Linear Algebra

Fall 2024, Quiz # 1

Oct. 4, 2024

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1. (30 pts) True or False? No explanation required. No penalty for wrong answer.

(1) Given a matrix A, if $Ax = 0$ only has a trivial solution (i.e., $x = 0$), then the linear transformation T_A associated with A is one-to-one.
Sol: True. If $Au = Av = b, u \neq v$, then $A(u - v) = 0$ and $u - v \neq 0$.

(2) If A is a 3×5 matrix, then the linear transformation T_A associated with A cannot be one-to-one.
Sol: True. $rank(A) + nullity(A) = 5; rank(A) \leq 3$, hence, $nullity(A) > 0$

(3) A square matrix A is invertible if and only if the rows of A are linearly independent.
Sol: True. $Ax = 0$ only has the trivial solution.

(4) A linear system with fewer equations than unknowns must have infinitely many solutions.
Sol: False. $x + y + z = 3; x + y + z = 2$ do not have a solution.

(5) If A in an $n \times m$ matrix, then $rank(A) \leq n$.
Sol: True. $Rank(A) =$ number of independent rows.

(6) If a square matrix has two equal rows, then it is not invertible.
Sol: True. RREF has a zero row.

(7) Suppose that A and B are $n \times n$ invertible matrices. Then $A^T B$ is invertible.
Sol: true. $(A^T B)^{-1} = B^{-1}(A^{-1})^T$

(8) Let A be a 3×3 singular matrix (i.e., not invertible). Then there exists a nonzero 3×3 matrix B such that $AB = O$, where O denotes the 3×3 zero matrix.
Sol: True. $Ax = 0$ has a nonzero solution x. Hence $A[x, 0, 0] = O$, where $B = [x, 0, 0]$ is a 3×3 matrix whose 1st column is x and 2nd and 3rd are 0.

(9) If v_1, v_2, v_3 are any three vectors in $\mathbb{R}^4$, then they cannot span $\mathbb{R}^4$.
Sol: True. R^4 needs at least 4 vectors to span.

(10) If A is any 5×4 matrix, then the equation $Ax = 0$ must have a non-trivial (i.e., $x \neq 0$) solution.
Sol: False. Might not have any nontrivial solution.

(11) There is a 2×3 matrix A and a 3×2 matrix B such that BA is the identity matrix I_3.
Sol: False. $R(BA) \leq min\{Rank(A), Rank(B)\} \leq 2$. $Rank(I_3) = 3$.

(12) Let A be an $n \times n$ matrix, and v_1 and v_2 be two $n \times 1$ vectors. If v_1 and v_2 are linearly independent, then Av_1 and Av_2 are also linearly independent.
Sol: False. Consider $A = O$, i.e., the zero matrix.

(13) The nullity of the matrix $\begin{bmatrix} 2 & 3 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 3.
Sol: True. There are only 2 pivots. Hence, nullity=3.

(14) If A and B are two $n \times n$ invertible matrices, then $(A + B)^{-1} = A^{-1} + B^{-1}$.
Sol: False. $A + -A = \mathrm{O}$ is not invertible.

(15) The matrix for a 90^o counter-clockwise rotation in the x-y plane is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
Sol: True. $M(e_1) = (0, 1)^T = e_2$, $M(e_2) = (-1, 0)^T = -e_1$

2. (20 pts) Consider the following system of linear equations.

$$\begin{aligned} 3x + 2y - 5z &= 1 \\ 4x - y + z &= 0 \\ x - z &= 2 \end{aligned}$$

Find all solutions by using Gauss elimination procedure to transform the augmented matrix of the above to its RREF. Show your work in sufficient detail.

Sol. Apply elementary row operations to the augmented matrix

$$A = \begin{bmatrix} 3 & 2 & -5 & 1 \\ 4 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} \Rightarrow \cdots \Rightarrow \begin{bmatrix} 1 & 2/3 & & -5/3 & 1/3 \\ 0 & 1 & -5 & 8 & \\ 0 & 0 & 1 & -21/8 & \end{bmatrix}, \text{ so } (x, y, z) = (\frac{-5}{8}, \frac{-41}{8}, \frac{-21}{8})$$

3. (10 pts) If u and v are two linearly independent vectors in $\mathbb{R}^n$, prove that the two vectors $u + v$ and $u - v$ are linearly independent.

Sol. Consider $a(u + v) + b(u - v) = 0$, which is $(a + b)u + (a - b)v = 0$. As u and v are linearly independent, we have $a+b = 0$ and $a-b = 0$. Hence $a = 0, b = 0$, which implies $u+v$ and $u-v$ are linearly independent.

4. (20 pts) Are the following 4 transformations linear? Give a brief explanation.

$T_1(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} x - y \\ 3 \end{bmatrix}$, $T_2(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} x - y \\ y \end{bmatrix}$, $T_3(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} \cos(45^o) \\ \sin(45^o) \end{bmatrix} + y \begin{bmatrix} -\sin(45^o) \\ \cos(45^o) \end{bmatrix}$

$T_4(\begin{bmatrix} x \\ y \end{bmatrix}) = x^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sol.

- T_1: No; $T_1(0) \neq 0$.
- T_2: Yes; Can be written in a matrix form. $= \begin{bmatrix} 8 & -7 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- T_3: Yes; Can be written in a matrix form. $= \begin{bmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
- T_4: Yes. Can be written in a matrix form. $= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

5. (20 pts) Is the following matrix invertible? If yes, compute its inverse. Show your work in detail.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

Sol. Apply elementary row operations to

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \cdots \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right]$$

Then we get $A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$