Linear Algebra

Fall 2024, Homework # 2 Reference Solutions

Due: Oct. 11, 2024

---

1. (20 pts) For which values of a, b, c is the following matrix invertible? Why?

$$A = \begin{bmatrix} 1 & 1 & a \\ 2 & 3 & b \\ 1 & 2 & c \end{bmatrix}$$

Sol. Apply cofactor expansion on column 3:

$$det(A) = a_{13}c_{13} + a_{23}c_{23} + a_{33}c_{33} = a - b + c$$

If $a - b + c \neq 0$ then $det(A) \neq 0$ and A is invertible.

2. (20 pts) Find the determinant of the following matrix. Show your work in sufficient detail.

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$$

Sol. Apply cofactor expansion on row 3 and then elementary row operations on rows 2 and 3:

$$det(A) = a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33} + a_{34}c_{34} = (-1)^{3+2} det(\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{bmatrix}) = -det(\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 2 & -2 \end{bmatrix}) = -det(\begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix}) = 4$$

.

3. (20 pts) Are there constants c_1 and c_2 such that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Why? Show your work in sufficient detail.

Sol.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 & 2c_1 \\ 0 & 2c_1 + c_2 \end{bmatrix}$$

which in turn yields the following 4 equations:

$$1 = c_1 + 2c_2 \quad (1)$$
$$2 = 2c_1 \quad (2)$$
$$0 = c_2 \quad (3)$$
$$1 = 2c_1 + c_2 \quad (4)$$

(2) and (3) contradict (4). So the answer is no.

4. (20 pts) Let A and B be two $n \times n$ matrices. Is the equation $(A+B)(A-B) = A^2 - B^2$ always true? Either prove it or find a counter-example.

Sol. $(A+B)(A-B) = A^2 - B^2 - AB + BA$. So if $AB \neq BA$, then $(A+B)(A-B) \neq A^2 - B^2$. For example, let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = BA$.

5. (20 pts) Prove that $det(\begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}) = (a-b)(b-c)(c-a)$

Sol. First perform elementary row operations on row 2 and 3. Then apply cofactor expansion on column 1:

$$det(\begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}) = det(\begin{bmatrix} 1 & a & bc \\ 0 & -(a-b) & c(a-b) \\ 0 & c-a & -b(c-a) \end{bmatrix}) = det(\begin{bmatrix} -(a-b) & c(a-b) \\ c-a & -b(c-a) \end{bmatrix}) = (a-b)(b-c)(c-a)$$