# Linear Algebra

Fall 2024, Homework # 4 Reference Solutions

Due: Dec. 6, 2024

1. (20 pts) The characteristic polynomial of a $4 \times 4$ matrix $A$ is $p(\lambda) = (1-\lambda)(\lambda^2 - 2)(3-\lambda)$. Compute $det(A)$ and $det(A^2 + A)$. (Hint: first argue that $A$ is diagonalizable, find a diagonal matrix $D$ similar to $A$, and then argue that $A^2 + A$ is similar to $D^2 + D$.)

   **Sol.** The characteristic polynomial of $A$ is $p(\lambda) = (1-\lambda)(\lambda^2-2)(3-\lambda)$, the four eigenvalues of $A$ are 1, $\pm\sqrt{2}$, and 3. So $det(A) = \lambda_1 \times \lambda_2 \times \lambda_3 \times \lambda_4 = 1 \times \sqrt{2} \times (-\sqrt{2}) \times 3 = -6$.

   Since the four eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ are distinct, $A$ is diagonalizable. Let $p_i$ be the eigenvector to eigenvalue $\lambda_i$.

   $$\text{Let } P = [p_1\ p_2\ p_3\ p_4] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}. \text{ Then } A = PDP^{-1}.$$

   $A^2 + A = (PD^2P^{-1} + PDP^{-1}) = P(D^2 + D)P^{-1}$ so $A^2 + A$ and $D^2 + D$ are similar and have the same determinant. $det(A^2+A) = det(D^2+D) = det(D(D+I)) = det(D) \times det(D+I) = 48$.

2. (20 pts) Let $P_2$ denote the vector space of polynomials of degree less than or equal to 2. Determine whether the linear transformation $T : P_2 \to P_2$ given by $T(p(x)) = p(x-3)$ is diagonalizable or not. (Note, for example, $T(x^2-1) = (x-3)^2 - 1$.)

   **Sol.** Let $p(x) = a_0 + a_1 x + a_2 x^2$.

   $$T(p(x)) = p(x-3) = a_0 + a_1(x-3) + a_2(x-3)^2 = (a_0 - 3a_1 + 9a_2) + (a_1 - 6a_2)x + a_2 x^2.$$

   $$\text{So } T = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \text{ and the characteristic polynomial is } f(\lambda) = det(T - \lambda I) = (1-\lambda)^3.$$

   There is one eigenvalue $\lambda_1 = 1$ with multiplicity 3. Let $p_1$ be the eigenvector corresponding to $\lambda_1$:

   $$(T - \lambda_1 I) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 9 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

   Since there is only one eigenvector, the dimension of the eigenspace (1) does not equal to the multiplicity of the eigenvalue (3). Therefore $T$ is not diagonalizable.

   *Note: You must state the correct reasons for your answer in order to get the full 20 points. If you omitted the reasoning or gave the wrong reasons (for example, using the sufficient conditions instead of the necessary conditions), some points will be de deducted.*

3. (20 pts) Suppose a square matrix $A$ satisfies $A^T = -A$.

   (a) Show that $I - A$ is always invertible.
   (Hint: Show that if $(I-A)x = 0$, then $x = 0$.)

   (b) Show that $Q = (I-A)^{-1}(I+A)$ is an orthogonal matrix.
   (Hint: Show that $QQ^T = I$. In your derivation, you may want to use the fact that $(I-A)(I+A) = (I+A)(I-A)$.)

   **Sol.**

(a) Suppose $(I-A)x=0$. Then $x-Ax=0 \Rightarrow x=Ax$
$x^Tx=(Ax)^Tx=x^TA^Tx=x^T(-A)x=-x^T(Ax)=-x^Tx \Rightarrow x^Tx=0 \Rightarrow x=\mathbf{0}$.
$\Rightarrow Null(I-A)=\{\mathbf{0}\} \Rightarrow I-A$ is invertible.

(b)

$$\begin{aligned}
QQ^T &= (I-A)^{-1}(I+A)\big((I-A)^{-1}(I+A)\big)^T \\
&= (I-A)^{-1}(I+A)(I+A)^T\big((I-A)^{-1}\big)^T \\
&= (I-A)^{-1}(I+A)(I+A^T)\big((I-A)^T\big)^{-1} \\
&= (I-A)^{-1}(I+A)(I-A)((I+A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\
&= I
\end{aligned}$$

4. (15 pts) Construct an orthogonal basis of $\mathbb{R}^2$ for the non-standard inner product

$$\langle x, y\rangle = x^T \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} y.$$

(Hint: Starting with $v_1=(1,0)^T, v_2=(0,1)^T$, apply Gram-Schmidt process.)

**Sol.**

Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $S=\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$. Apply Gram-Schimidt process:

$u_1 = v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1; \quad \langle v_2, u_1\rangle = [0\ 1] \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0; \quad u_2 = v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$S' = \{u_1, u_2\}$ is an orthogonal basis.

5. (25 pts) Let $T$ be a linear transformation from $P_2$ to $P_2$ defined as

$$T(f) = 2f' + f'',$$

where $f'$ and $f''$ are the first and second derivatives of $f$, respectively.

(a) Let $\mathbb{B} = (1, t, t^2)$ be the standard basis of $P_2$. Find the matrix representation $B$ of $T$ with respect to the basis $\mathbb{B}$.

(b) Find a basis for the null space of $T$ and a basis for the range (i.e., image) of $T$.

(c) Write down the characteristic equation for the matrix $B$. Find the eigenvalues of $B$.

(d) For each eigenvalue, find a basis for the corresponding eigenspace.

(e) Is the matrix $B$ diagonalizable?

**Sol.**

(a) $T(1)=0$, $T(t)=2$, $T(t^2)=4t+2$. So $B = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(b) Suppose $Bx = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So $\mathbb{B}' = \{1\}$, i.e., $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ is a basis for Null $T$.

The range of $T$ is $P_1$, for which $\{1, t\}$, i.e., $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ is a basis.

(c) The characteristic equation for $B$ is :

$$0 = det(B - \lambda I) = det\,(\begin{bmatrix} -\lambda & 2 & 2 \\ 0 & -\lambda & 4 \\ 0 & 0 & -\lambda \end{bmatrix}) = -\lambda^3$$

B has only one eigenvalue $\lambda_1 = 0$.

(d) Since $B - \lambda_1 I = B$, the null space of $B - \lambda_1 I$ is the same as that of $B$. From (b), $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$ is a basis for the eigenspace corresponding to $\lambda_1$ and the dimension of the eigenspace is 1.

(e) Since the multiplexity of $\lambda_1$ (which is 3) does not equal to the dimension of the corresponding eigenspace (which is 1), $B$ is not diagonalizable.