Linear Algebra

Fall 2024, Homework # 4

Due: Dec. 6, 2024

1. (20 pts) The characteristic polynomial of a 4×4 matrix A is $p(\lambda) = (1-\lambda)(\lambda^2 - 2)(3-\lambda)$. Compute $det(A)$ and $det(A^2 + A)$. (Hint: first argue that A is diagonalizable, find a diagonal matrix D similar to A, and then argue that $A^2 + A$ is similar to $D^2 + D$.)

2. (20 pts) Let P_2 denote the vector space of polynomials of degree less than or equal to 2. Determine whether the linear transformation $T : P_2 \to P_2$ given by $T(p(x)) = p(x-3)$ is diagonalizable or not. (Note, for example, $T(x^2 - 1) = (x-3)^2 - 1$.)

3. (20 pts) Suppose a square matrix A satisfies $A^T = -A$.

   (a) Show that $I - A$ is always invertible.
   (Hint: Show that if $(I - A)x = 0$, then $x = 0$.)

   (b) Show that $Q = (I - A)^{-1}(I + A)$ is an orthogonal matrix.
   (Hint: Show that $QQ^T = I$. In your derivation, you may want to use the fact that $(I - A)(I + A) = (I + A)(I - A)$.)

4. (15 pts) Construct an orthogonal basis of $\mathbb{R}^2$ for the non-standard inner product

$$\langle x, y \rangle = x^T \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} y.$$

   (Hint: Starting with $v_1 = (1, 0)^T, v_2 = (0, 1)^T$, apply Gram-Schmidt process.)

5. (25 pts) Let T be a linear transformation from P_2 to P_2 defined as

$$T(f) = 2f' + f'',$$

   where f' and f'' are the first and second derivatives of f, respectively.

   (a) Let $\mathbb{B} = (1, t, t^2)$ be the standard basis of P_2. Find the matrix representation B of T with respect to the basis $\mathbb{B}$.

   (b) Find a basis for the null space of T and a basis for the range (i.e., image) of T.

   (c) Write down the characteristic equation for the matrix B. Find the eigenvalues of B.

   (d) For each eigenvalue, find a basis for the corresponding eigenspace.

   (e) Is the matrix B diagonalizable?