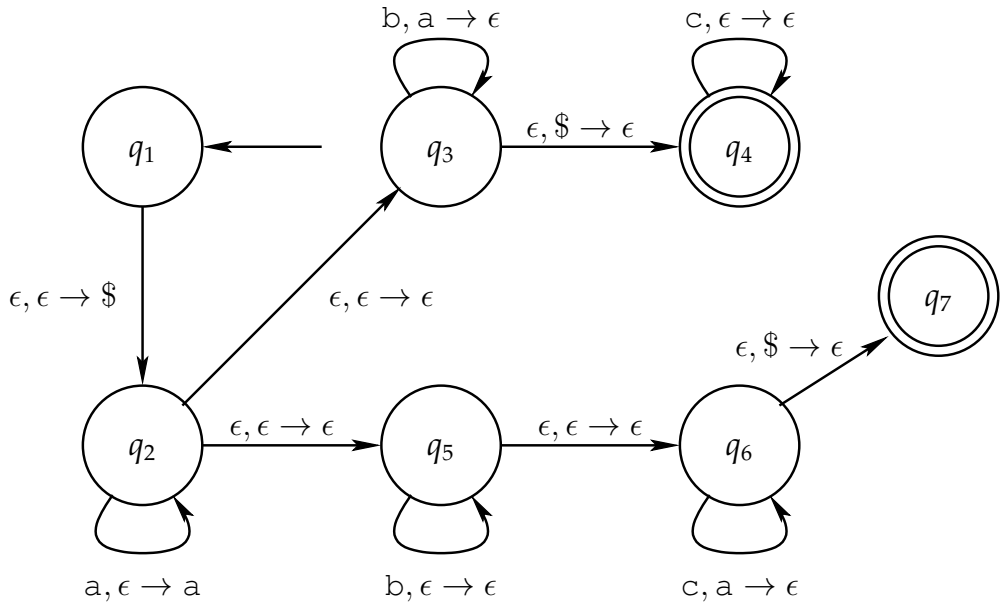
- Consider the following pushdown automaton M_2:



- $L(M_2) = \{a^i b^j c^k : i, j, k \geq 0 \text{ and, } i = j \text{ or } i = k\}$

Pushdown Automata – Example

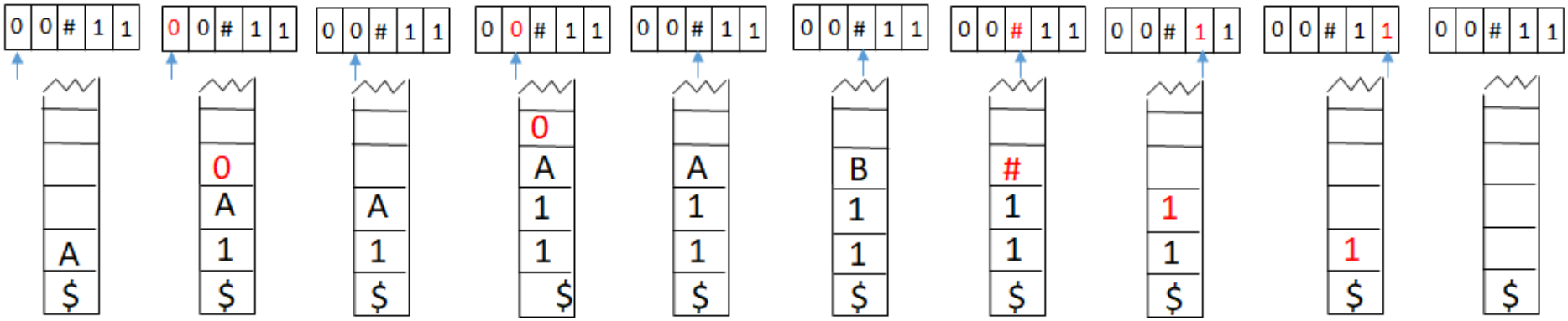
- Consider the following pushdown automaton M_2:



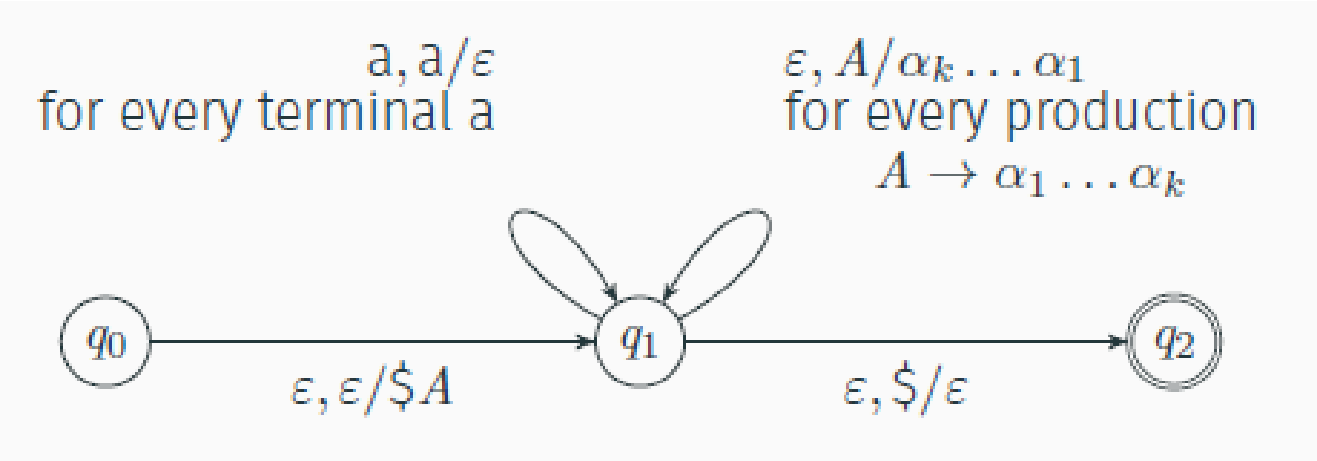
- $L(M_2) = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k : i, j, k \geq 0 \text{ and, } i = j \text{ or } i = k\}$

Context-Free Grammars ⇒ Pushdown Automata

- Idea: Use PDA to simulate derivations
- Example: $G : A \to 0A1 \mid B;\ \ B \to \#$
- Derivation: $A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 00B11 \Rightarrow 00\#11$
- Rule:
  - Write the start symbol A onto the stack
  - Rewrite variable on top of stack (in reverse) according to production
  - Pop top terminal if it matches input



Context-Free Grammars ⇒ Pushdown Automata



Note

- The above construction seems to suggest that the number of states in a PDA is not very ”important”.
- In fact, we can turn an arbitrary PDA into an equivalence one with a single state, such PDA are sometimes called ”stateless” PDA.
  - The ”state” information can be incorporated into a ”stack symbol”, in a way how a programming language handles the ”call-return” mechanism in a function call.

Context-Free Grammars ⇒ Pushdown Automata

Lemma

If a language is context-free, some pushdown automaton recognizes it.

Proof.

Let $G = (V, \Sigma, R, S)$ be a context-free grammar generating the language. Define
$P = (\{q_{\text{start}}, q_{\text{loop}}, q_{\text{accept}}, \ldots\}, \Sigma, V \cup \Sigma \cup \{\$\}, \delta, q_{\text{start}}, \{q_{\text{accept}}\})$ where

- $\delta(q_{\text{start}}, \epsilon, \epsilon) = \{(q_{\text{loop}}, S\$)\}$
- $\delta(q_{\text{loop}}, \epsilon, A) = \{(q_{\text{loop}}, w) : A \longrightarrow w \in R\}$
- $\delta(q_{\text{loop}}, a, a) = \{(q_{\text{loop}}, \epsilon)\}$
- $\delta(q_{\text{loop}}, \epsilon, \$) = \{(q_{\text{accept}}, \epsilon)\}$

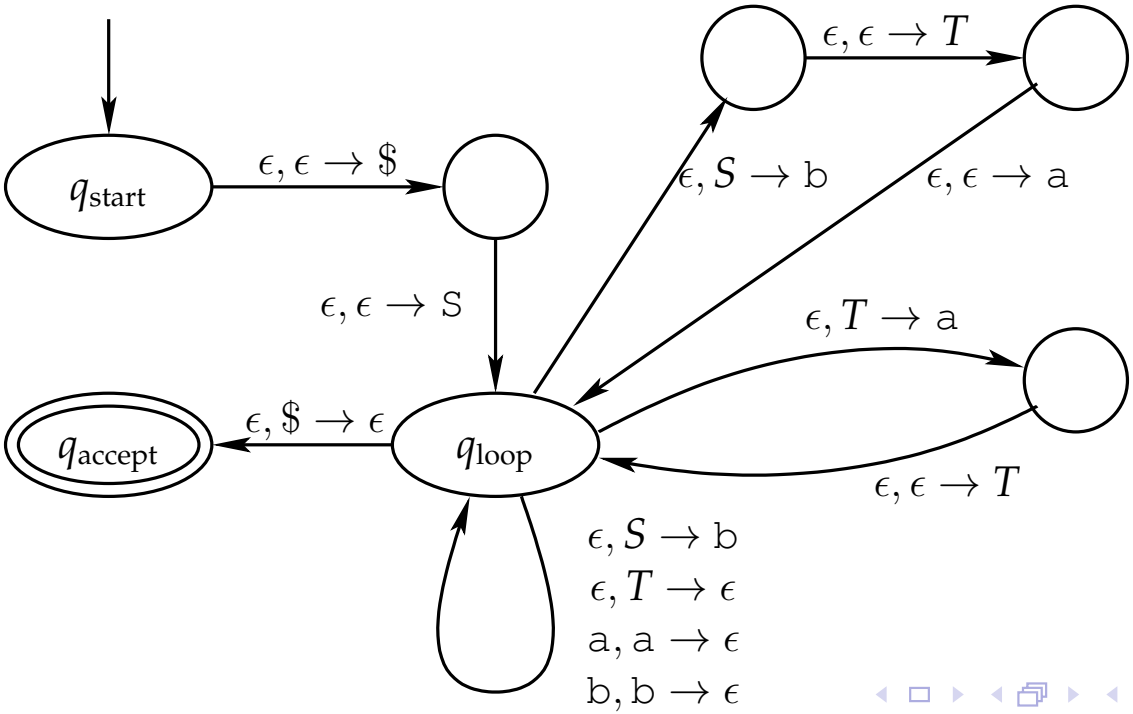
Note that $(r, u_1 u_2 \cdots u_l) \in \delta(q, a, s)$ is simulated by $(q_1, u_l) \in \delta(q, a, s)$, $\delta(q_1, \epsilon, \epsilon) = \{(q_2, u_{l-1})\}, \ldots, \delta(q_{l-1}, \epsilon, \epsilon) = \{(r, u_1)\}$. □

Example

Example

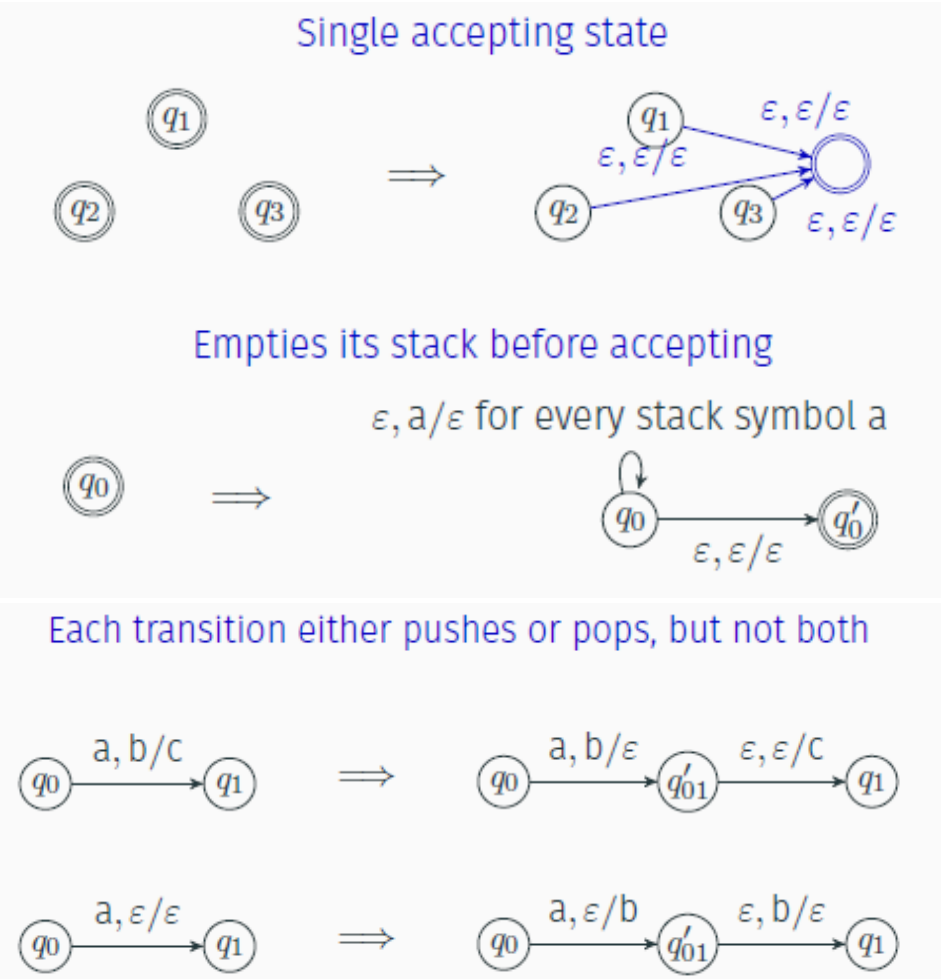
Find a pushdown automaton recognizing the language of the following context-free grammar:

$$\begin{aligned} S &\longrightarrow \mathtt{a}T\mathtt{b} \mid \mathtt{b} \\ T &\longrightarrow T\mathtt{a} \mid \epsilon \end{aligned}$$



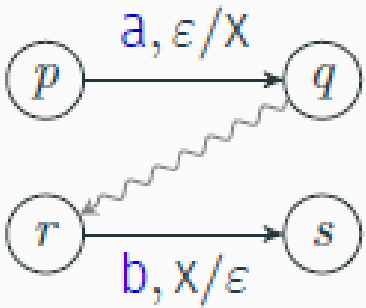
Simplified PDA

- Has a single accepting state
- Empties its stack before accepting
- Each transition is either a push, or a pop, but not both



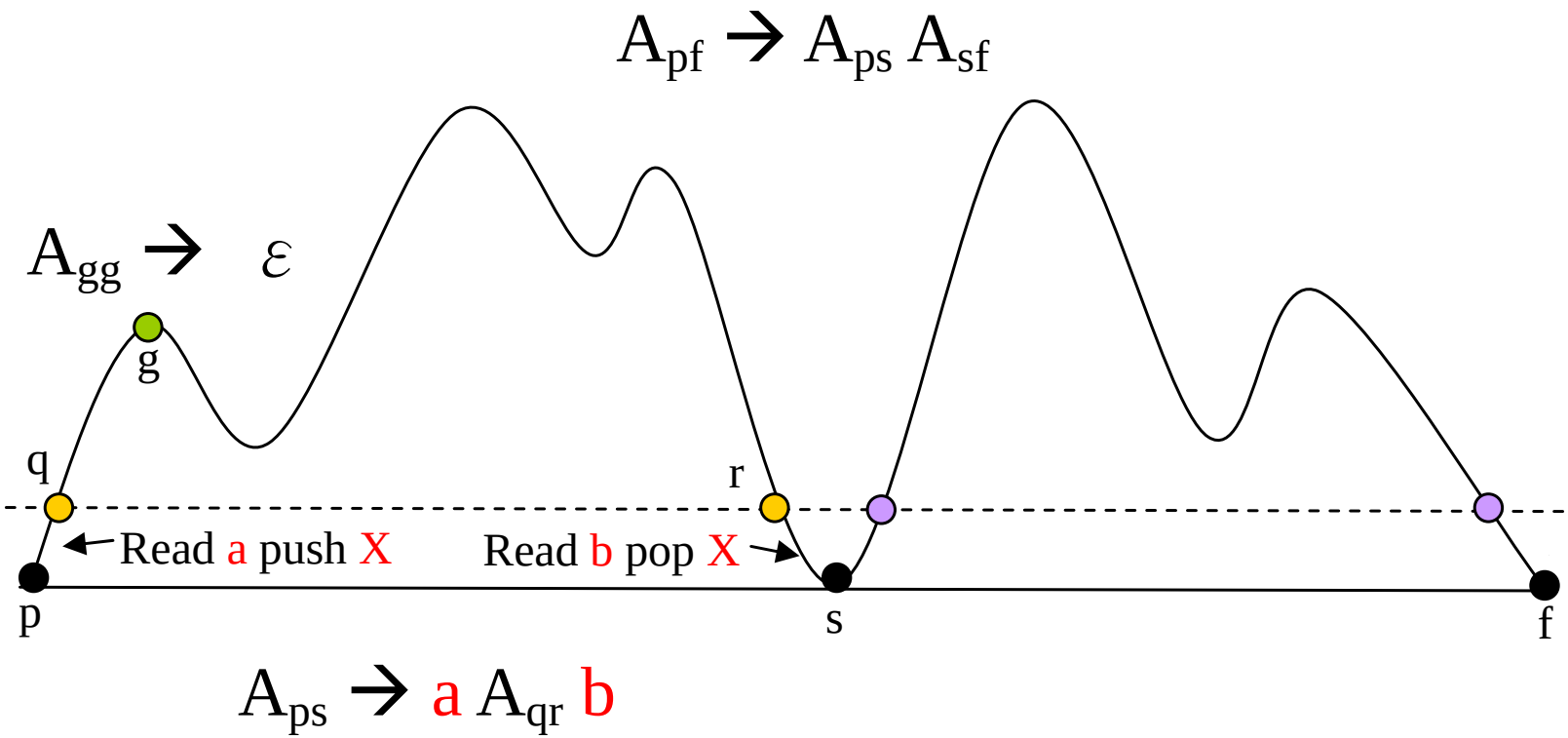
Pushdown Automata ⇒ Context-Free Grammars

- Key Idea: For every pair (q, r) of states in PDA, introduce variable A_{qr} in CFG so that
  - $A_{qr} \overset{*}{\Longrightarrow} w$ iff PDA goes from q to r reading w (with empty stack both at q and at r)

PDA	CFG
	$A_{qq} \to \varepsilon$
	$A_{pr} \to A_{pq}A_{qr}$
	$A_{ps} \to \mathrm{a}A_{qr}\mathrm{b}$ a = ε or b = ε allowed

Pushdown Automata ⇒ Context-Free Grammars

- Type 1: $A_{ps} \rightarrow aA_{qr}b$
- Type 2: $A_{pf} \rightarrow A_{ps}A_{sf}$
- Type 3: $A_{gg} \rightarrow \epsilon$



Pushdown Automata ⇒ Context-Free Grammars

Lemma

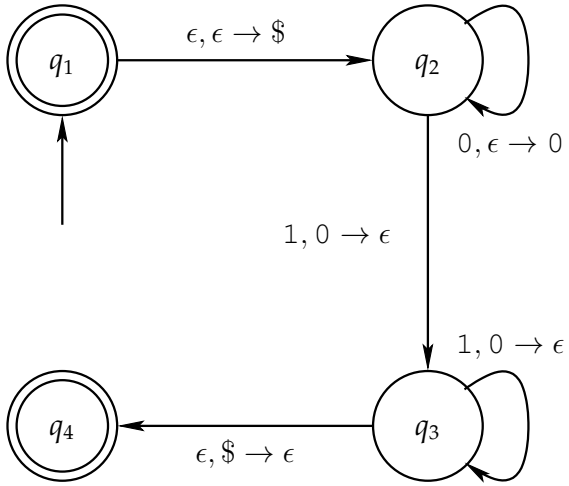
If a pushdown automaton recognizes a language, the language is context-free.

Proof.

Without loss of generality, we consider a pushdown automaton that has a single accept state q_{accept} and empties the stack before accepting. Moreover, its transition either pushes or pops a stack symbol at any time. Let $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$. Define the context-free grammar $G = (V, \Sigma, R, S)$ where

- $V = \{A_{pq} : p, q \in Q\}$, $S = A_{q_0, q_{\text{accept}}}$; and
- R has the following rules:
  - For each $p, q, r, s \in Q$, $t \in \Gamma$, and $a, b \in \Sigma_\epsilon$, if $(r, t) \in \delta(p, a, \epsilon)$ and $(q, \epsilon) \in \delta(s, b, t)$, then $A_{pq} \longrightarrow aA_{rs}b \in R$.
  - For each $p, q, r \in Q$, $A_{pq} \longrightarrow A_{pr}A_{rq} \in R$.
  - For each $p \in Q$, $A_{pp} \longrightarrow \epsilon \in R$.

Example



- We write $A_{i,j}$ for $A_{q_i q_j}$.
- Consider the following context-free grammar:

$$
\begin{array}{lcll}
A_{14} & \to & A_{23} & \text{since } (q_2, \$) \in \delta(q_1, \epsilon, \epsilon) \text{ and } (q_4, \epsilon) \in \delta(q_3, \epsilon, \$) \\
A_{23} & \to & 0A_{23}1 & \text{since } (q_2, 0) \in \delta(q_2, 0, \epsilon) \text{ and } (q_3, \epsilon) \in \delta(q_3, 1, 0) \\
A_{23} & \to & 0A_{22}1 & \text{since } (q_2, 0) \in \delta(q_2, 0, \epsilon) \text{ and } (q_3, \epsilon) \in \delta(q_2, 1, 0) \\
A_{22} & \to & \epsilon &
\end{array}
$$

Pushdown Automata $\Rightarrow$ Context-Free Grammars

Lemma

If A_{pq} generates x in G, then x can bring P from p with empty stack to q with empty stack.

Proof.

Prove by induction on the length k of derivation.

- $k = 1$. The only possible derivation of length 1 is $A_{pp} \Rightarrow \epsilon$.
- Consider $A_{pq} \overset{*}{\Longrightarrow} x$ of length $k+1$. Two cases for the first step:
  - $A_{pq} \Rightarrow aA_{rs}b$. Then $x = ayb$ with $A_{rs} \overset{*}{\Longrightarrow} y$. By IH, y brings P from r to s with empty stack. Moreover, $(r, t) \in \delta(p, a, \epsilon)$ and $(q, \epsilon) \in \delta(s, b, t)$ since $A_{pq} \longrightarrow aA_{rs}b \in R$. Let P start from p with empty stack, P first moves to r and pushes t to the stack after reading a. It then moves to s with t in the stack. Finally, P moves to q with empty stack after reading b and popping t.
  - $A_{pq} \Rightarrow A_{pr}A_{rq}$. Then $x = yz$ with $A_{pr} \overset{*}{\Longrightarrow} y$ and $A_{rq} \overset{*}{\Longrightarrow} z$. By IH, P moves from p to r, and then r to q. $\square$

Pushdown Automata $\Rightarrow$ Context-Free Grammars

Lemma

If x can bring P from p with empty stack to q with empty stack, A_{pq} generates x in G.

Proof.

Prove by induction on the length k of computation.

- $k = 0$. The only possible 0-step computation is to stay at the same state while reading ϵ. Hence $x = \epsilon$. Clearly, $A_{pp} \stackrel{*}{\Longrightarrow} \epsilon$ in G.
- Two possible cases for computation of length $k + 1$.
  - The stack is empty only at the beginning and end of the computation. If P reads a, pushes t, and moves to r from p at step 1, $(r, t) \in \delta(q, a, \epsilon)$. Similarly, if P reads b, pops t, and moves to q from s at step $k + 1$, $(q, \epsilon) \in \delta(s, b, t)$. Hence $A_{pq} \longrightarrow aA_{rs}b \in G$. Let $x = ayb$. By IH, $A_{rs} \stackrel{*}{\Longrightarrow} y$. We have $A_{pq} \stackrel{*}{\Longrightarrow} x$.
  - The stack is empty elsewhere. Let r be a state where the stack becomes empty. Say y and z are the inputs read during the computation from p to r and r to q respectively. Hence $x = yz$. By IH, $A_{pr} \stackrel{*}{\Longrightarrow} y$ and $A_{rq} \stackrel{*}{\Longrightarrow} z$. Since $A_{pq} \longrightarrow A_{pr}A_{rq} \in G$. We have $A_{pq} \stackrel{*}{\Longrightarrow} x$. $\square$

Context-Free Grammars and Pushdown Automata

Theorem

A language is context-free if and only if some pushdown automaton recognizes it.

Corollary

Every regular language is context-free.

- When we say PDA, we mean "nondeterministic PDA"
- Deterministic PDA (DPDA) are less powerful, they only accept deterministic context-free languages (DCFL).
- DPDA cannot accept $\{ww^R \mid w \in \{0,1\}^*\}$ or $\overline{\{a^n b^n c^n \mid n \geq 0\}}$.
- The equivalence problem is undecidable for PDA, yet it is decidable for DPDA.
- $\textit{Regular} \subsetneq \textit{DCFL} \subsetneq \textit{CFL}$
- DCFLs are not closed under union, intersection, concatenation, but are closed under complement.

Pumping Lemma for CFLs

Theorem

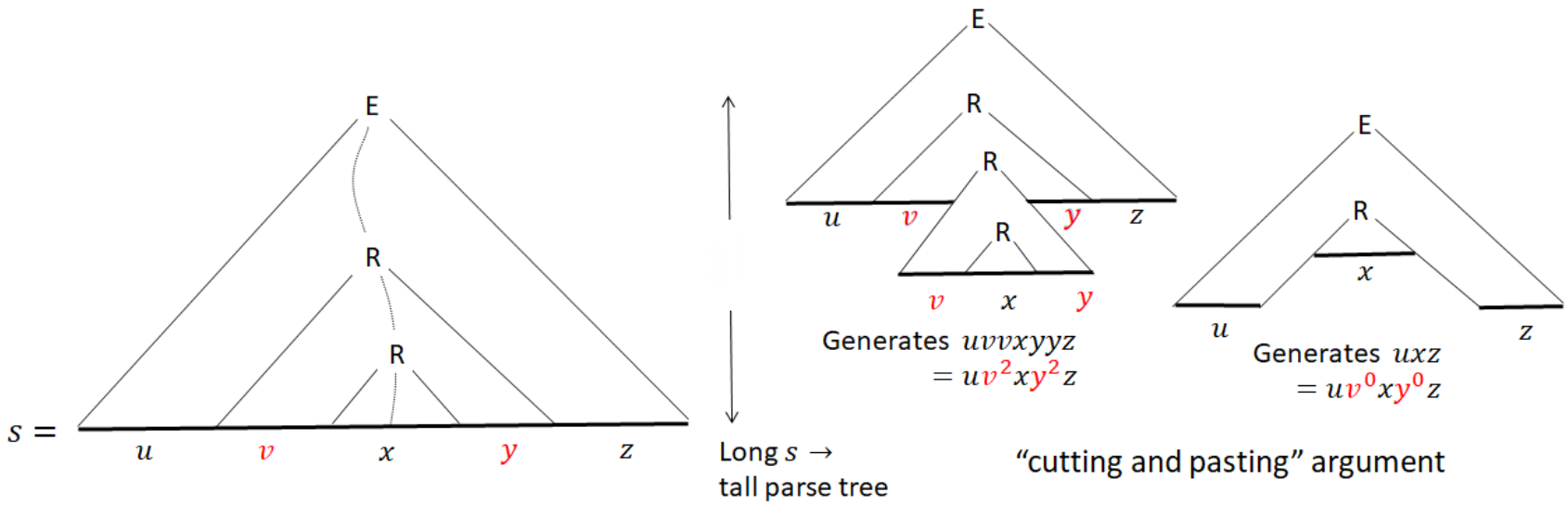
If A is a context-free language, then there is a number p (the puming length) such that for every $s \in A$ with $|s| \geq p$, there is a partition $s = uvxyz$ that

1. *for each $i \geq 0$, $uv^i xy^i z \in A$;*
2. *$|vy| > 0$; and*
3. *$|vxy| \leq p$.*

Proof.

Let $G = (V, \Sigma, R, T)$ be a context-free grammar for A. Define b to be the maximum number of symbols in the right-hand side of a rule. Observe that a parse tree of height h has at most b^h leaves and hence can generate strings of length at most b^h.
Choose $p = b^{|V|+1}$. Let $s \in A$ with $|s| \geq p$ and τ the smallest parse tree for s. Then the height of $\tau \geq |V| + 1$. There are $|V| + 1$ variables along the longest branch. A variable R must appear twice.

Pumping Lemma for CFLs



(Fig. from M. Sipser's class notes)

Proof. (cont'd).

From Figure (a), we see $uv^ixy^iz \in A$ for $i \geq 0$.
Suppose $|vy| = 0$. Then Figure (b) is a smaller parse tree than τ. A contradiction. Hence $|vy| > 0$.
Finally, recall R is in the longest branch of length $|V| + 1$. Hence the subtree R generating vxy has height $\leq |V| + 1$. $|vxy| \leq b^{|V|+1} = p$. □

Pumping Lemma – Examples

Example

Show $B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n : n \geq 0\}$ is not a context-free language.

Proof.

Let p be the pumping length. $s = \overbrace{\mathtt{a}^p\mathtt{b}^p\mathtt{c}^p}^{uvxyz} \in B$. Consider a partition $s = uvxyz$ with $|vy| > 0$. There are two cases:

- v or y contain more than one type of symbol, e.g., $\overbrace{aaaa}^{u}\overbrace{aab}^{v}\overbrace{bbbbbccccc}^{xyz}$. Then $uv^2xy^2z \notin B$.
- v and y contain only one type of symbol, e.g., $\overbrace{aaa}^{u}\overbrace{aa}^{v}\overbrace{ab}^{x}\overbrace{bb}^{y}\overbrace{bbbccccc}^{z}$. Then one of $\mathtt{a}$, $\mathtt{b}$, or $\mathtt{c}$ does not appear in v nor y (pigeon hole principle). Hence $uv^2xy^2z \notin B$ for $|vy| > 0$. □

Pumping Lemma – Examples

Example

Show $C = \{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k : 0 \le i \le j \le k\}$ is not a context-free language.

Proof.

Let p be the pumping length and $s = \mathtt{a}^p\mathtt{b}^p\mathtt{c}^p \in C$. Consider any partition $s = uvxyz$ with $|vy| > 0$. There are two cases:

- v or y contain more than one type of symbol. Then $uv^2xy^2z \notin C$.
- v and y contain only one type of symbol. Then one of $\mathtt{a}$, $\mathtt{b}$, or $\mathtt{c}$ does not appear in v nor y. We have three subcases:
  - $\mathtt{a}$ does not appear in v nor y. $uxz \notin C$ for it has more $\mathtt{a}$ then $\mathtt{b}$ or $\mathtt{c}$.
  - $\mathtt{b}$ does not appear in v nor y. Since $|vy| > 0$, $\mathtt{a}$ or $\mathtt{c}$ must appear in v or y. If $\mathtt{a}$ appears, $uv^2xy^2z \notin C$ for it has more $\mathtt{a}$ than $\mathtt{b}$. If $\mathtt{c}$ appears, $uxy \notin C$ for it has more $\mathtt{b}$ than $\mathtt{c}$.
  - $\mathtt{c}$ does not appear in v nor y. $uv^2xy^2z \notin C$ for it has less $\mathtt{c}$ than $\mathtt{a}$ or $\mathtt{b}$.

□

Pumping Lemma – Examples

Example

Show $D = \{ww : w \in \{0,1\}^*\}$ is not a context-free language.

Proof.

Let p be the pumping length and $s = 0^p 1^p 0^p 1^p$. Consider a partition $s = uvxyz$ with $|vy| > 0$ and $|vxy| \leq p$.

If $0 \cdots 0 \overbrace{0 \cdots 0 1 \cdots 1}^{vxy} 1 \cdots 1 0^p 1^p$, uv^2xy^2z moves 1 into the second half and thus $uv^2xy^2z \notin D$. Similarly, uv^2xy^2z moves 0 into the first half if $0^p 1^p 0 \cdots 0 \overbrace{0 \cdots 0 1 \cdots 1}^{vxy} 1 \cdots 1$.

It remains to consider $0^p 1 \cdots 1 \overbrace{1 \cdots 1 0 \cdots 0}^{vxy} 0 \cdots 0 1^p$. But then $uxz = 0^p 1^i 0^j 1^p$ with $i < p$ or $j < p$ for $|vy| > 0$. $uxz \notin D$. □

Non-Decision Properties

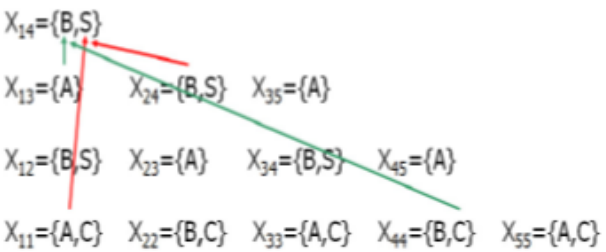
- Many questions that can be decided for regular sets cannot be decided for CFLs.
- Example: Are two CFLs the same?
- Example: Are two CFLs disjoint?
- Need theory of Turing machines and decidability to prove no algorithm exists.

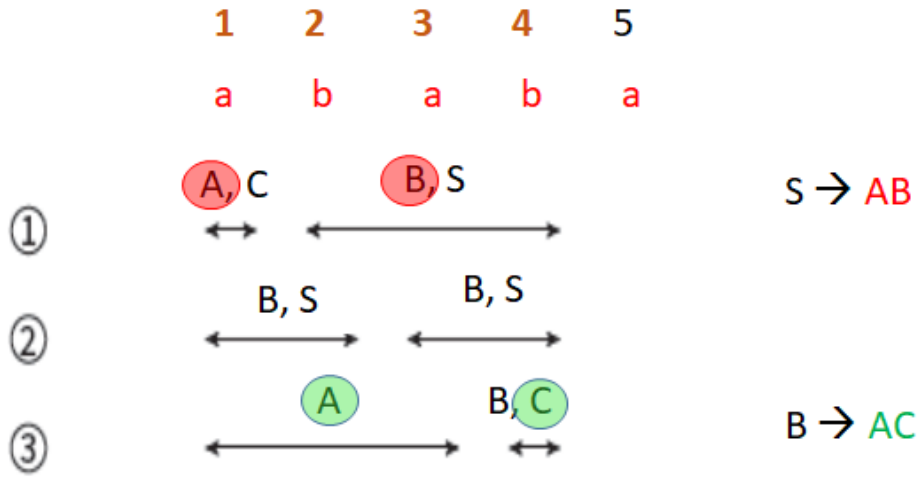
Testing Membership (Cocke-Younger-Kasami Algo.)

- Test "$w = a_1 \ldots a_n \in L(G)$?", assuming G in CNF.
- Algorithm (CYK) is a good example of dynamic programming and runs in time $O(n^3)$, where $n = |w|$.
  - We construct an n-by-n lower triangular array of sets of variables.
  - $X_{ij} = \{A \mid A \stackrel{*}{\Longrightarrow} w_{i,j}\}$, where $w_{i,j} = a_i \cdots a_j$. Finally, ask if $S \in X_{1n}$.
- Basis: $X_{ii} = \{A \mid A \to a_i \text{ is a production}\}$
- To compute X_{ij} inductively, try all possible ways of splitting $a_i \ldots a_j$ into substrings.

Example

Grammar: $S \to AB$, $A \to BC \mid a$, $B \to AC \mid b$, $C \to a \mid b$
String $w = ababa$





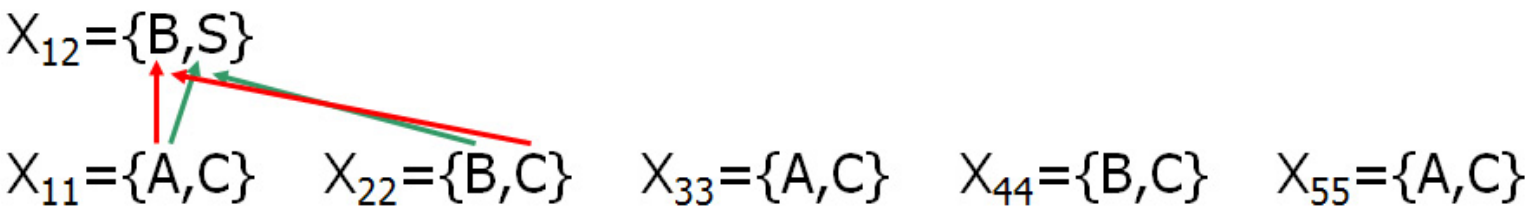
CYK Algorithm V (2)

- Basis: $X_{ii} = \{A \mid A \to a_i \text{ is a production }\}$.
- Induction: $X_{ij} = \{A \mid \text{ there is a production } A \to BC \text{ and an integer } k, i < k < j, B \in X_{ik}, C \in X_{k+1,j}\}$.

Example

Grammar: $S \to AB, \quad A \to BC \mid a, \quad B \to AC \mid b, \quad C \to a \mid b$
String $w = ababa$

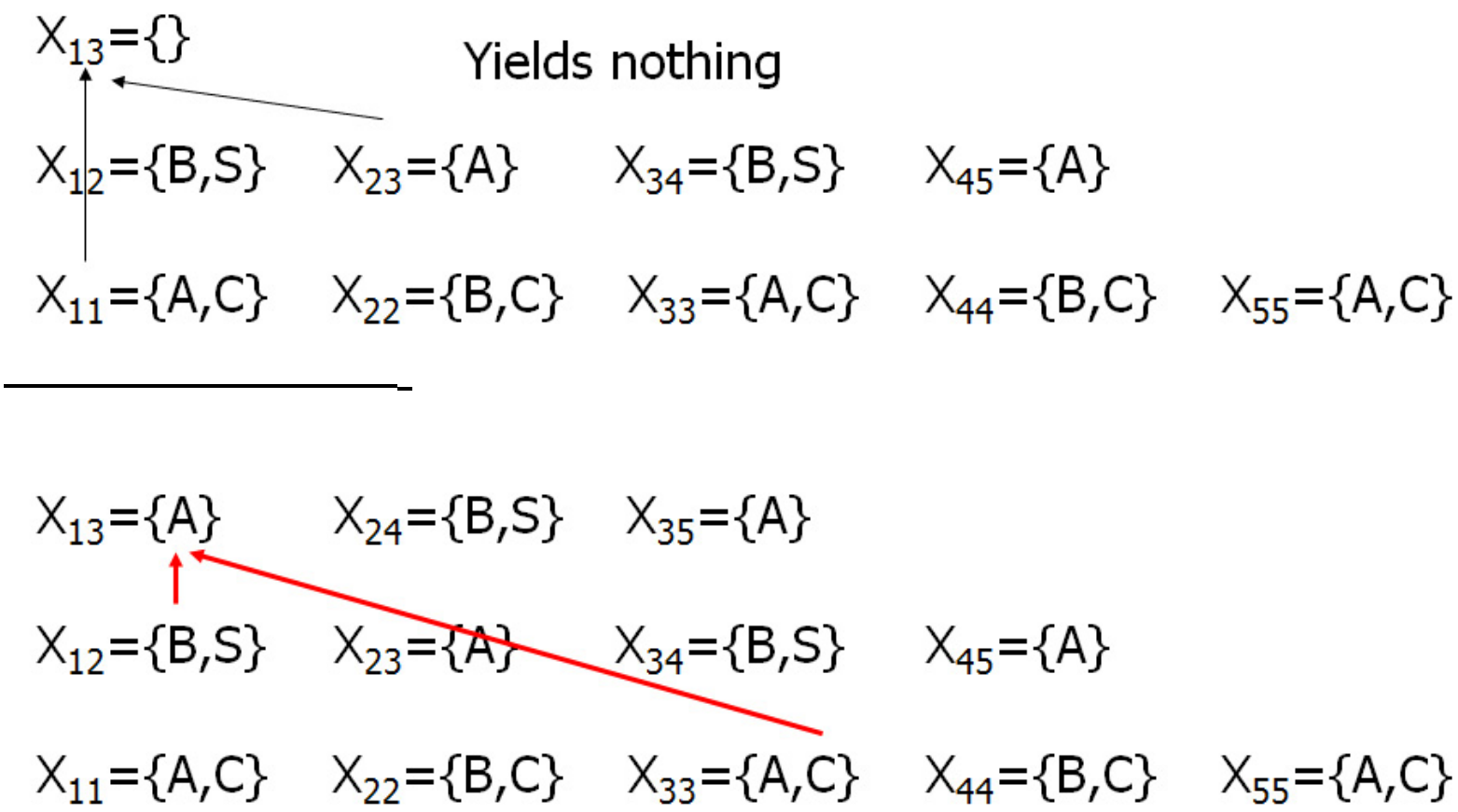
X_{11}={A,C} X_{22}={B,C} X_{33}={A,C} X_{44}={B,C} X_{55}={A,C}



Example (cont'd)

Example

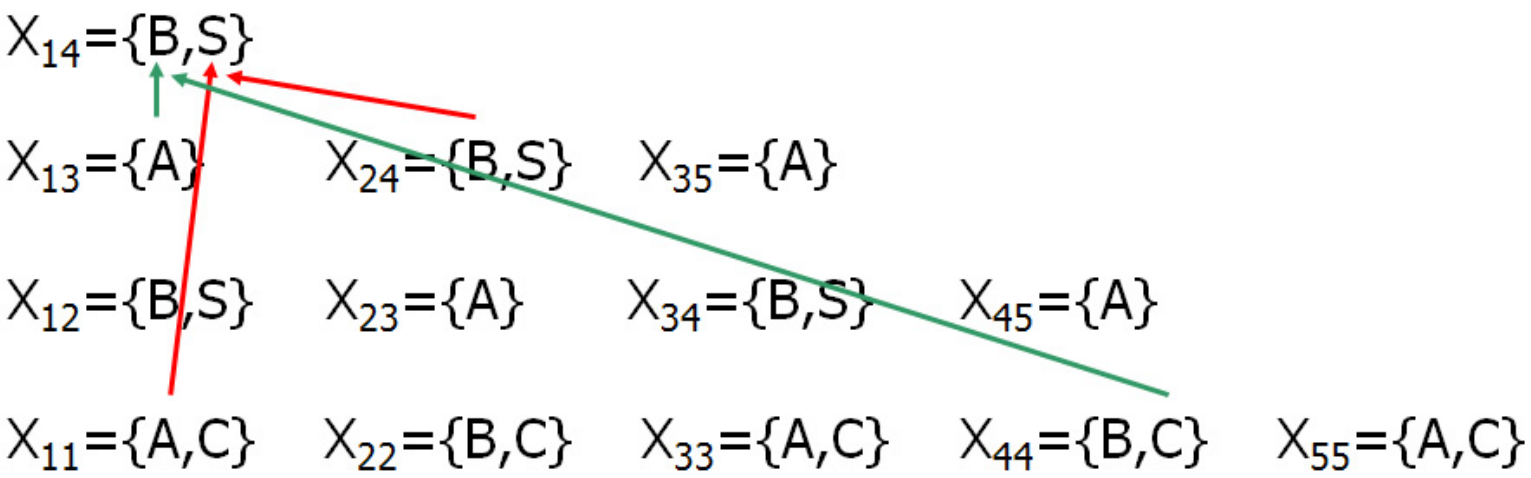
Grammar: $S \to AB, \quad A \to BC \mid a, \quad B \to AC \mid b, \quad C \to a \mid b$
String $w = ababa$



Example (cont'd)

Example

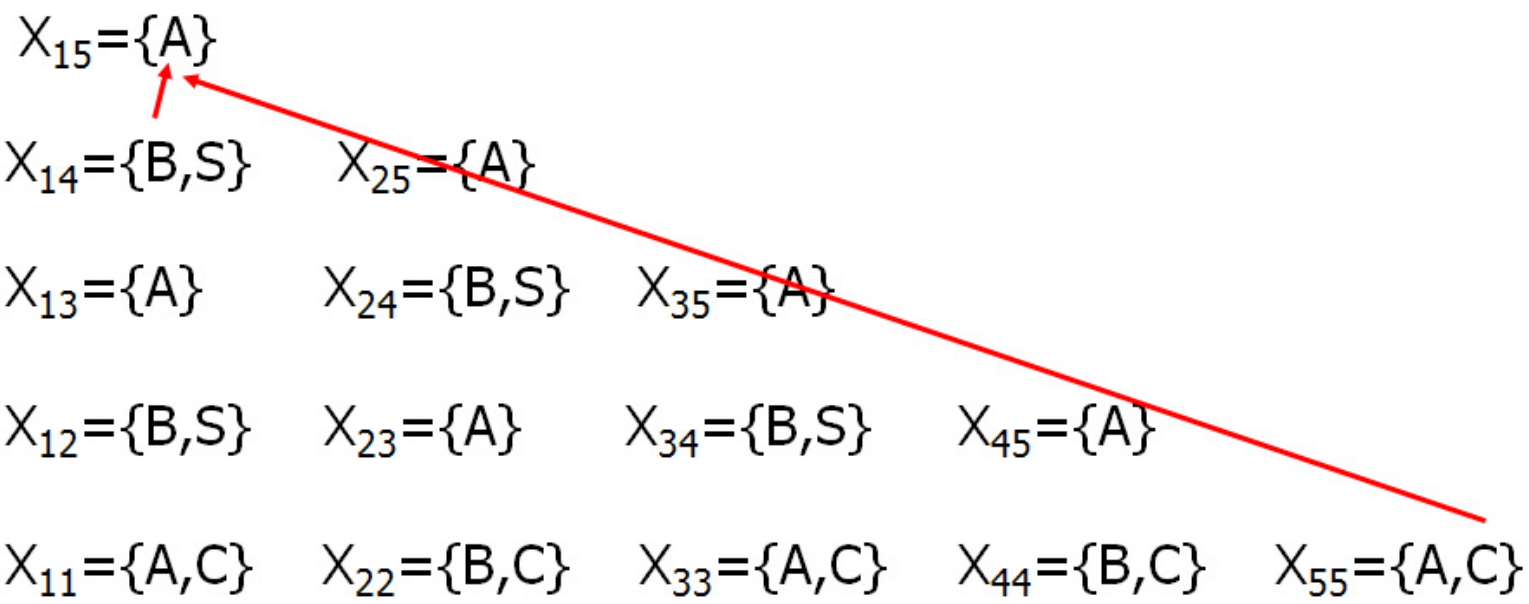
Grammar: $S \to AB, \quad A \to BC \mid a, \quad B \to AC \mid b, \quad C \to a \mid b$
String $w = ababa$



Example (cont'd)

Example

Grammar: $S \to AB, \quad A \to BC \mid a, \quad B \to AC \mid b, \quad C \to a \mid b$
String $w = ababa$



Testing Emptiness

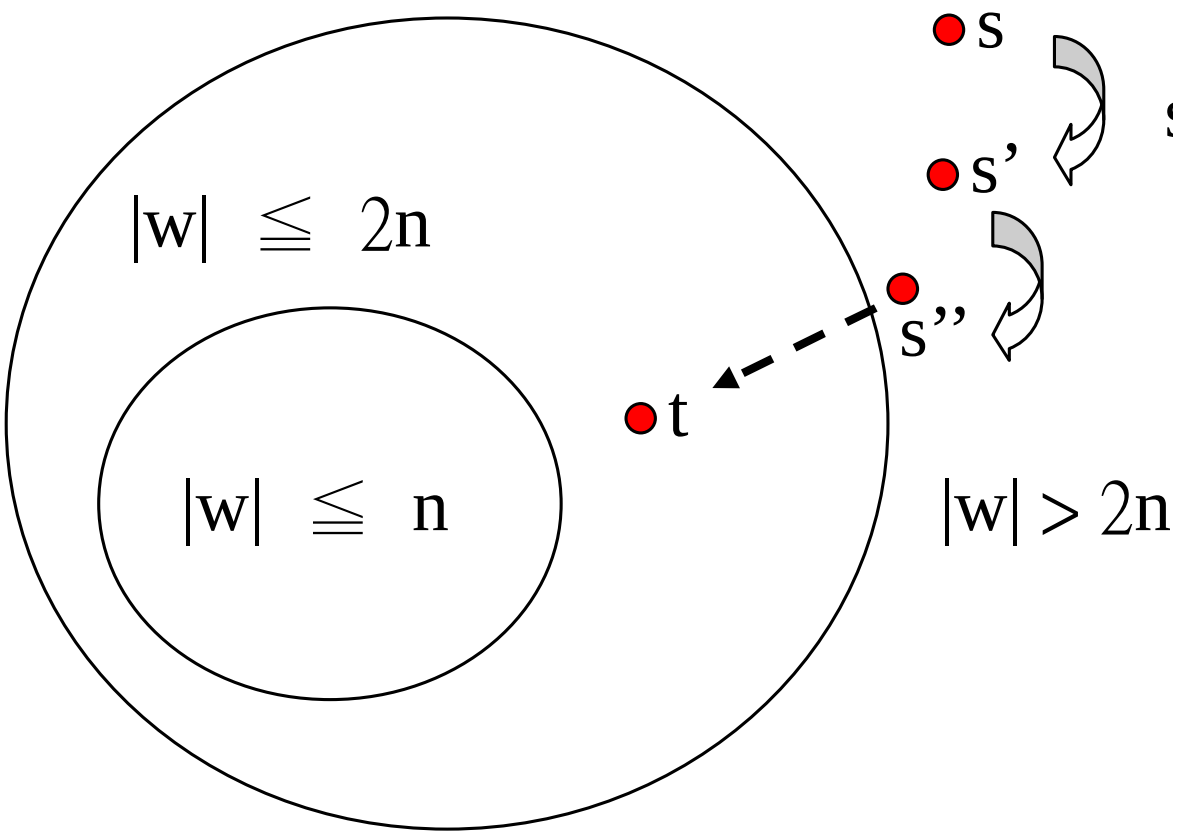
Given a CFG $G = (V, \Sigma, P, S)$ in CNF, construct a set $T = \{A \mid A \overset{*}{\Longrightarrow} w, w \in \Sigma^*\}$ iteratively in the following way:

1. Let $T = \{A \mid A \to a \in P, a \in \Sigma_\epsilon\}$.
2. For all rules $B \to CD \in P$, if $C, D \in T$, then $T = T \cup \{B\}$.
3. Repeat Step (2) until no more variable is added to T.

Claim: $S \in T$ iff $L(G) \neq \emptyset$.

Testing Infiniteness

- The idea is essentially the same as for regular languages.
- Use the pumping lemma constant n. If there is a string in the language of length between n and $2n - 1$, then the language is infinite; otherwise not.



Closure Properties of CFLs

- CFLs are closed under union, concatenation, and Kleene closure.
- Also, under reversal, homomorphisms and inverse homomorphisms.
- But not under intersection or difference.

Closure of CFLs Under Reversal

- If L is a CFL with grammar G, form a grammar for L^R by reversing the right side of every production.
- Example: Let G have $S \to 0S1 \mid 01$.
- The reversal of $L(G)$ has grammar $S \to 1S0 \mid 10$.

Closure of CFLs Under Homomorphism

- Let L be a CFL with grammar G.
- Let h be a homomorphism on the terminal symbols of G.
- Construct a grammar for $h(L)$ by replacing each terminal symbol a by $h(a)$.

Example

G has productions $S \to 0S1 \mid 01$. h is defined by $h(0) = ab, h(1) = \epsilon$. $h(L(G))$ has the grammar with productions $S \to abS \mid ab$.

Closure of CFLs Under Inverse Homomorphism

- Here, grammars don't help us.
- But a PDA construction serves nicely.
- Intuition: Let $L = L(P)$ for some PDA P.
- Construct PDA P' to accept $h^{-1}(L)$.
- P' simulates P, but keeps, as one component of a two-component state a buffer that holds the result of applying h to one input symbol.

Closure of CFLs Under Inverse Homomorphism

Consider a homomorphism $h(0) = aba, h(1) = bc$. Suppose PDA P accepts $ababc$. The following is the way how P' accepts $h^{-1}(ababc) = 01$.

- Each state of P' is of the form $[q, z]$, where q is a state of P and $z \in \{a, b, c\}^*$.
- P' starts in state $[q_0, \epsilon]$, upon reading input 0, P' moves to $[q_0, aba]$; then simulate P's computation on aba as follows
  - $[q_0, \epsilon] \overset{0,\epsilon\to\epsilon}{\to} [q_0, aba] \overset{\epsilon,\alpha\to\beta}{\to} [q_1, ba] \overset{\epsilon}{\to} [q_2, a] \overset{\epsilon,-}{\to} [q_3, \epsilon]$
  - in the above, $[q_0, aba] \overset{\epsilon,\alpha\to\beta}{\to} [q_1, ba]$ simulates $(q_1, \alpha \to \beta) \in \delta(q_0, a)$ (i.e., specified in the transition function of P).
- in state $[q_3, \epsilon]$, upon reading input 1, P' moves to $[q_3, bc]$; then simulate P's computation on bc as follows
  - $[q_3, \epsilon] \overset{1,\epsilon\to\epsilon}{\to} [q_3, bc] \overset{\epsilon,-}{\to} [q_4, c] \overset{\epsilon,-}{\to} [q_5, \epsilon]$, where q_5 is an accept state.
- Hence, P' accepts 01.
- P' updates the stack in the same way as P does.

Construction of P'

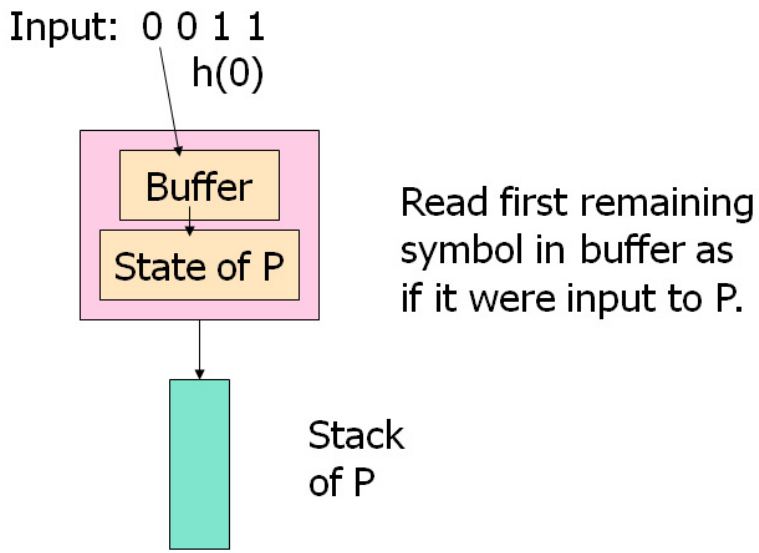
- States are pairs $[q, b]$, where:
  1. q is a state of P.
  2. b is a suffix of $h(a)$ for some symbol a.

  Thus, only a finite number of possible values for b.

- Stack symbols of P' are those of P.

- Start state of P' is $[q_0, \epsilon]$.
- Input symbols of P' are the symbols to which h applies.

- Final states of P' are the states $[q, \epsilon]$ such that q is a final state of P.



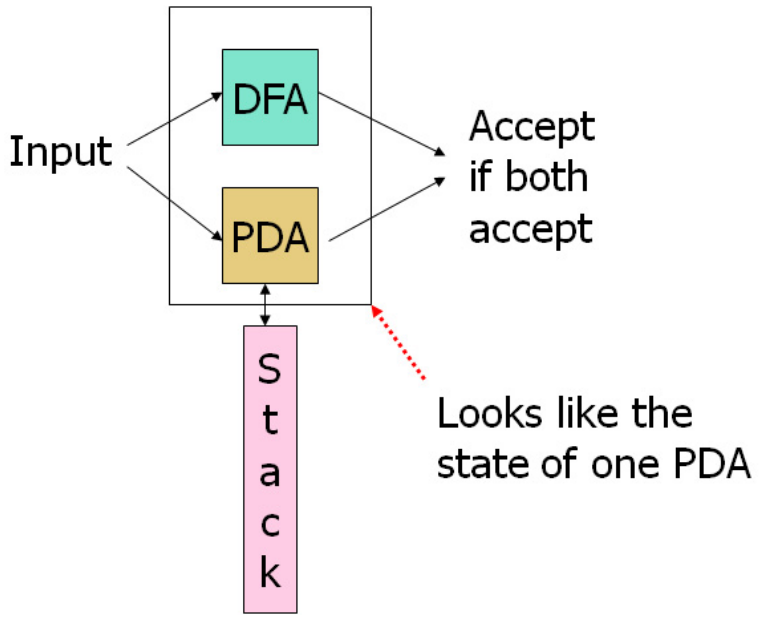
Transitions of P'

1. $\delta'(([q, \epsilon], a, X) = \{([q, h(a)], X)\}$ for any input symbol a of P' and any stack symbol X.
   - When the buffer is empty, P' can reload it.

2. $\delta'([q, bw], \epsilon, X)$ contains $([p, w], \alpha)$ if $\delta(q, b, X)$ contains (p, α), where b is either an input symbol of P or ϵ.
   - Simulate P from the buffer.

Intersection with a Regular Language

- Intersection of two CFL's need not be context free.
- But the intersection of a CFL with a regular language is always a CFL.
- Proof involves running a DFA in parallel with a PDA, and noting that the combination is a PDA. (PDAs accept by final state.)



Formal Construction

- Let the DFA A have transition function δ_A.
- Let the PDA P have transition function δ_P.
- States of combined PDA are $[q, p]$, where q is a state of A and p a state of P.
- $\delta([q, p], a, X)$ contains $([\delta_A(q, a), r], \alpha)$ if $\delta_P(p, a, X)$ contains (r, α). Note a could be ϵ, in which case $\delta_A(q, a) = q$.
- Accepting states of combined PDA are those $[q, p]$ such that q is an accepting state of A and p is an accepting state of P.
- Easy induction: $([q_0, p_0], w, Z_0) \overset{*}{\vdash} ([q, p], \epsilon, \alpha)$ if and only if $\delta_A(q_0, w) = q$ and in P : $(p_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, \alpha)$.

$\{x^n y^n \mid n \geq 0\} \cup \{x^n y^{2n} \mid n \geq 0\}$ Not a DPDA Language

Theorem

CFL $L = \{x^n y^n \mid n \geq 0\} \cup \{x^n y^{2n} \mid n \geq 0\}$ *cannot be acceptable by a DPDA.*

Proof.

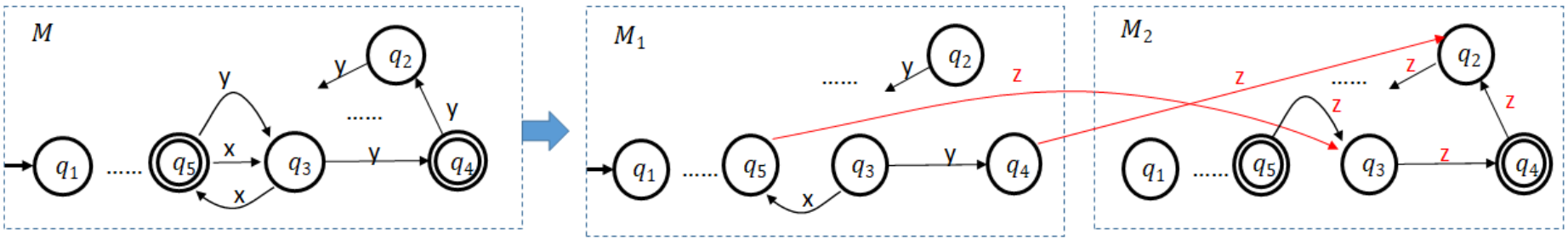
Assume, otherwise, that DPDA M accepts L. We construct a new DPDA M_0 which consists of "two modified copies" M_1 and M_2 of M in the following way:

- the initial state of M_0 is the initial state of M_1, and the final states of M_0 are the final states of M_2,
- remove all x transitions from M_2,
- replace all the y transitions of M_2 with z transitions,
- for each y transition emanating from an accept state of M_1, remove that transition and add a z transition to its "copy" in M_2,
- remove all x transitions emanating from accept states of M_1,
- update on the stack remains unchanged.

$\{x^n y^n \mid n \geq 0\} \cup \{x^n y^{2n} \mid n \geq 0\}$ Not a DPDA Language

Proof.

- Claim: $L(M_0) \subseteq x^* y^* z^*$
  - The prefix before entering M_2 must be accepted by M_1. Hence, the prefix must be of the form $x^n y^n$ or $x^n y^{2n}$.
- If M_0 accepts $x^n y^n z^i$ $(i \geq 1)$, then M must accept $x^n y^{n+i}$, only possible if $i = n$.
- Hence, M_0 accepts $\{x^n y^n z^n \mid n \geq 1\}$ – a contradiction.



Deterministic Context-Free Languages

- Deterministic PDA: in state q reading a with x at top of stack, at most one transition can apply.
  - if $q \overset{\epsilon,\epsilon\to\epsilon}{\to} p$ exists, it is the only transition in q;
  - $q \overset{a,x\to y}{\to} p$ cannot co-exist with either $q \overset{a,\epsilon\to z}{\to} p'$ or $q \overset{a,x\to z}{\to} p''$.

Theorem

$Reg \subsetneq DCFL \subsetneq CFL$

Theorem

- *DCFLs are closed under complement, union/intersection with regular languages.*
- *DCFLs are not closed under union, intersection, concatenation.*

Closure Properties of DCFL

Proof.

- (Complementation - YES) [Proof Idea] swap accept/non-accept states but need to make sure that DPDA reads the entire string.
  - $L = \{a^n b^n c^n \mid n \geq 0\}$ is not CF, yet its complement $\overline{L}$ is CF (Why?). So $\overline{\{a^n b^n c^n \mid n \geq 0\}}$ is NOT a DCFL.
- (Union/Concatenation with Regular - YES): Proof IdeaDPDA $\times$ DFA $\rightarrow$ DPDA.
- (Union - NO) $\overline{L}$ =
  1. $\{a^i b^j c^k \mid i \neq j\} \cup \{a^i b^j c^k \mid i \neq k\} \cup \{a^i b^j c^k \mid j \neq k\} \cup$
  2. $\{$anything with $ba, cb, ca\}$

  Each of the above four sub-languages is DCFL. However, (1) is not DCFL; otherwise, $\overline{L}$ is DCFL.
- (Intersection - NO): $L \cup M = \overline{\overline{L} \cap \overline{M}}$.

Closure Properties of DCFL

Proof.

- (Concatenation - NO):
  - Let $L_1 = \{a^i b^j c^k \mid i \neq j\}$ and $L_2 = \{a^i b^j c^k \mid j \neq k\}$; both are DCFLs.
  - $L_3 = 0L_1 \cup L_2$ is a DCFL.
  - Claim: $A = 0^* L_3$ is not a DCFL.
    - Suppose A is a DCFL. Then $A \cap 0a^*b^*c^*$ is a DCFL.
    - $A \cap 0a^*b^*c^* = 0L_1 \cup 0L_2$ (a DCFL) implies $L_1 \cup L_2$ is a DCFL. However, $L_1 \cup L_2 = \overline{\overline{L_1} \cap \overline{L_2}} = \overline{\{a^n b^n c^n \mid n \geq 0\}} - \{\text{anything with } ba, cb, ca\}$
    - As $\overline{\{a^n b^n c^n \mid n \geq 0\}}$ is not a DCFL, $L_1 \cup L_2$ is not a DCFL – a contradiction.

□

Note:

- If L is a DCFL and R is regular, LR is always a DCFL but RL may not be a DCFL
- DCFLs are interesting as they are closed under complementation, but not closed under union, intersection, concatenation.

Using Idea behind PDA $\Rightarrow$ CFG to Show Closure with Regular Sets

Theorem

CFLs are closed under intersection with regular languages.

Proof.

Let $G = (V, \Sigma, R, S)$ be a CFG in CNF and $N = (Q, \Sigma, \delta, q_0, \{q_f\})$ be an NFA with a unique accept state. We construct $G' = (V', \Sigma, R', S')$ as follows. A variable in V' is of the form (q_i, A, q_j), where $q_i, q_j \in Q, A \in V$

- $S' = (q_0, S, q_f)$,
- $(q_i, S, q_j) \to \epsilon$ if $S \to \epsilon$ in R, and $q_j \in \delta(q_i, \epsilon)$,
- $(q_i, A, q_j) \to a$ if $A \to a$ in R, and $q_j \in \delta(q_i, a)$,
- $(q_i, A, q_j) \to (q_i, B, q_k)(q_k, C, q_j)$ if $A \to BC$ in R, $\forall q_k \in Q$.

Claim: $(q_i, A, q_j) \overset{*}{\Longrightarrow} w$ in G' iff $q_i \overset{w}{\to} q_j$ in N and $A \overset{*}{\Longrightarrow} w$ in G. $\square$