Linear Algebra

Fall 2024, Homework # 1 Reference Solutions

Due: Oct. 4, 2024

1. (10 pts) Consider the following system of linear equations, where c is a real number:

$$\begin{aligned} x_1 + x_2 \; + cx_3 + x_4 \; &= c \\ - x_2 \; + x_3 \; + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 \; - x_4 \; &= -c \end{aligned}$$

Apply elementary row operations to the augmented matrix of the above to yield its REF (Row Echelon Form). Note that the REF contains c as a parameter. For what c, does the linear system have a solution? Show your derivation in detail.

Sol. Apply the following elementary row operations to the augmented matrix

$$A = \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 1 & 2 & 1 & -1 & -c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1-c & -2 & -2c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2-c & 0 & -2c \end{bmatrix}$$

When $c = 2$, the REF of A becomes $\begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$, which is inconsistent.

So if $c \neq 2$, the linear system is consistent and therefore has a solution.

2. (15 pts) Find all possible values of $rank(A)$ as a varies

$$A = \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix}$$

(Hint:) Use elementary row operations ($R_2 + 2R_1$; $R_3 - aR_1$, where R_i represents row i) we obtain:

$$B = \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix}$$

Then show your argument for each of the following cases:

- (Case 1): $a = -1$, What is the rank? Why?
- (Case 2): $a \neq -1$
  - (Case 2-1) $a = 2$, What is the rank? Why?
  - (Case 2-2) $a \neq 2$, What is the rank? Why?

Note that in some of the above cases, you might have to apply elementary row operations again. Show your work in sufficient detail.

Sol.

- (Case 1): $a = -1$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is RREF(A). Since it has *one* non-zero row, $rank(A) = 1$.

- (Case 2): $a \neq -1$
  Apply elementary row operations to B:

$$B = \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 0 & 2+a-a^2 \end{bmatrix} = C$$

  - (Case 2-1) $a = 2$, $C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$. Since it has *two* non-zero rows, $rank(A) = 2$.
  - (Case 2-2) $a \neq 2$ and $a \neq -1$. So $2 + a - a^2 = -(a-2)(a+1) \neq 0$. Divide the 3rd row by $2 + a - a^2$ and we get $\begin{bmatrix} 1 & 2 & a \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \text{RREF}(A)$. So $rank(A) = 3$.

3. (10 pts) Find a quadratic polynomial, say $f(x) = ax^2 + bx + c$, such that $f(1) = 1, f(2) = 9, f(3) = 27$. To this end, find the values of a, b and c. Show your derivation in sufficient detail.

Sol. Suppose

$$\begin{aligned} f(1) &= a \;\; + b \;\; + c = 1 \\ f(2) &= 4a + 2b + c = 9 \\ f(3) &= 9a + 3b + c = 27 \end{aligned}$$

Apply the following elementary row operations to the augmented matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 9 \\ 9 & 3 & 1 & 27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & 5 \\ 0 & -6 & -8 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3/2 & -5/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

So $c = 3$, $b = -7$, $a = 5$.

4. (10 pts) Given an $n \times n$ matrix $A = I - uu^T$, where I is the $n \times n$ identity matrix and u is a column vector in $\mathbb{R}^n$ with $u^T u = 1$ (note that this is just the number 1), prove that $AA = A$.

Sol.

$$\begin{aligned} AA &= (I - uu^T)(I - uu^T) \\ &= I - 2uu^T + uu^T uu^T \\ &= I - 2uu^T + u(u^T u)u^T \\ &= I - 2uu^T + u(1)u^T \\ &= I - uu^T = A \end{aligned}$$

5. (10 pts) For what values of k, if any, is the vector b in the span of the columns of A?

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \\ -1 & 0 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 1 \\ k-2 \\ 1 \end{bmatrix}$$

Show your work in sufficient detail.

Sol.
For b to be in the span of the columns of A, $Ax = b$ must have non-zero solutions. Consider the augmented matrix

$$B = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & k-2 \\ -1 & 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{RREF}(B)$$

If $k \neq 0$, then RREF(B) is inconsistent and $Ax = b$ does not have solutions.
So b is in the span of the columns of A if $k = 0$.

6. (15 pts) Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one vector in the set as a linear combination of the others.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix} \right\}$$

Sol. Suppose there exist scalars x_1, x_2, x_3, x_4 such that $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & -2 & -2 \\ -1 & 3 & 0 & 7 \\ 0 & 4 & 1 & 11 \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Apply elementary row operations to the augmented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & -2 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ -1 & 3 & 0 & 7 & 0 \\ 0 & 4 & 1 & 11 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 4 & -1 & 5 & 0 \\ 0 & 0 & 5 & 15 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 9 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & -2 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly x_4 is free. Let $x_4 = -1$. Then we have $x_3 = 3$, $x_2 = 2$, $x_1 = -1$. Therefore the set of vectors is linearly dependent and

$$-1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 7 \\ 11 \end{bmatrix}$$

7. (10 pts) Let A and B be two $n \times n$ matrices such that AB is invertible.
   (a) Prove that B is invertible.
   (b) Prove that A is invertible.

   (Hint: You may use the following facts. A matrix C is NOT invertible if and only if $Cx = 0$ has a non-zero solution. Furthermore, if C and D are invertible, CD is also invertible.)

Sol.

(a) Suppose B is not invertible, then there exists an $x \neq 0$ such that $Bx = 0$. In this case, $(AB)x = A(Bx) = 0$. Since $x \neq 0$, AB is not invertible – a contradiction.

(b) From (a), B is invertible, so B^{-1} exists. We then have $AB(B^{-1}) = A(BB^{-1}) = A$. Since AB and B^{-1} are both invertible, A is also invertible.

8. (10 pts) Let
$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

(a) Find a 4×2 matrix B such that $AB = I_2$, where I_2 is the 2×2 identity matrix.

(b) Explain why there is no 4×2 matrix C such that $CA = I_4$, where I_4 is the 4×4 identity matrix.

Sol.

(a) Let $B = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$ such that $AB = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then we find the solution for x_i and y_i:

For x_i: $\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$, so $(x_1, x_2, x_3, x_4) = (a, b, 1-a-b, a-1)$.

For y_i: $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$, so $(y_1, y_2, y_3, y_4) = (c, d, -c-d, c+1)$.

So $B = \begin{bmatrix} a & c \\ b & d \\ 1-a-b & -c-d \\ a-1 & c+1 \end{bmatrix}$. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Suppose there exists a 4×2 matrix C such that $CA = I_4$. $rank(CA) = rank(I_4) = 4$ and $rank(CA) \leq rank(A) = 2$ – a contradiction.

9. (10 pts) Find the inverse of the following matrix A by row reducing $[A|I]$.
$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 2 & 7 \end{bmatrix}.$$

Sol.

$$[A|I] = \begin{bmatrix} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 11 & -7 \\ 0 & 1 & 0 & 0 & -7 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} 1 & 11 & -7 \\ 0 & -7 & 4 \\ 0 & 2 & -1 \end{bmatrix}$.