Chapter 7
Vector Space

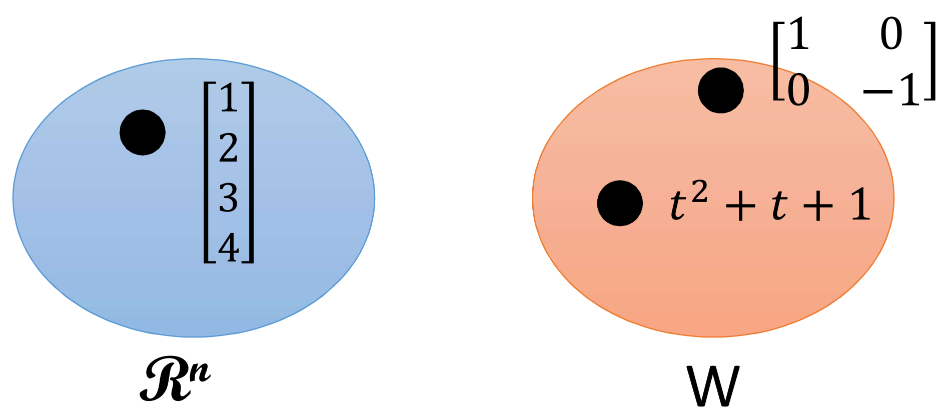
(除了標註※之簡報外，其餘採用李宏毅教授之投影片 教材)

Vector Spaces and Their Subspaces

(Chap. 7.1)

Introduction

- Many things can be considered as “vectors”.
  - E.g. a function can be regarded as a vector
- We can apply the concept we learned on those “vectors”.
  - Linear combination
  - Span
  - Basis
  - Orthogonal ……



(Abstract) Vector Space

For any vectors **u**, **v** and **w** in $\mathcal{V}$, and scalars a and b in $\mathcal{R}$, **u** + **v** and a**u** are in $\mathcal{V}$, and the following axioms hold

1. **u + v = v + u**
2. (**u** + **v**) + **w** = **u** + (**v** + **w**)
3. There is an element $\mathbf{0}$ in $\mathcal{V}$ such that $\mathbf{0}$ + **u** = **u**
4. There is an element –**u** in $\mathcal{V}$ such that –**u** + **u** = $\mathbf{0}$
5. 1**u** = **u**
6. (ab)**u** = a(b**u**)
7. a(**u**+**v**) = a**u** + a**v**
8. (a+b)**u** = a**u** + b**u**

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ zero vector

Why $0u = 0$ and $(-1)u = -u$?

- Can you prove that $0\mathbf{u} = \mathbf{0}$ (i.e., zero vector)?
  - $0\mathbf{u}$
  - $= \mathbf{0} + 0\mathbf{u}$ (from (3))
  - $= (-0\mathbf{u} + 0\mathbf{u}) + 0\mathbf{u}$ (from (4))
  - $= -0\mathbf{u} + (0\mathbf{u} + 0\mathbf{u})$ (from (2))
  - $= -0\mathbf{u} + ((0+0)\mathbf{u})$ (from (8))
  - $= -0\mathbf{u} + 0\mathbf{u}$ ($0+0=0$ as 0 is in $\mathcal{R}$)
  - $= \mathbf{0}$ (from (4))
- Can you prove that $(-1)\mathbf{u} = -\mathbf{u}$ (i.e., inverse of $\mathbf{u}$)?

Are they vectors?

Are they vectors?

- A matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- A linear transform
- A polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Choose a basis

$$1, x, \cdots, x^n$$

Are they vectors?

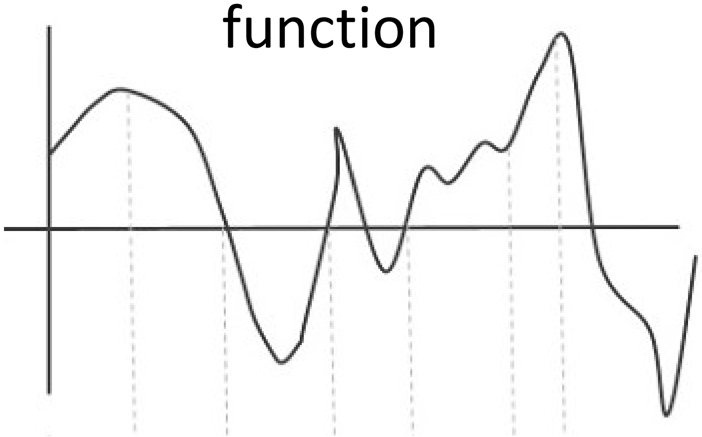
What is the zero vector?

$$z(t) = 0$$

- Any function from S to R is a vector?

Infinite

function



$$v(t) = e^t \Rightarrow v = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

$$g(t) = t^2 - 1 \Rightarrow g = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

$$\mathrm{h}(t) = e^t + t^2 - 1 \Rightarrow v + g$$

$$\mathrm{r}(t) = ce^t \Rightarrow cv$$

Vector?

Objects in Different Vector Spaces

All the polynomials with degree less than or equal to 2 form a vector space (often denoted as P_2)

w.r.t. Basis $\{1, t, t^2\}$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$f(t) = 1 \qquad g(t) = t + 1 \qquad h(t) = t^2 + t + 1$$

Subspaces

Chap. 6.1

Review: Subspace

- A vector set V is called a subspace if it has the following three properties:
- 1. The zero vector **0** belongs to V
- 2. If **u** and **w** belong to V, then **u+w** belongs to V

Closed under (vector) addition

- 3. If **u** belongs to V, and c is a scalar, then c**u** belongs to V

Closed under scalar multiplication

Are they subspaces?

$$trace\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

- All the functions pass 0 at t_0
- All the matrices whose trace equal to zero
- All the matrices of the form

$$\begin{bmatrix} a & a+b \\ b & 0 \end{bmatrix}$$

- All the continuous functions
- All the polynomials with degree n ✗ $t^n, -t^n$
- All the polynomials with degree less than or equal to n

P: all polynomials, P_n: all polynomials with degree less than or equal to n

Linear Combination and Span

Chap. 7.2

Linear Combination and Span

- Matrices

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Linear combination with coefficient a, b, c

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

What is Span S?

All 2x2 matrices whose trace equal to zero

Linear Combination and Span

- Polynomials

$$S = \{1, x, x^2, x^3\}$$

Is $f(x) = 2 + 3x - x^2$ linear combination of the "vectors" in S?

$$f(x) = 2 \cdot 1 + 3 \cdot x + (-1) \cdot x^2$$

$$Span\{1, x, x^2, x^3\} \qquad = P_3$$

$$Span\{1, x, \cdots, x^n, \cdots\} \qquad = P$$

Linear Transformations

(Chap. 7.2)

Linear transformation

- A mapping (function) T is called linear if for all "vectors" u, v and scalars c:
- Preserving vector addition:

$$T(u+v) = T(u) + T(v)$$

- Preserving vector multiplication: $T(cu) = cT(u)$

Is matrix transpose linear?

Input: m x n matrices, output: n x m matrices

Linear transformation

- Derivative: **linear?**

function f
e.g. x^2 → **Derivative** → function f'
e.g. 2x

- Integral from a to b **linear?**

function f
e.g. x^2 → **Integral** (from a to b) → scalar

$$\int_a^b f(t)dt$$

e.g. $\frac{1}{3}(b^3 - a^3)$

Null Space and Range

- Null Space
  - The null space of T: V → W is the set of all "vectors" in V such that T(v)=0, where 0 is the zero vector in W.
  - What is the null space of matrix transpose ?

$$T: \mathcal{M}_{m\times n} \rightarrow \mathcal{M}_{n\times m}$$

$0_{m\times n}$

- Range (or Image)
  - The range of T is the set of all images of T.
  - That is, the set of all "vectors" T(v) for all v in the domain
  - What is the range of matrix transpose?

$\mathcal{M}_{n\times m}$

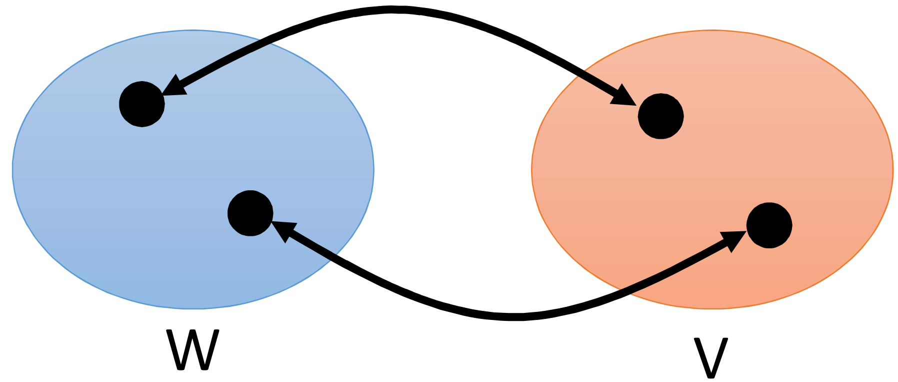
One-to-one and Onto

- U: $\mathcal{M}_{m\times n} \to \mathcal{M}_{n\times m}$ defined by $U(A) = A^T$.
  - Is U one-to-one? yes
  - Is U onto? yes

- D: $\mathcal{P}_3 \to \mathcal{P}_3$ defined by $D(f) = f'$
  - Is D one-to-one? no
  - Is D onto? no

$$x^3 + 2x + 3 \to 3x^2 + 2x$$
$$x^3 + 2x \to 3x^2 + 2x$$

Isomorphism (同構)



- Let V and W be vector space.
- A linear transformation T: V→W is called an isomorphism if it is one-to-one and onto
  - Invertible linear transform
  - W and V are isomorphic.

Example 1: U: $\mathcal{M}_{m\times n} \rightarrow \mathcal{M}_{n\times m}$ defined by $U(A) = A^T$.

Example 2: T: $\mathcal{P}_2 \rightarrow \mathcal{R}^3$

$$T\left(a + bx + \frac{c}{2}x^2\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Basis and Dimension

(Chap. 7.3)

Independent

A basis for subspace V is a linearly independent generation set of V.

- Example

$S = \{x^2 - 3x + 2,\ 3x^2 - 5x,\ 2x - 3\}$ is a subset of $\mathscr{P}_2$.

Is it linearly independent?

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = \mathbf{0}$$

No

- Example

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$ is a subset of 2x2 matrices.

Is it linearly independent?

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

implies that $a = b = c = 0$

Yes

Independent

If $\{v_1, v_2, \ldots\ldots, v_k\}$ are independent, and T is an isomorphism, $\{T(v_1), T(v_2), \ldots\ldots, T(v_k)\}$ are independent

- Example

The infinite vector set $\{1, x, x^2, \cdots, x^n, \cdots\}$

Is it linearly independent?

$\Sigma_i c_i x^i = 0$ implies $c_i = 0$ for all i.

Yes

- Example

$S = \{e^t, e^{2t}, e^{3t}\}$ Is it linearly independent?

Yes

$$ae^t + be^{2t} + ce^{3t} = 0 \qquad a + b + c = 0$$

$$ae^t + 2be^{2t} + 3ce^{3t} = 0 \qquad a + 2b + 3c = 0$$

$$ae^t + 4be^{2t} + 9ce^{3t} = 0 \qquad a + 4b + 9c = 0$$

Independent

Theorem:
If $\{v_1, v_2, \ldots\ldots, v_k\}$ are independent, and T is an isomorphism, $\{T(v_1), T(v_2), \ldots\ldots, T(v_k)\}$ are independent

(Proof)

Suppose $a_1 T(v_1) + a_2 T(v_2) + \ldots\ldots + a_k T(v_k) = 0$

$\Rightarrow T(a_1 v_1) + T(a_2 v_2) + \ldots\ldots + T(a_k v_k) = 0$

$\Rightarrow T(a_1 v_1 + a_2 v_2 + \ldots\ldots + a_k v_k) = 0$

$\Rightarrow a_1 v_1 + a_2 v_2 + \ldots\ldots + a_k v_k = 0$ ----- (one-to-one)

$\Rightarrow a_1 = a_2 = \ldots\ldots = a_k = 0$

※

Basis

- Example

For the subspace of all 2 x 2 matrices,

The basis is

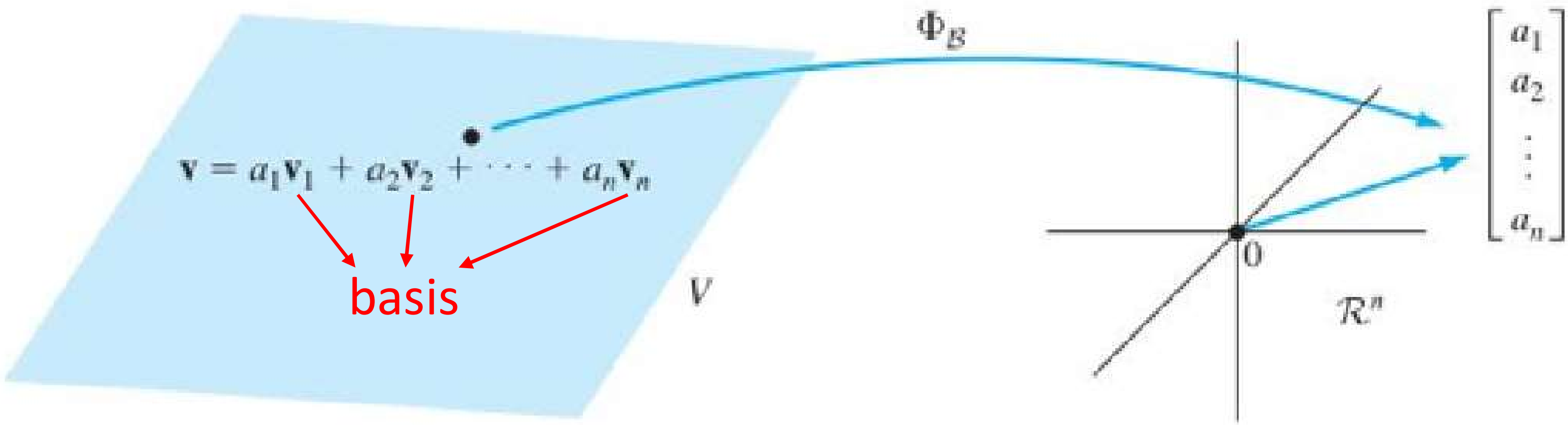
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$ Dim = 4

- Example

$S = \{1, x, x^2, \cdots, x^n, \cdots\}$ is a basis of $\mathscr{P}$. Dim = inf

Vector Representation of Object

- Coordinate Transformation



P_n: Basis: $\{1, x, x^2, \cdots, x^n\}$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Coordinate Vectors

Definition

Let $\mathscr{B}$ be a basis for V, $\Phi_{\mathscr{B}} : V \to \mathscr{R}^n$ (coordinate transformation) is an isomorphism. Any vector v in V, $\Phi_{\mathscr{B}}(v)$ is called the coordinate vector of v relative to $\mathscr{B}$, written as $[v]_{\mathscr{B}}$

- Example

$\mathscr{B} = \{1, x, x^2, \cdots, x^n\}$ is a basis for $\mathscr{P}_n$.

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$[p(x)]_{\mathscr{B}} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Coordinate Vectors

Let $\mathcal{B} = \left\{\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix}\right\}$

be two bases of $\mathcal{R}^2$

What is the matrix M such that

$[x]_{\mathcal{C}} = \text{M}\ [x]_{\mathcal{B}}$?

$$\text{P}_{\mathcal{B}} = \begin{bmatrix} 3 & 4 \\ -5 & -6 \end{bmatrix} \qquad \text{P}_{\mathcal{C}} = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$$

$$[x] = \text{P}_{\mathcal{B}}\ [x]_{\mathcal{B}} \qquad [x] = \text{P}_{\mathcal{C}}\ [x]_{\mathcal{C}}$$

w.r.t. standard basis

$$[x]_{\mathcal{C}} = \text{P}^{-1}{}_{\mathcal{C}}\, \text{P}_{\mathcal{B}}\ [x]_{\mathcal{B}}$$

Matrix Representations of Linear Operators

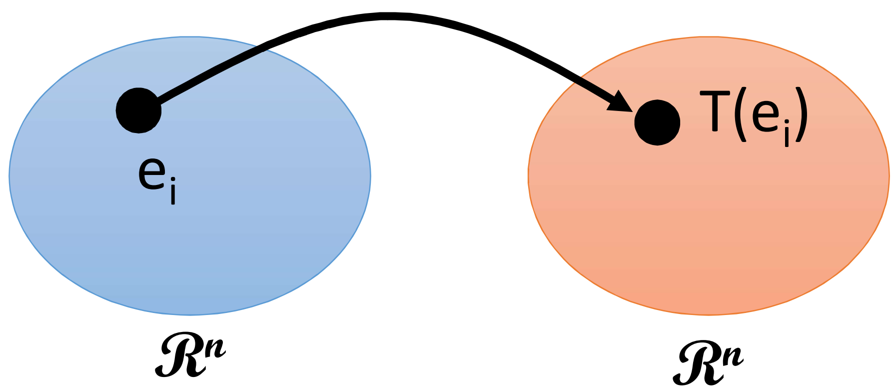
(Chap. 7.4)

Matrix Representation of Linear Operator

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ i-th coordinate}$$

Let T be a linear operator from $\mathcal{R}^n$ to $\mathcal{R}^n$ with standard basis $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$. Then the matrix representation of T w.r.t. $\mathcal{E}$ is

$$[T]_{\mathcal{E}} = [\ [T(e_1)_{\mathcal{E}}\ \ T(e_2)_{\mathcal{E}}\ \ \ldots\ \ T(e_n)_{\mathcal{E}}\ ]$$



Linear Transformation and Matrix

$$T(x) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix}$$

$$\text{Ax} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matrix Representation of Linear Operator

Let T be a linear operator on vector space V with basis $\mathcal{B}$ =$\{v_1, v_2, \ldots, v_n\}$. Then the matrix representation of T w.r.t. $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = [\ [T(v_1)]_{\mathcal{B}}\ \ T(v_2)_{\mathcal{B}}\ \ \ldots\ \ T(v_n)_{\mathcal{B}}\ ]$$

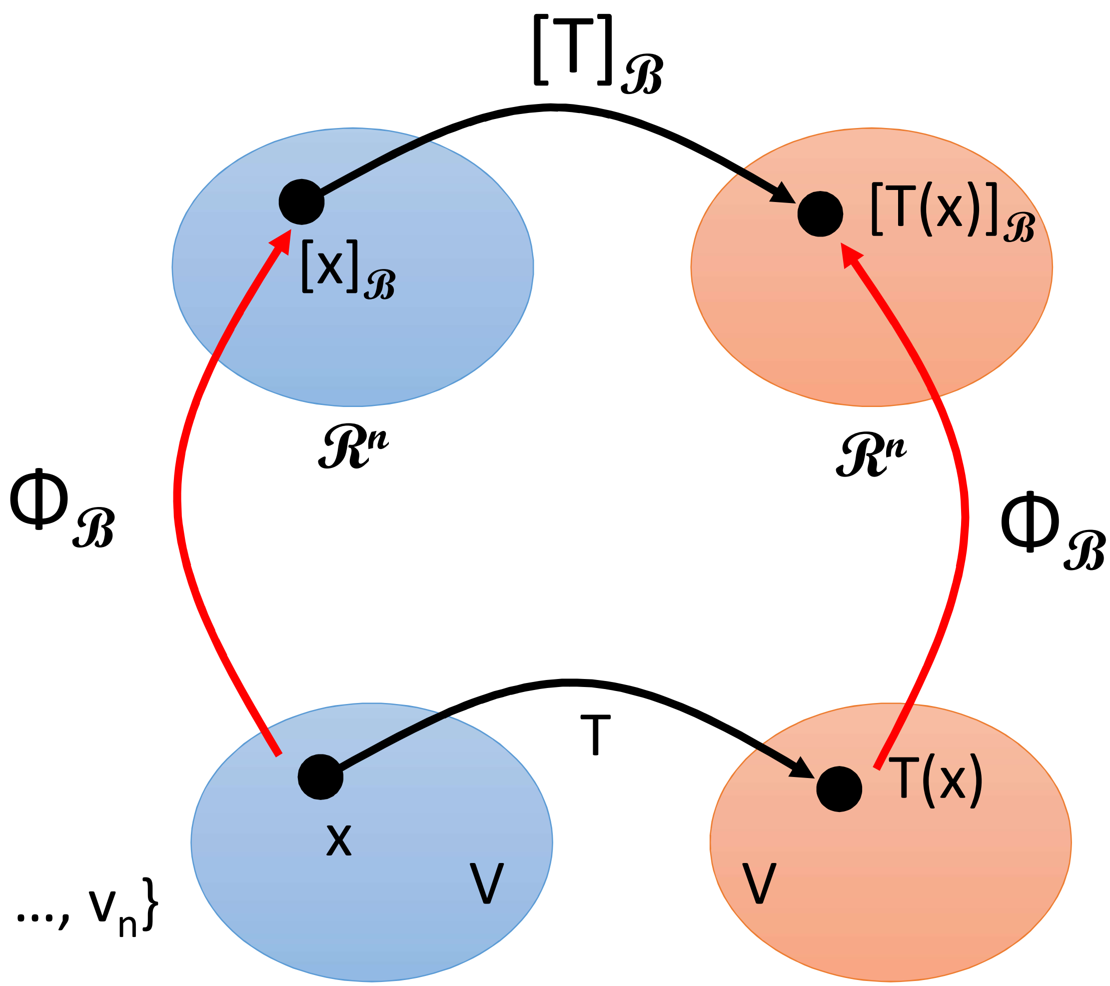
Example: $\mathcal{B}$ =$\{1, x\}$.

T(1) = 1+2x T(x)= 3x $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

T(2-x) = 2(1+2x)-(3x) = 2+x $[T(2\text{-}x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Linear Operator T: V → V

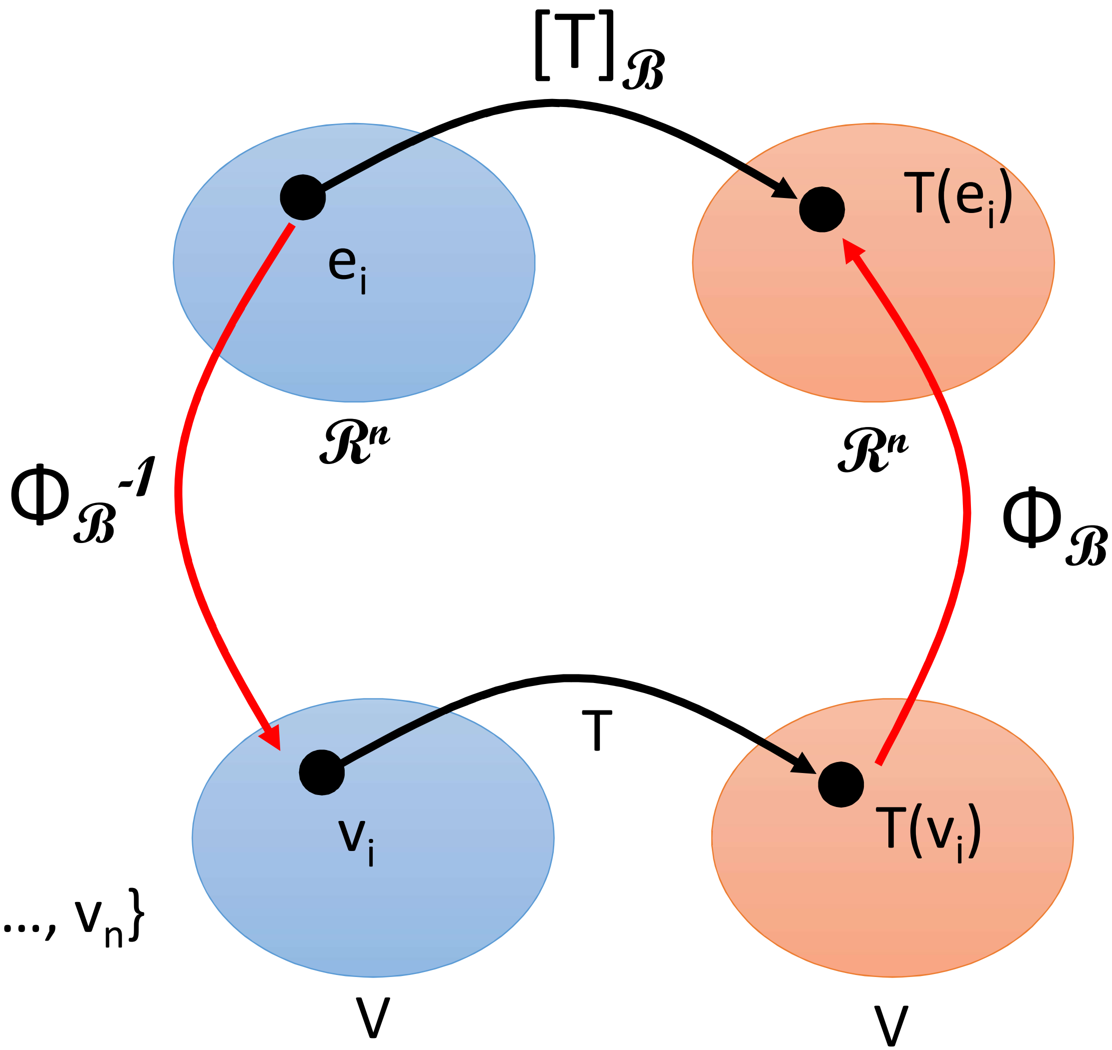
What is the matrix for T?



$\mathcal{B} = \{v_1, v_2, \dots, v_n\}$

$[T]_{\mathcal{B}} = [\ [T(v_1)]_{\mathcal{B}}\ \ T(v_2)_{\mathcal{B}}\ \dots\ T(v_n)_{\mathcal{B}}\ ]$

How to compute $[T]_{\mathscr{B}}$?



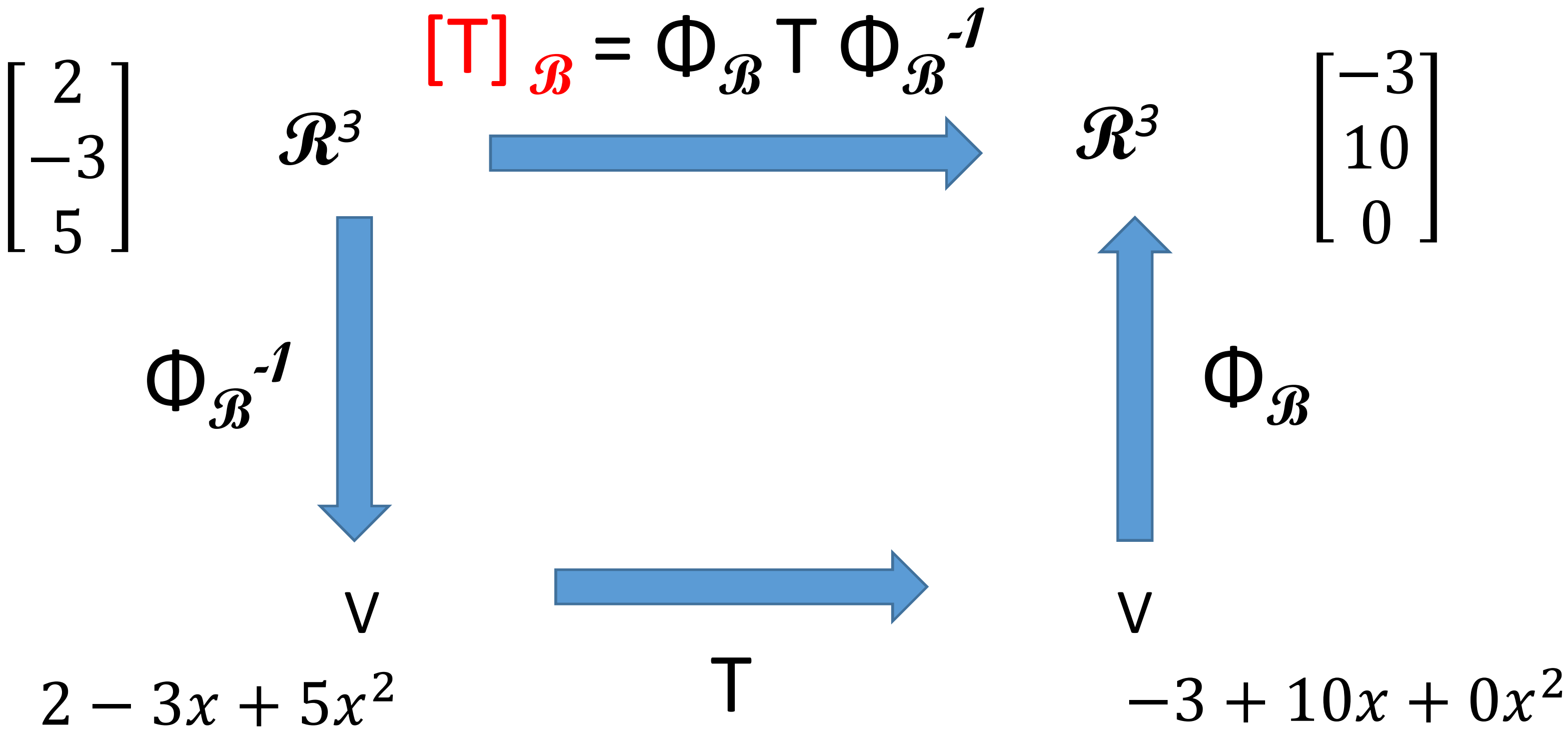
$\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$

※

Matrix Representation of Linear Operator

Represent it as a matrix

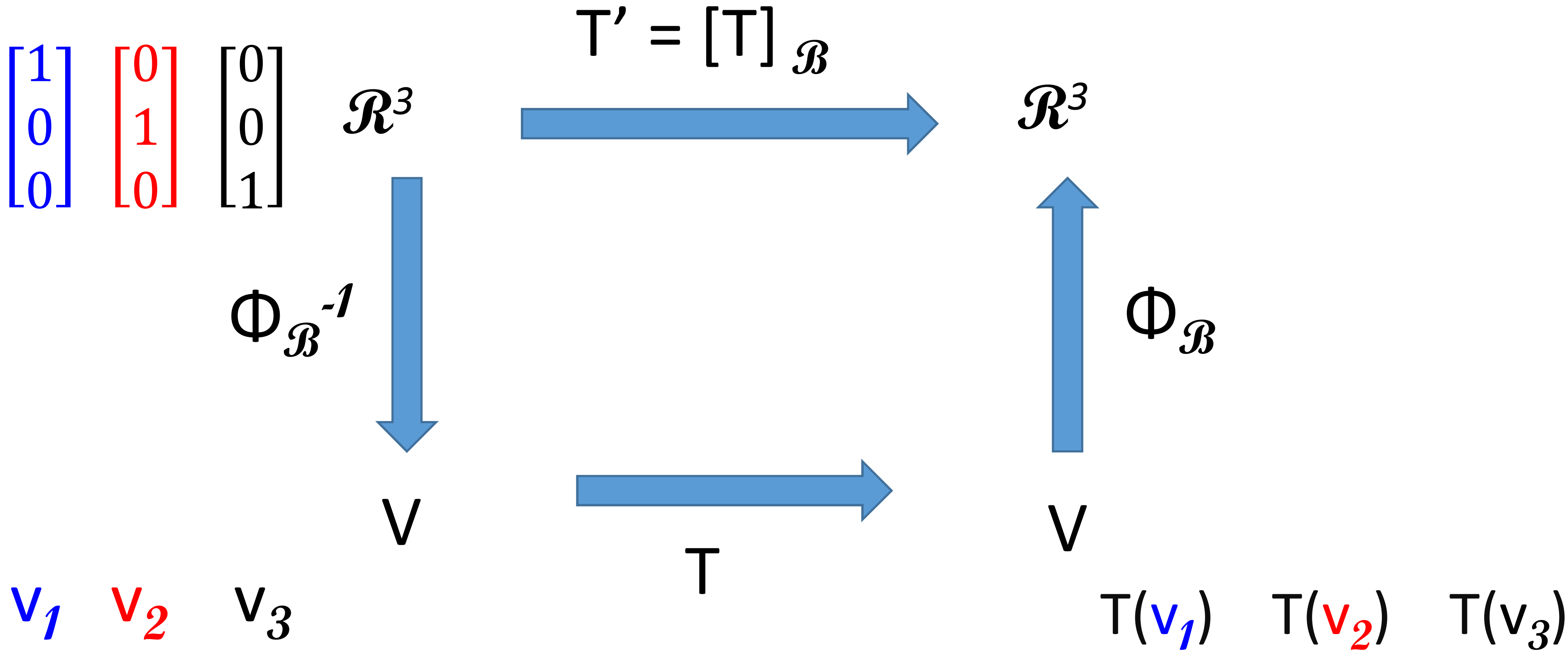
$\mathscr{B} = \{1, x, x^2\}$　　$\Phi_{\mathscr{B}}(2 - 3x + 5x^2) = (2, -3, 5)^T$



How to find $[T]_{\mathscr{B}}$?

$[T(v_1)]_{\mathscr{B}}$ = 1st col of $[T]_{\mathscr{B}}$

$T'\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ $T'\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$ $T'\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$



※ Basis

Example

Let T : $\mathscr{P}_2 \to \mathscr{P}_2$ defined as

$$T(p(x)) = p(0) + 3p(1)\,x + p(2)x^2$$

$\mathscr{B} = \{1, x, x^2\}$ is a basis for $\mathscr{P}_2$.

$T(1) = 1 + 3x + x^2 \quad \Rightarrow \quad [T(1)]_{\mathscr{B}} = (1, 3, 1)^T$

$T(x) = 0 + 3x + 2x^2 \quad \Rightarrow \quad [T(x)]_{\mathscr{B}} = (0, 3, 2)^T$

$T(x^2) = 0 + 3x + 4x^2 \quad \Rightarrow \quad [T(x)]_{\mathscr{B}} = (0, 3, 4)^T$

$$[T]_{\mathscr{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

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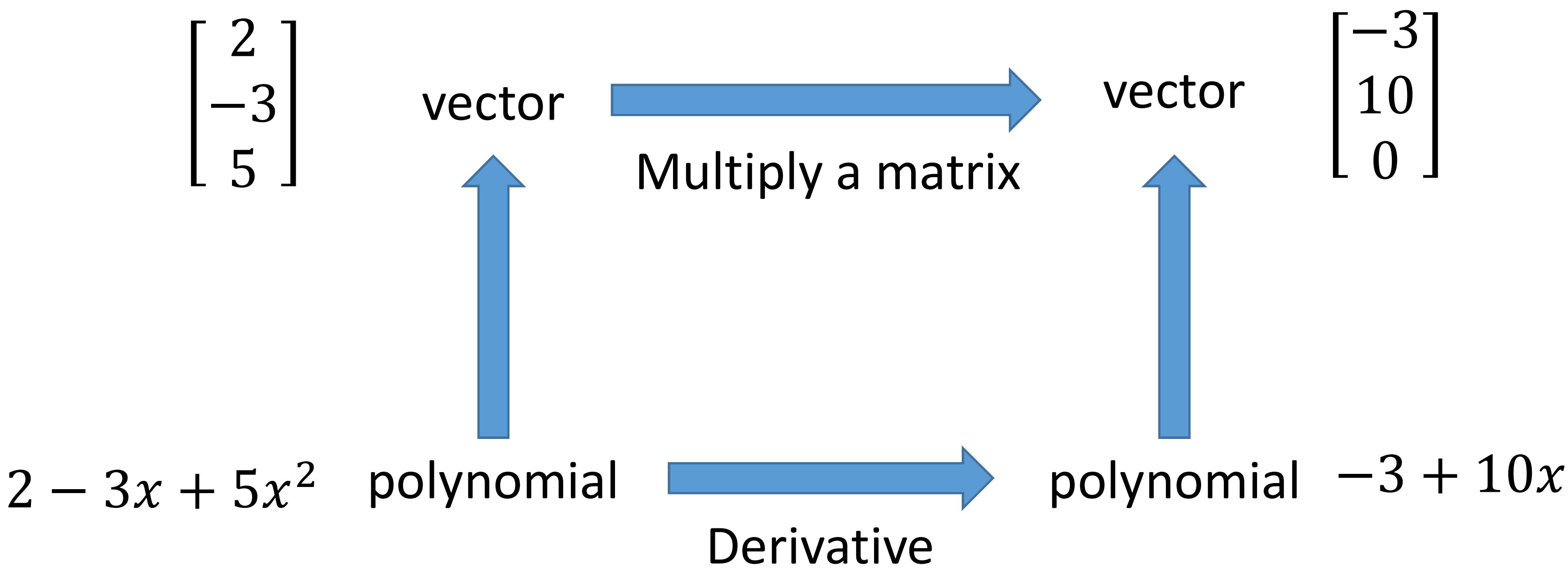
Matrix Representation of Linear Operator

- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix



Matrix Representation of Linear Operator

- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

vector

Multiply a matrix

$$\begin{bmatrix}0 & 1 & 0\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

vector

$$\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\2\\0\end{bmatrix}$$

1

x

x^2

polynomial

Derivative

polynomial

0

1

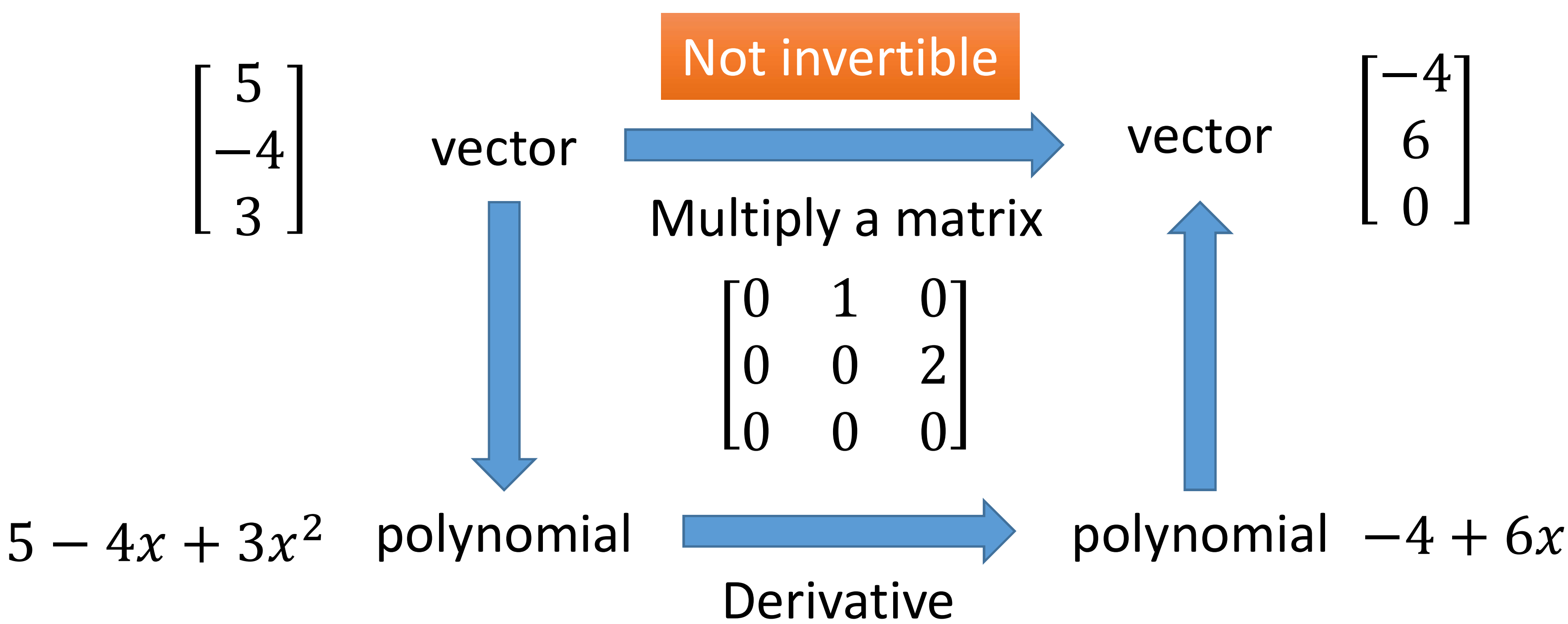
$2x$

Matrix Representation of Linear Operator

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

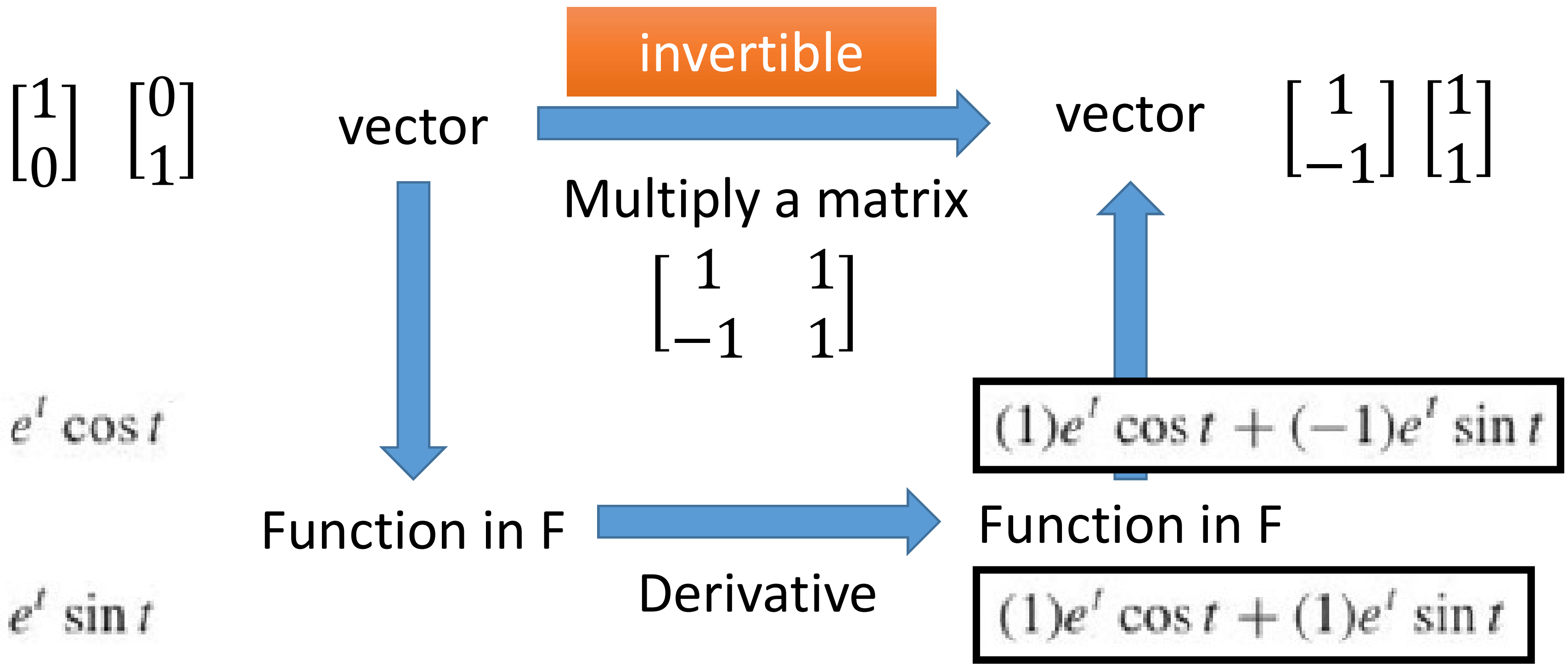
- Example:
  - D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix



Matrix Representation of Linear Operator

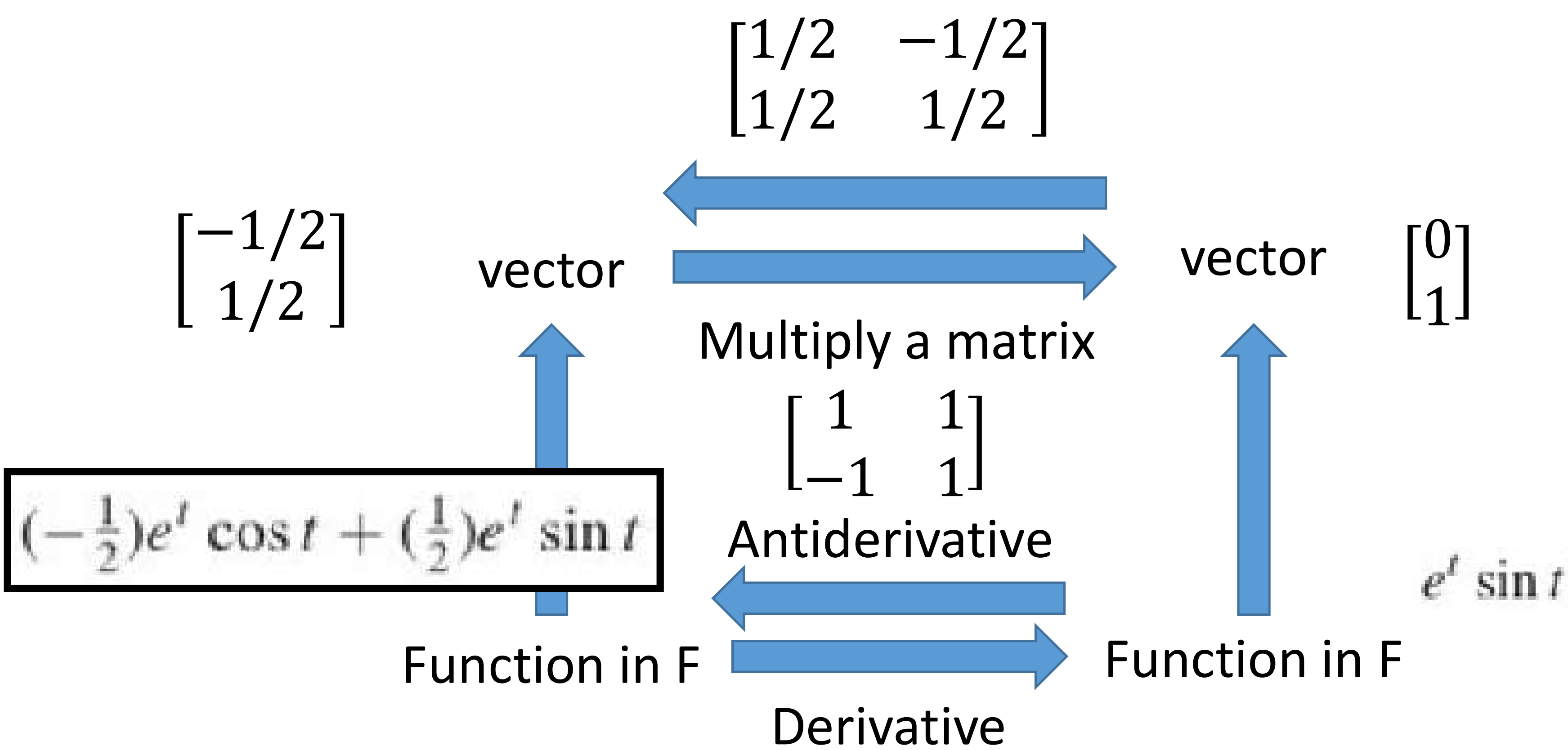
- Example:
  - D (derivative): Function set F → Function set F
  - Basis of F is $\{e^t \cos t, e^t \sin t\}$



Matrix Representation of Linear Operator

Basis of F is

$\{e^t \cos t, e^t \sin t\}$



Linear operator between two bases

Let T : $\mathcal{V} \rightarrow \mathcal{W}$ be a linear operator between $\mathcal{V}$ and $\mathcal{W}$.
$\mathcal{B}$ ={b_1 b_2 ... b_n} and $\mathcal{B}'$={b'_1 b'_2 ... b'_n} bases of $\mathcal{V}$ and $\mathcal{W}$.

What is matrix representation $\mathcal{M}$ of T w.r.t. $\mathcal{B}$ and $\mathcal{B}'$
i.e., $[T(x)]_{\mathcal{B}'} = \mathcal{M}\,[x]_{\mathcal{B}}$?

$$\mathcal{M} = [\, T[b_1]_{\mathcal{B}'} ,\ T[b_2]_{\mathcal{B}'} ,\ \ldots ,T[b_n]_{\mathcal{B}'} \,]$$

Example (P279) Determine T

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\1\end{bmatrix} \quad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-1\\1\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$\mathbf{b}_1$ $\mathbf{c}_1$ $\mathbf{b}_2$ $\mathbf{c}_2$ $\mathbf{b}_3$ $\mathbf{c}_3$

$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $[v]_{\mathcal{B}}$ → $[T]_{\mathcal{B}}$ → $[T(v)]_{\mathcal{B}}$ $\mathbf{B}^{-1}\mathbf{c}_1$ $\mathbf{B}^{-1}\mathbf{c}_2$ $\mathbf{B}^{-1}\mathbf{c}_3$

$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ as a coordinate system

{$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$} is a basis of R^3

B^{-1} ↑ B^{-1} ↑

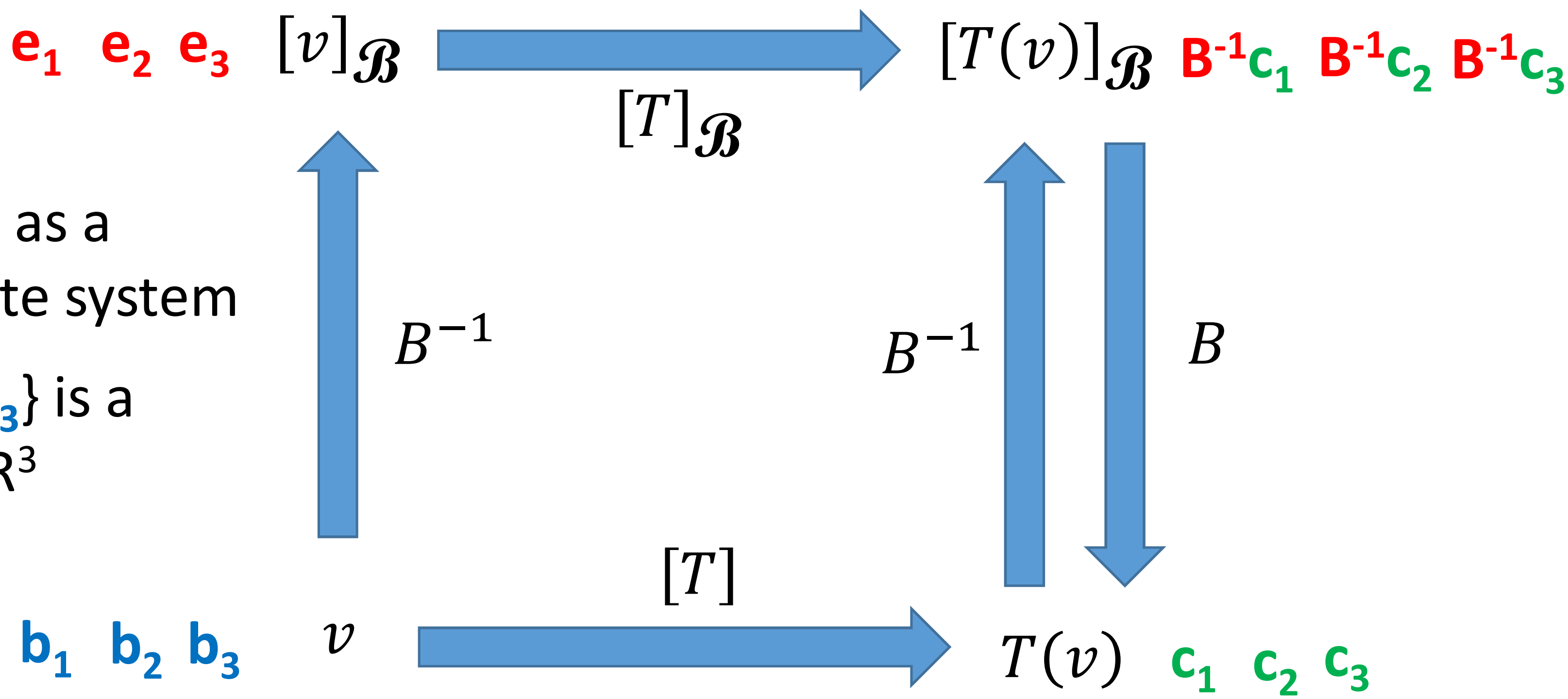
$$B=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$

$\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ v → $[T]$ → $T(v)$ $\mathbf{c}_1$ $\mathbf{c}_2$ $\mathbf{c}_3$

Example (P279) Determine T

$$[T]_{\mathcal{B}} = [B^{-1}c_1 \quad B^{-1}c_2 \quad B^{-1}c_3] \; = B^{-1}C$$

$$[T] = B[T]_{\mathcal{B}}B^{-1} = BB^{-1}CB^{-1} \; = CB^{-1}$$



b$_1$, **b**$_2$, **b**$_3$ as a coordinate system

{**b**$_1$, **b**$_2$, **b**$_3$} is a basis of R^3

Eigenvalue and Eigenvector

$T(\boldsymbol{v}) = \lambda \boldsymbol{v}$, $\boldsymbol{v} \neq \mathbf{0}$, $\boldsymbol{v}$ is eigenvector, λ is eigenvalue

Chap. 7.4

Eigenvalue and Eigenvector

- Consider derivative (linear transformation, input & output are functions)

Is e^{at} an "eigenvector"? ae^{at} What is the "eigenvalue"? a

Every scalar is an eigenvalue of derivative.

- Consider Transpose (also linear transformation, input & output are matrices)

Is $\lambda = 1$ an eigenvalue?

Symmetric matrices form the eigenspace

Is $\lambda = -1$ an eigenvalue?

Skew-symmetric matrices form the eigenspace.

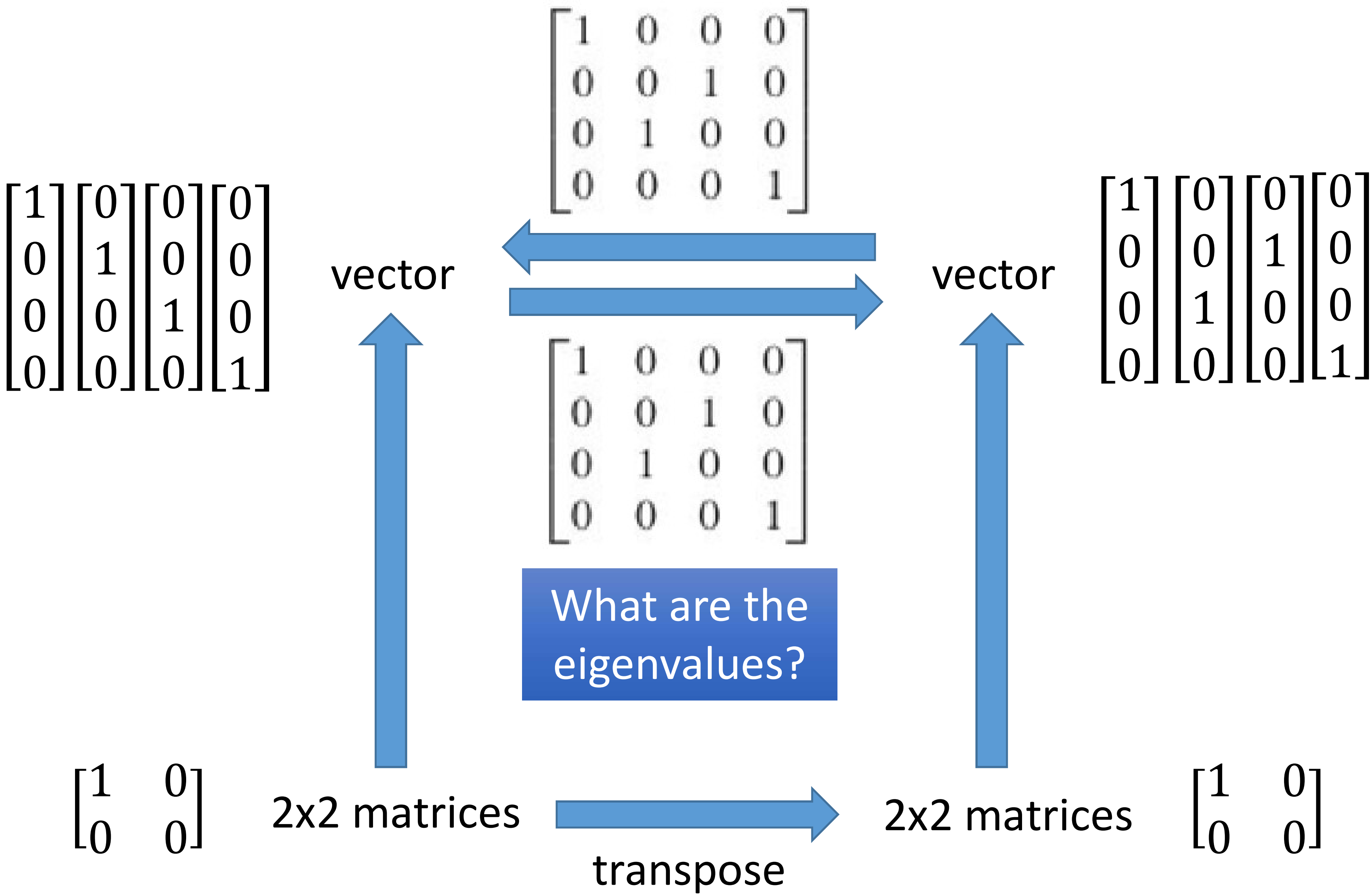
Symmetric:

$$A^T = A$$

Skew-symmetric:

$$A^T = -A$$

Consider Transpose of 2x2 matrices



Eigenvalue and Eigenvector

- ***<u>Consider Transpose of 2x2 matrices</u>***

Matrix representation of transpose

$$det \begin{bmatrix} 1-t & 0 & 0 & 0 \\ 0 & 0-t & 1 & 0 \\ 0 & 1 & 0-t & 0 \\ 0 & 0 & 0 & 1-t \end{bmatrix}$$

Characteristic polynomial

$$(t-1)^3(t+1)$$

$\lambda = 1$

Symmetric matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Dim=3

$\lambda = -1$

Skew-symmetric matrices

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

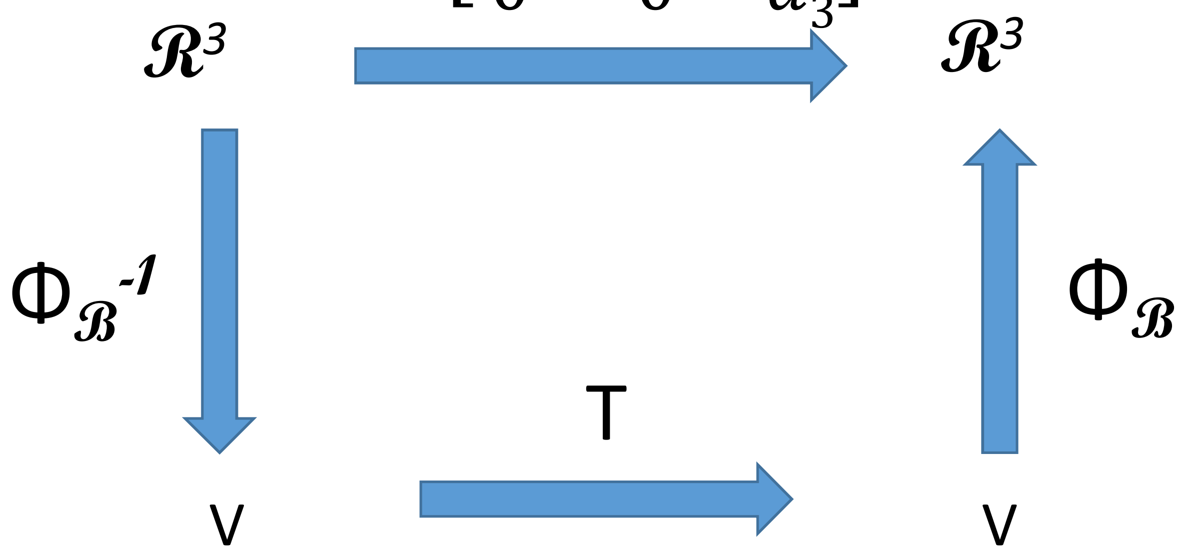
Dim=1

Diagonalizable Linear Operator

T is diagoalizable if there is a basis

$\mathcal{B} = \{v_1, v_2 \ \ldots \ v_n\}$ s.t. $[T]_{\mathcal{B}}$ is diagonal

$$[T]_{\mathcal{B}} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$



Claim:
$T(v_i) = d_i v_i$

Diagonalizable Linear Operator

The following statements are equivalent

- T is diagoalizable
- There is a basis $\mathcal{B} = \{v_1, v_2 \ \ldots v_n\}$ s.t. $[T]_{\mathcal{B}}$ is diagonal
- T has a basis consisting of eigenvectors
- For every basis $\mathcal{B}$, $[T]_{\mathcal{B}}$ is diagoalizable

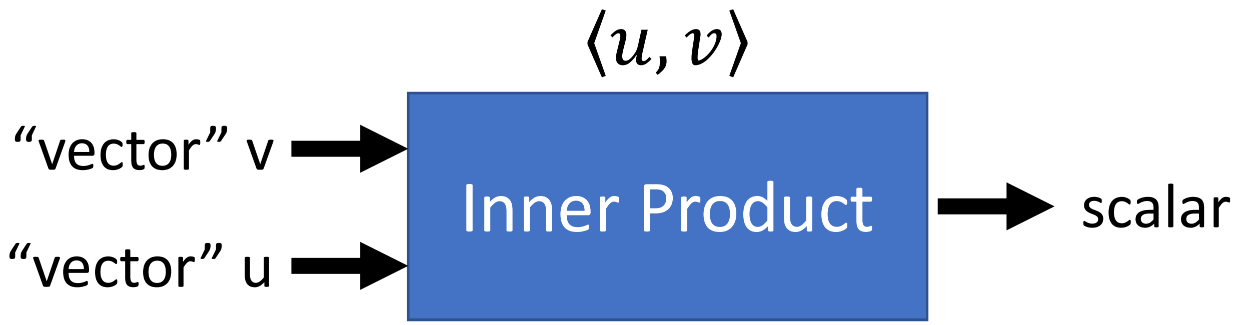
Inner Product Spaces

(Chap. 7.5)

Inner Product

Norm (length): $\|v\| = \sqrt{\langle v, v\rangle}$

Orthogonal: Inner product is zero



For any vectors u, v and w, and any scalar a, the following axioms hold:

1. $\langle u, u\rangle > 0$ if $u \neq 0$
2. $\langle u, v\rangle = \langle v, u\rangle$
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
4. $\langle au, v\rangle = a\langle u, v\rangle$

Dot product is a special case of **inner product** $c(u \cdot v)\ \ c > 0$

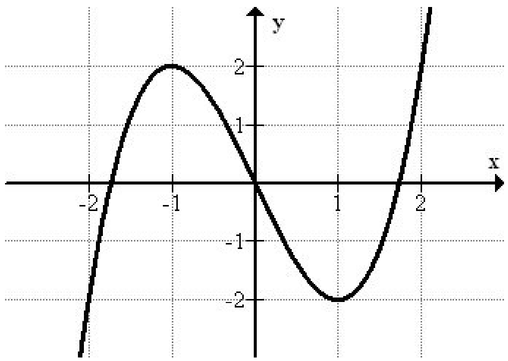
Can you define other inner product for normal vectors?

Inner Product

1. $\langle u, u\rangle > 0$ if $u \neq 0$
2. $\langle u, v\rangle = \langle v, u\rangle$
3. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
4. $\langle au, v\rangle = a\langle u, v\rangle$

- Continuous functions in C[-1, 1]

$$\langle g, h\rangle = \int_{-1}^{1} g(x)h(x)dx$$



$Axiom$ (1):

$$f \neq 0 \Rightarrow f^2(\mathrm{c}) > 0\,,\ \text{for some c in } [-1, 1].$$

$$\Rightarrow f^2(\mathrm{t}) > \mathrm{d} > 0\,,\ \mathrm{t}\ \text{ in } [\mathrm{c}-\varepsilon,\ \mathrm{c}+\varepsilon].$$

$$\Rightarrow \langle f, f\rangle = \int_{-1}^{1} f^2(\mathrm{t})dt = f^2(\mathrm{t}) > 2\varepsilon\mathrm{d} > 0$$

$Axioms$ (2 – 4): Easy to check

Inner Product

1. $\langle u,u\rangle > 0$ if $u \neq 0$
2. $\langle u,v\rangle = \langle v,u\rangle$
3. $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$
4. $\langle au,v\rangle = a\langle u,v\rangle$

- Inner product for any function with input [-1, 1]

$$\langle g,h\rangle = \int_{-1}^{1} g(x)h(x)dx$$

$$= \int_{-1}^{1} xdx = 0$$

Is $g(x) = 1$ and $h(x) = x$ orthogonal?

yes

$$\langle g,h\rangle = \sum_{i=-10}^{10} g\left(\frac{i}{10}\right) h\left(\frac{i}{10}\right)$$

Can it be inner product for general functions?

no

$$u\left(\frac{i}{10}\right) = 0, otherwise \neq 0 \qquad \langle u,u\rangle = 0, but\ u \neq 0$$

Inner Product

Check Axioms (1-4)

- Inner Product of Matrix

Frobenius inner product

$$\langle A, B \rangle = trace(AB^T)$$
$$= trace(BA^T)$$

$$\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right\rangle = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

Element-wise multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad ||A|| = \sqrt{1^2 + 2^2 + 3^2 + 4^2}$$

Orthogonal Projection

- Consider a subspace W with orthogonal basis $S = \{v_1, v_2, \cdots, v_k\}$ of an n-dim vector space V. Let u be any vector in V, which can be written as

$$u = \underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k}_{\in W} + \underbrace{c_{k+1} w_{k+1} + \cdots + c_n w_n}_{\in W^{\perp}}$$

$$c_1 = \frac{u \cdot v_1}{\|v_1\|^2} \quad c_2 = \frac{u \cdot v_2}{\|v_2\|^2} \quad c_k = \frac{u \cdot v_k}{\|v_k\|^2}$$

orthogonal projection of u on W

- If S an orthonormal basis of W.

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$c_1 = u \cdot v_1 \quad c_2 = u \cdot v_2 \quad c_k = u \cdot v_k$$

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Orthogonal Projection

- Consider a subspace W with orthonormal basis $S = \{v_1, v_2, \cdots, v_k\}$ of an *n*-dim vector space V. The orthogonal projection of u on W can be written as

$$(u \cdot v_1)v_1 + (u \cdot v_2)v_2 + \cdots + (u \cdot k)v_k$$

$$= v_1(v_1 \cdot u) + v_2(v_2 \cdot u) + \cdots + v_k(v_k \cdot u)$$

$$= v_1({v_1}^T u) + v_2({v_2}^T u) + \cdots + v_k({v_k}^T u)$$

$$= (v_1 {v_1}^T)u + (v_2 {v_2}^T)u + \cdots + (v_k {v_k}^T)u$$

$$= (v_1 {v_1}^T + v_2 {v_2}^T + \cdots + v_k {v_k}^T)u$$

$$= [v_1\ v_2\ \cdots v_k] \begin{bmatrix} {v_1}^T \\ {v_2}^T \\ \vdots \\ \vdots \\ {v_k}^T \end{bmatrix} u = CCT\ u$$

Recall that is S is an arbitrary basis, the projection matrix is $C(C^T C)^{-1} C^T$

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Orthogonal Basis

Let $\{u_1, u_2, \cdots, u_k\}$ be a basis of a subspace V. How to transform $\{u_1, u_2, \cdots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \cdots, v_k\}$?

Gram-Schmidt Process

Then $\{v_1, v_2, \cdots, v_k\}$ is an orthogonal basis for W

After normalization, you can get orthonormal basis.

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
  - Define an inner product of P_2 by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

  - Find a basis $\{1, x, x^2\}$ u_1, u_2, u_3 ➡ v_1, v_2, v_3

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = x^2 - \frac{1}{3}$$

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
  - Define an inner product of P_2 by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

  - Get an orthogonal basis $\{1, x, x^2-1/3\}$

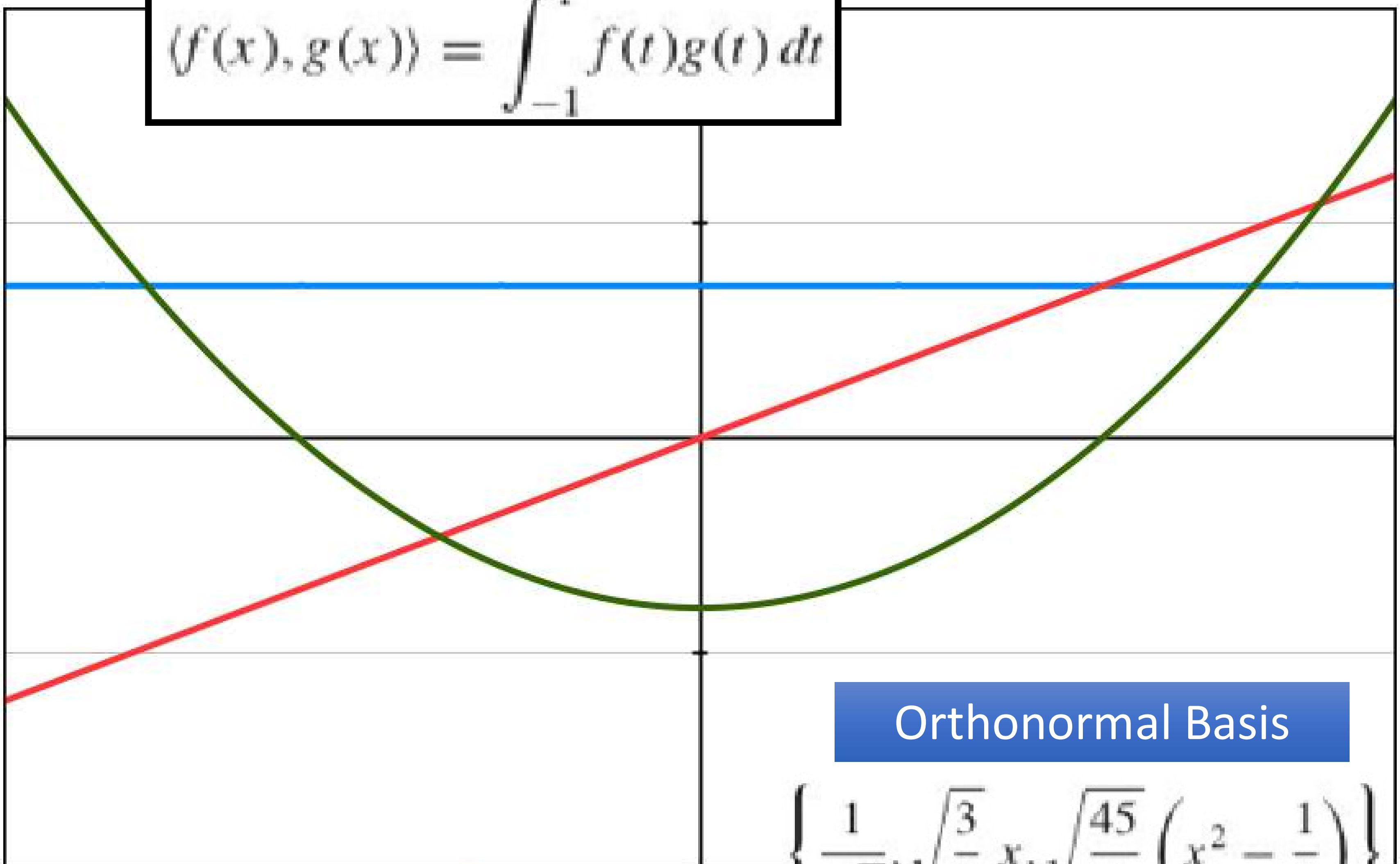
$$\|\mathbf{v}_1\| = \sqrt{\int_{-1}^{1} 1^2\,dx} = \sqrt{2} \qquad \|\mathbf{v}_2\| = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{v}_3\| = \sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} = \sqrt{\frac{8}{45}}$$

Orthonormal Basis

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right\}$$

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$$



Orthonormal Basis

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

Least square approx. of x^3?

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \right\}$$

v_1 v_2 v_3

Orthonormal Basis

Compute orthogonal projection of $f = x^3$ on P_2

Orthogonal projection is

$\langle f, v_1 \rangle\, v_1 + \langle f, v_2 \rangle\, v_2 + \langle f, v_3 \rangle\, v_3$

Hence, the l.s.a. of x^3 is 2/5 x

$\langle f, v_1 \rangle = \int_{-1}^{1} 1/\sqrt{2}\, t^3 dt$ = 0

$\langle f, v_2 \rangle = \int_{-1}^{1} \sqrt{3/2}\, t^4 dt$ = (2/5)$\sqrt{3/2}$

$$\langle f, v_3 \rangle = \int_{-1}^{1} \sqrt{45/8}\,(t^2 - 1/3) t^3 dt = 0$$