Linear Algebra

Fall 2024, Homework # 1

Due: Oct. 4, 2024

1. (10 pts) Consider the following system of linear equations, where c is a real number:

$$
\begin{aligned}
x_1 + x_2 \;\; + cx_3 + x_4 \;\; &= c \\
- x_2 \;\; + x_3 \;\; + 2x_4 &= 0 \\
x_1 + 2x_2 + x_3 \;\; - x_4 \;\; &= -c
\end{aligned}
$$

   Apply elementary row operations to the augmented matrix of the above to yield its REF (Row Echelon Form). Note that the REF contains c as a parameter. For what c, does the linear system have a solution? Show your derivation in detail.

2. (15 pts) Find all possible values of $rank(A)$ as a varies

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

   (Hint:) Use elementary row operations ($R_2 + 2R_1$; $R_3 - aR_1$, where R_i represents row i) we obtain:

$$B = \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix}$$

   Then show your argument for each of the following cases:

   - (Case 1): $a = -1$, What is the rank? Why?
   - (Case 2): $a \neq -1$
     - (Case 2-1) $a = 2$, What is the rank? Why?
     - (Case 2-2) $a \neq 2$, What is the rank? Why?

   Note that in some of the above cases, you might have to apply elementary row operations again. Show your work in sufficient detail.

3. (10 pts) Find a quadratic polynomial, say $f(x) = ax^2 + bx + c$, such that $f(1) = 1, f(2) = 9, f(3) = 27$. To this end, find the values of a, b and c. Show your derivation in sufficient detail.

4. (10 pts) Given an $n \times n$ matrix $A = I - uu^T$, where I is the $n \times n$ identity matrix and u is a column vector in $\mathbb{R}^n$ with $u^T u = 1$ (note that this is just the number 1), prove that $AA = A$.

5. (10 pts) For what values of k, if any, is the vector b in the span of the columns of A?

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \\ -1 & 0 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \\ k-2 \\ 1 \end{bmatrix}$$

   Show your work in sufficient detail.

6. (15 pts) Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one vector in the set as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}$$

7. (10 pts) Let A and B be two $n \times n$ matrices such that AB is invertible.

   (a) Prove that B is invertible.

   (b) Prove that A is invertible.

   (Hint: You may use the following facts. A matrix C is NOT invertible if and only if $Cx = 0$ has a non-zero solution. Furthermore, if C and D are invertible, CD is also invertible.)

8. (10 pts) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

   (a) Find a 4×2 matrix B such that $AB = I_2$, where I_2 is the 2×2 identity matrix.

   (b) Explain why there is no 4×2 matrix C such that $CA = I_4$, where I_4 is the 4×4 identity matrix.

9. (10 pts) Find the inverse of the following matrix A by row reducing $[A|I]$.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{bmatrix}.$$