Linear Algebra

Fall 2024, Homework # 3 Reference Solutions

Due: Nov. 15, 2024

1. (20 pts) For the following matrix,

   (a) (6 pts) calculate the characteristic polynomial of A,

   (b) (4 pts) find the eigenvalues of A,

   (c) (5 pts) find a basis for each eigenspace of A, (For simplicity, choose a basis not to have a fraction)

   (d) (5 pts) determine whether A is diagonizable. If yes, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

   Show your work in sufficient detail.

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 3 & 5 & 3 \\ -1 & -1 & 1 \end{bmatrix}$$

Sol:

(a) The characteristic polynomial of A is

$$det(A-\lambda I_3) = det \begin{bmatrix} -1-\lambda & -3 & -3 \\ 3 & 5-\lambda & 3 \\ -1 & -1 & 1-\lambda \end{bmatrix} = det \begin{bmatrix} -1-\lambda & -3 & -3 \\ 0 & 2-\lambda & 6-3\lambda \\ -1 & -1 & 1-\lambda \end{bmatrix} = -(\lambda-1)(\lambda-2)^2$$

(b) The eigenvalues of A are the roots of the characteristic equation $det(A - \lambda I_3) = 0$. Therefore the two eigenvalues are 1 and 2.

(c)
- $\lambda = 1$:

$$A - \lambda I_3 = \begin{bmatrix} -2 & -3 & -3 \\ 3 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow \text{RREF} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3\,x_3 \\ -3\,x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

  So $[\,3, -3, 1\,]^T$ is a basis for the eigenspace of 1.

- $\lambda = 2$

$$A-\lambda I_3 = \begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \text{RREF} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

  So $\{\ [-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a basis for the eigenspace of 2.

(d) $P = \begin{bmatrix} 3 & -1 & -1 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. (20 pts) Suppose that $\{v_1, v_2, ..., v_k\}$ form a basis for the null space of a square matrix A and that $C = B^{-1}AB$ for some invertible matrix B. Find a basis for the null space of C. (Hint: you must show your "basis" to be linearly independent, and also spans the null space of C.)

Sol. $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ is a basis for the null space of C.

- $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ is linearly indepedent.
  Proof: Suppose $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ is linearly dependent, there exists a set of scalars $a_1, a_2, \ldots, a_k$ that are not all zeros such that

  $$\sum_{i=1}^{k} a_i B^{-1} v_i = \mathbf{0} \Rightarrow B^{-1} \sum_{i=1}^{k} a_i v_i = \mathbf{0} \Rightarrow x = \sum_{i=1}^{k} a_i v_i \text{ is in the null space of } B^{-1}.$$

  Since B^{-1} is invertible, the dimension of the null space of B^{-1} is 0 so $x = \mathbf{0}$. Since $\{v_1, v_2, ..., v_k\}$ form a basis, $v_1, v_2, ..., v_k$ are linearly independent. So $a_i = 0$ for all $i = 1, 2, \ldots, k$. A contradiction.

- $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ spans the null space of C.
  Proof:
  We first show that Span $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\} \subseteq$ Null C.
  Let $y = a_1 B^{-1} v_1 + \cdots + a_k B^{-1} v_k$ be a vector in the span of $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ where $a_i \in \mathbb{R}$ for $i = 1, \ldots k$. Let $x = a_1 v_1 + \cdots + a_k v_k$ then $y = B^{-1}x$. Clearly $Ax = \mathbf{0}$ since $x \in$ Null A.

  $$CB^{-1}x = (B^{-1}AB)B^{-1}x = B^{-1}A\,x = B^{-1}\mathbf{0} = \mathbf{0} \Rightarrow y = B^{-1}x \in \text{Null } C.$$

  We then show that Null $C \subseteq$ Span$\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$.
  Let $u \in$ Null C. Then $Cu = B^{-1}ABu = \mathbf{0}$ and $Bu \in$ Null A. Then

  $$u = B^{-1}Bu = B^{-1}(b_1 v_1 + \cdots b_k v_k) = b_1 B^{-1} v_1 + \cdots + b_k B^{-1} v_k \in \text{Span } \{B^{-1}v_1, \ldots, B^{-1}v_k\}$$

  Therefore $\{B^{-1}v_1, B^{-1}v_2, ..., B^{-1}v_k\}$ spans the null space of C.

3. (20 pts)

   (a) (10 pts) Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Recall that these are the roots of the characteristic polynomial of A, defined as $f(\lambda) = det(A - \lambda I)$. Show that the determinant of A is equal to the product of its eigenvalues, i.e.

   $$det(A) = \lambda_1 \times \lambda_2 ... \times \lambda_n.$$

   (Hint: Consider the highest order term of the characteristic polynomial and $f(0)$. Pay special attention to the +/- sign of the highest order term, i.e., λ^n, of the characteristic polynomial.)

   (b) (10 pts) Given an $n \times n$ matrix $A = vv^T$, where v is an $n \times 1$ column vector. Find an eigenvalue and an eigenvector of A.

   Sol.

   (a) The highest order term of $det(A - \lambda I)$ comes from the product of the n elements along the diagonal of the matrix $A - \lambda I$, that is, $(a_{11} - \lambda_1)(a_{22} - \lambda_2) \cdots (a_{nn} - \lambda_n)$. So the sign of the highest order term is $(-1)^n$. We write $f(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$. If we evaluate the characteristic polynomial at zero we get $f(0) = det(A - 0I) = detA = (-1)^n (0 - \lambda_1) \cdots (0 - \lambda_n) = \lambda_1 \times \lambda_2 ... \times \lambda_n$.

(b) Consider $(vv^T)v = v(v^Tv)$. Since v^Tv is a scalar, we have $Av = (v^Tv)v$. So v is an eigenvector of A with v^Tv being the corresponding eigenvalue.

4. (20 pts) For the space $\mathbb{R}^4$, let $w_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $w_2 = \begin{bmatrix}3\\3\\-1\\-1\end{bmatrix}$ $y = \begin{bmatrix}6\\0\\2\\0\end{bmatrix}$ and let $W = span\{w_1, w_2\}$.

(a) Find a basis for W consisting of two orthogonal vectors

(b) Express y as the sum of a vector in W and a vector in $W^\perp$. That is, find a $w \in W$ and $w' \in W^\perp$ such that $y = w + w'$.

Show your work in sufficient detail.

Sol.

(a) Let $\{v_1, v_2\}$ be a basis for W where $v_1 \perp v_2$. Use Gram-Schmidt Process:

$$v_1 = w_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad v_2 = w_2 - \frac{w_2 \cdot v_1}{||v_1||^2} v_1 = \begin{bmatrix}3\\3\\-1\\-1\end{bmatrix} - \frac{4}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\-2\\-2\end{bmatrix}$$

(b) Let w be the orthogonal projection of y on W. Then $w' = y - w \in W^\perp$.

$$w = U_W(y) = \frac{y \cdot v_1}{||v_1||^2} v_1 + \frac{y \cdot v_2}{||v_2||^2} v_2 = \frac{8}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{8}{16}\begin{bmatrix}2\\2\\-2\\-2\end{bmatrix} = \begin{bmatrix}3\\3\\1\\1\end{bmatrix}, \quad w' = y - 2 = \begin{bmatrix}3\\-3\\1\\-1\end{bmatrix}$$

5. (20 pts) Consider the following matrix $A = \begin{bmatrix}1 & 2\\-1 & 4\\1 & 2\end{bmatrix}$ and vector $b = \begin{bmatrix}3\\-1\\5\end{bmatrix}$.

(a) Find the orthogonal projection of b onto $Col(A)$.

(b) Find a least square solution of $Ax = b$.

Show your work in sufficient detail.

Sol:

(a) Let $W = Col(A)$. The orthogonal projection matrix $P_W = A(A^TA)^{-1}A^T = \begin{bmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\0 & 1 & 0\\\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$.

The orthogonal projection of b onto $Col(A)$ is $P_Wb = \begin{bmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\0 & 1 & 0\\\frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}3\\-1\\5\end{bmatrix} = \begin{bmatrix}4\\-1\\4\end{bmatrix}$.

(b) $(A^TA)^{-1}A^Tb = \begin{bmatrix}\frac{1}{3} & \frac{-1}{3} & \frac{1}{2}\\\frac{1}{12} & \frac{1}{6} & \frac{1}{12}\end{bmatrix}\begin{bmatrix}3\\-1\\5\end{bmatrix} = \begin{bmatrix}3\\\frac{1}{2}\end{bmatrix}$

(Alternative Solution)
A least square solution of $Ax = b$ is the solution of $Ax = P_Wb$.

$$\begin{bmatrix}1 & 2\\-1 & 4\\1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}4\\-1\\4\end{bmatrix} \Rightarrow \begin{cases}x_1 + 2x_2 = 4\\-x_1 + 4x_2 = -1\end{cases} \Rightarrow \begin{bmatrix}3\\\frac{1}{2}\end{bmatrix} \text{ is a least square solution.}$$