Linear Algebra

Fall 2024, Quiz # 2 Reference Solutions

Dec. 6, 2024

---

1. (30 pts) True or False. (No explanations needed. No penalty for incorrect answer.)

   (1) A matrix whose rows form an orthonormal basis for $\mathbb{R}^n$ is an orthogonal matrix.
   Sol. True. If A's rows form an orthonormal basis, so do A^T's columns. Thus A^T is an orthogonal matrix, which implies $(A^T)^T = (A^T)^{-1} = (A^{-1})^T$. So $A^T = A^{-1}$, A is orthogonal.

   (2) The following matrix is diagonalizable $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$
   Sol. True.

   (3) If a matrix A has trivial null space, $Null(A) = \{0\}$, then as a function A is one-to-one.
   Sol. True.

   (4) If λ is an eigenvalue for matrix A, then the eigenvectors with eigenvalue λ are scalar multiples of each other.
   Sol. False.

   (5) If matrix A is diagonalizable, then A is invertible.
   Sol. False. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   (6) Suppose v is an eigenvector of matrix A with eigenvalue λ, then v is also an eigenvector of matrix A^2 with eigenvalue λ^2
   Sol. True.

   (7) If A and B are similar matrices, then they have the same eigenvectors.
   Sol. False. They have the same eigenvalues, but not necessarily the same eigenvectors. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

   (8) In $\mathbb{R}^3$, if v (a 3×1 column vector) is a unit vector on a line L, then vv^T is the projection matrix onto line L.
   Sol. True.

   (9) If P is an orthogonal projection matrix onto a subspace W, then the eigenvalue of P is either 0 or 1.
   Sol. True.

   (10) If A and B are matrices whose eigenvalues, counted with their algebraic multiplicities, are the same, then A and B are similar.
   Sol: False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (11) If 0 is an eigenvalue of an $n \times n$ matrix A, then $rank(A) < n$.
   Sol: True

   (12) Given $n \times n$ symmetric matrices A and B, then AB is always a symmetric matrix.
   Sol: False $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (13) Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is similar to matrix $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$
   Sol: False. Determinants are different.

   (14) If 1 is the only eigenvalue of a diagonalizable $n \times n$ matrix A, then A must be the identity matrix I_n.
   Sol: True. $A = PI_nP^{-1} = PP^{-1} = I_n$

(15) If A^2 is the zero matrix, then the only possible eigenvalue is 0.
Sol: True.

2. (10 pts) Let A be a 2×2 matrix such that $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for A with eigenvalue 2, and $\begin{bmatrix}2\\3\end{bmatrix}$ is an eigenvector for A with eigenvalue 1. If $v = \begin{bmatrix}3\\4\end{bmatrix}$, compute A^3v.

Sol. Let $\lambda_1 = 2$, $p_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\lambda_2 = 1$, and $p_2 = \begin{bmatrix}2\\3\end{bmatrix}$. Then $A = PDP^{-1}$ where

$$D = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix},\ P = [\,p_1\ p_2\,] = \begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix},\ P^{-1} = \begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}^T = \begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}$$

$$A^3v = PD^3P^{-1}v = \begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}2^3 & 0\\ 0 & 1^3\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix}$$

3. (20 pts) Let W be the subspace spanned by the vectors $w_1 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$ and $w_1 = \begin{bmatrix}4\\0\\1\end{bmatrix}$. Find a basis for $W^\perp$. (Hint: Let $A = \begin{bmatrix}2 & 4\\ 1 & 0\\ -2 & 1\end{bmatrix}$. $W^\perp = null(A^T)$)

Sol. Since $W^\perp = null(A^T)$,

$$A^Tx = 0 \Rightarrow \begin{bmatrix}2 & 1 & -1\\ 4 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}-1\\10\\4\end{bmatrix}.\ \text{So}\ \begin{bmatrix}-1\\10\\4\end{bmatrix}\ \text{is a basis for}\ W^\perp.$$

4. (20 pts) Let $B = \{\begin{bmatrix}-2\\2\\1\\4\end{bmatrix}, \begin{bmatrix}4\\1\\-2\\2\end{bmatrix}, \begin{bmatrix}1\\4\\2\\-2\end{bmatrix}\}$ be an orthogonal basis, and W is a subspace spanned by B. Find the orthogonal projection of $\begin{bmatrix}-13\\18\\9\\1\end{bmatrix}$ into W. Show your work in sufficient detail.

Sol. Since B is an orthogonal basis, we can convert it into an orthonormal basis $B' = \{\frac{1}{5}\begin{bmatrix}-2\\2\\1\\4\end{bmatrix}, \frac{1}{5}\begin{bmatrix}4\\1\\-2\\2\end{bmatrix}, \frac{1}{5}\begin{bmatrix}1\\4\\2\\-2\end{bmatrix}\}$. Then the orthogonal projection operator $P_W = CC^T$ where $C = \frac{1}{5}\begin{bmatrix}-2 & 4 & 1\\ 2 & 1 & 4\\ 1 & -2 & 2\\ 4 & 2 & -1\end{bmatrix}$. The orthogonal projection of $v = \begin{bmatrix}-13\\18\\9\\1\end{bmatrix}$ is $P_Wv = CC^Tv = \begin{bmatrix}-11\\16\\13\\2\end{bmatrix}$.

5. (20 pts) Let $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$.

(a) What is the characteristic polynomial of A?

(b) Find the eigenvalues of A.

(c) For each eigenvalue, find the eigenvector(s) that generates the eigenspace corresponding to the eigenvalue.

(d) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Sol.

(a) The characteristic polynomial of A is

$$f(\lambda) = det(A - \lambda I) = det(\begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix}) = (-2-\lambda)(5-\lambda) + 12 = (\lambda - 1)(\lambda - 2)$$

(b) Two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$.

(c) $(A - \lambda_1 I)x = 0 \Rightarrow \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow p_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$(A - \lambda_2 I)x = 0 \Rightarrow \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow p_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$P = [\, p_1 \; p_2 \,] = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$