Linear Algebra

Fall 2024, Homework # 3

Due: Nov. 15, 2024

1. (20 pts) For the following matrix,

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{bmatrix}$$

   (a) (6 pts) calculate the characteristic polynomial of A,

   (b) (4 pts) find the eigenvalues of A,

   (c) (5 pts) find a basis for each eigenspace of A, (For simplicity, choose a basis not to have a fraction)

   (d) (5 pts) determine whether A is diagonizable. If yes, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

   Show your work in sufficient detail.

2. (20 pts) Suppose that $\{v_1, v_2, ..., v_k\}$ form a basis for the null space of a square matrix A and that $C = B^{-1}AB$ for some invertible matrix B. Find a basis for the null space of C. (Hint: you must show your "basis" to be linearly independent, and also spans the null space of C.)

3. (20 pts)

   (a) (10 pts) Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Recall that these are the roots of the characteristic polynomial of A, defined as $f(\lambda) = det(A - \lambda I)$. Show that the determinant of A is equal to the product of its eigenvalues, i.e.

$$det(A) = \lambda_1 \times \lambda_2 ... \times \lambda_n.$$

   (Hint: Consider the highest order term of the characteristic polynomial and $f(0)$. Pay special attention to the +/- sign of the highest order term, i.e., λ^n, of the characteristic polynomial.)

   (b) (10 pts) Given an $n \times n$ matrix $A = vv^T$, where v is an $n \times 1$ column vector. Find an eigenvalue and an eigenvector of A.

4. (20 pts) For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}$ $\quad y = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and let $W = span\{w_1, w_2\}$.

   (a) Find a basis for W consisting of two orthogonal vectors

   (b) Express y as the sum of a vector in W and a vector in $W^\perp$. That is, find a $w \in W$ and $w' \in W^\perp$ such that $y = w + w'$.

   Show your work in sufficient detail.

5. (20 pts) Consider the following matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and vector $b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$.

   (a) Find the orthogonal projection of b onto $Col(A)$.

   (b) Find a least square solution of $Ax = b$.

   Show your work in sufficient detail.