Chapter 5

Eigenvalues, Eigenvectors, and Diagonozation

(除了標註※之簡報外，其餘採用李宏毅教授之投影片教材)

Eigenvalues and eigenvectors

(Chapter 5.1)

Eigenvalues and Eigenvectors

$$A\mathbf{0} = \lambda\mathbf{0}$$

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A **excluding zero vector**
  - λ is an eigenvalue of A that corresponds to v

A must be square

Eigen value

$$\begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

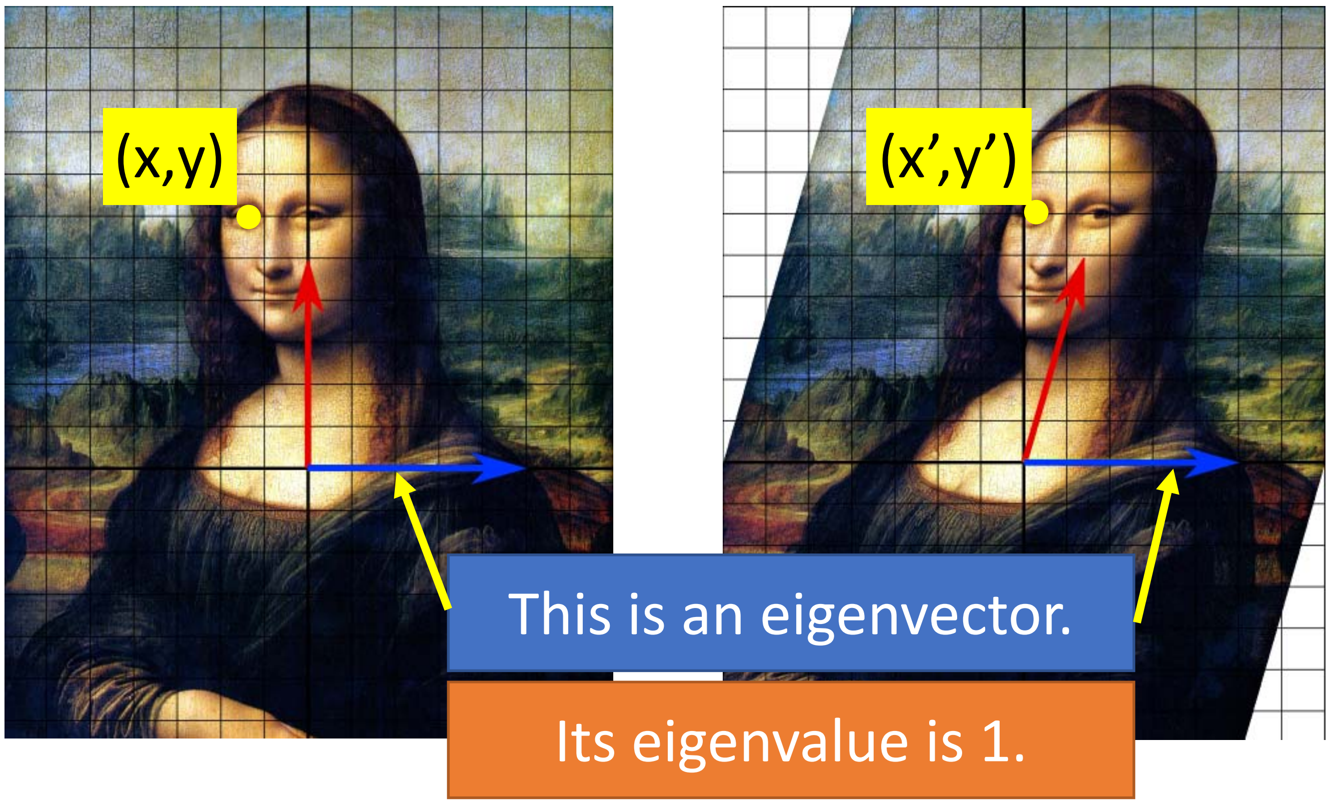
Eigen vector

Eigenvalues and Eigenvectors

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- T is a ***linear operator.*** If $\mathrm{T}(v) = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of T excluding zero vector
  - λ is an eigenvalue of T that corresponds to v

Eigenvalues and Eigenvectors

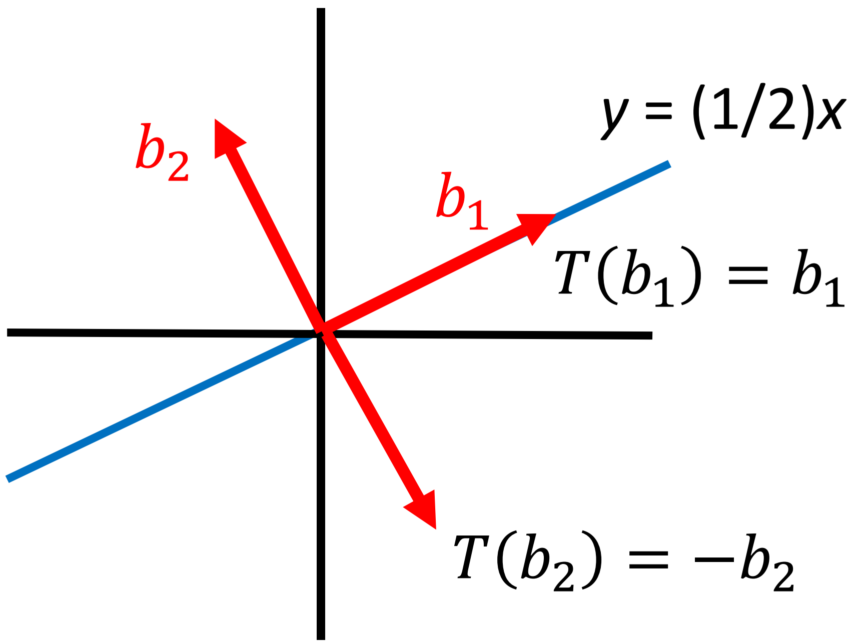
- Example: Shear Transform $\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$



Eigenvalues and Eigenvectors

- Example: Reflection

reflection operator T about the line $y = (1/2)x$



$\mathbf{b}_1$ is an eigenvector of T

Its eigenvalue is 1.

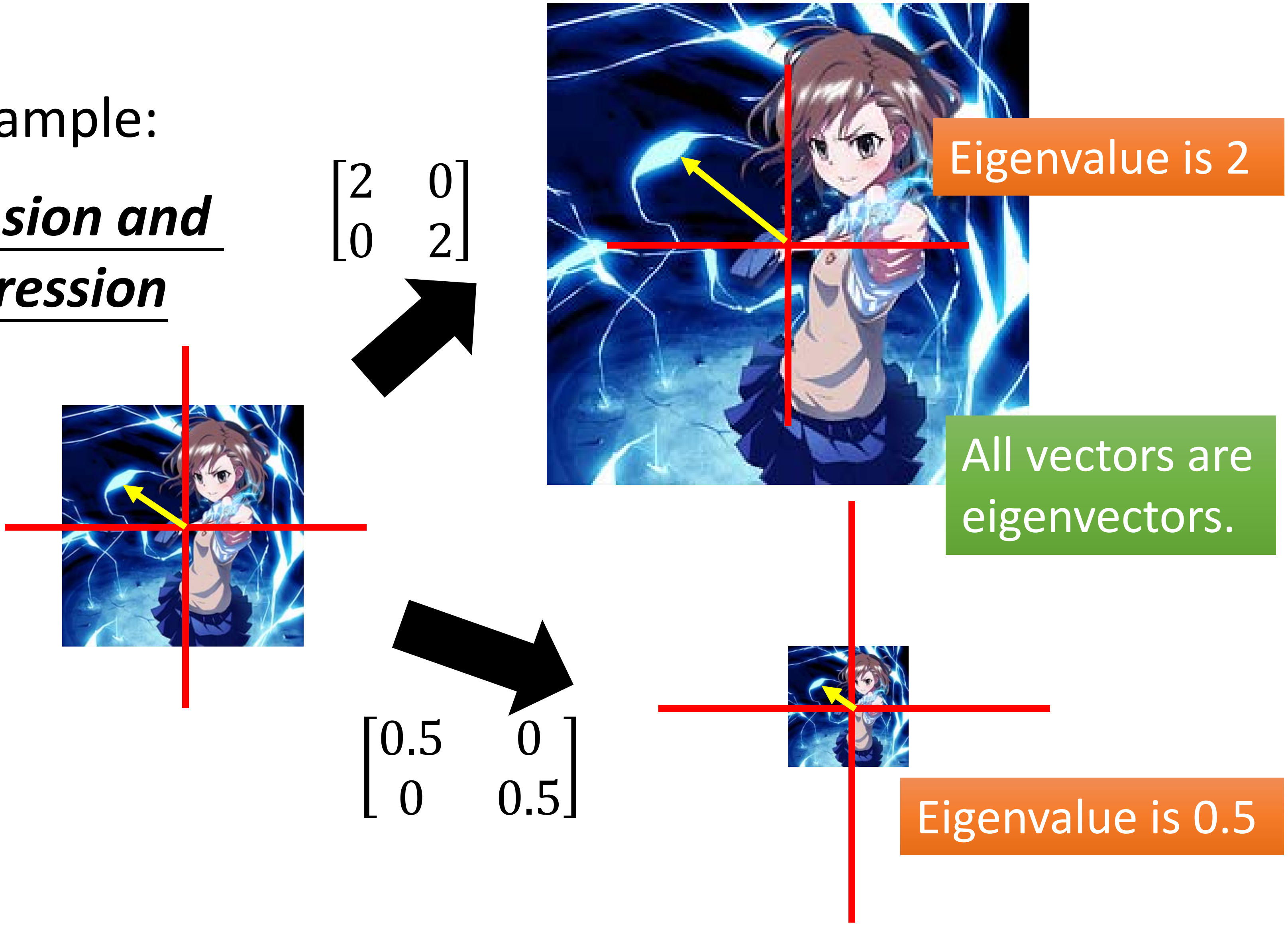
$\mathbf{b}_2$ is an eigenvector of T

Its eigenvalue is -1.

Eigenvalues and Eigenvectors

- Example:

<u>Expansion and Compression</u>



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Eigenvalues and Eigenvectors

- Example: Rotation

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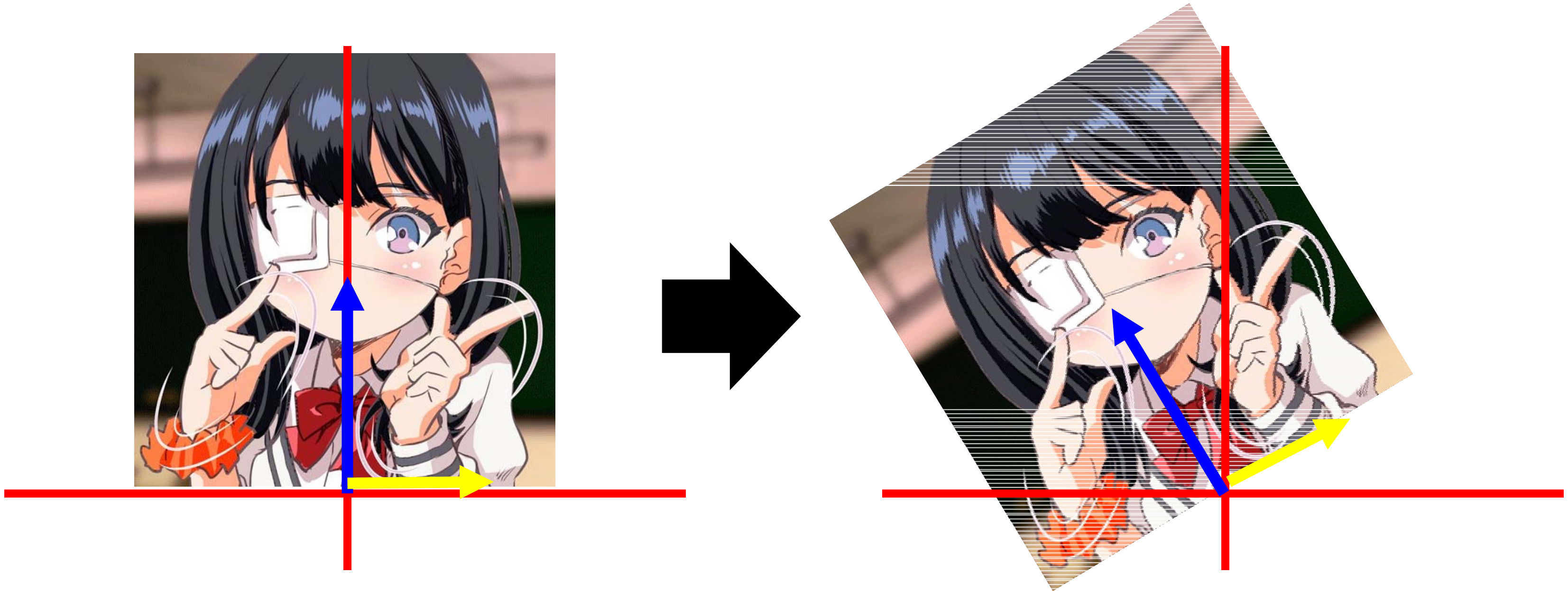


Do any n x n matrix or linear operator have eigenvalues?

How to find eigenvectors (given eigenvalues)

(Chapter 5.1)

Eigenvalues and Eigenvectors

- An eigenvector of *A* corresponds to a unique eigenvalue.
- An eigenvalue of *A* has infinitely many eigenvectors.

Example:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvalue= -1

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Eigenvalue= -1

Do the eigenvectors correspond to the same eigenvalue form a subspace?

$$Av = \lambda v \qquad A(cv) = \lambda(cv)$$

$$Au = \lambda u \qquad A(u+v) = \lambda(u+v)$$

Eigenspace

- Assume we know λ is the eigenvalue of matrix A
- Eigenvectors corresponding to λ

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$A\mathbf{v} - \lambda I_n\mathbf{v} = \mathbf{0}$$

$$\underbrace{(A - \lambda I_n)}_{\text{matrix}}\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ are nonzero solution of

$$(A - \lambda I_n)\mathbf{v} = \mathbf{0}$$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

Check whether a scalar is an eigenvalue

(Chapter 5.1)

Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- How to know whether a scalar λ is the eigenvalue of A?

Check the dimension of eigenspace of λ

If the dimension is 0

Eigenspace only contains {0}

No eigenvector

λ is not eigenvalue

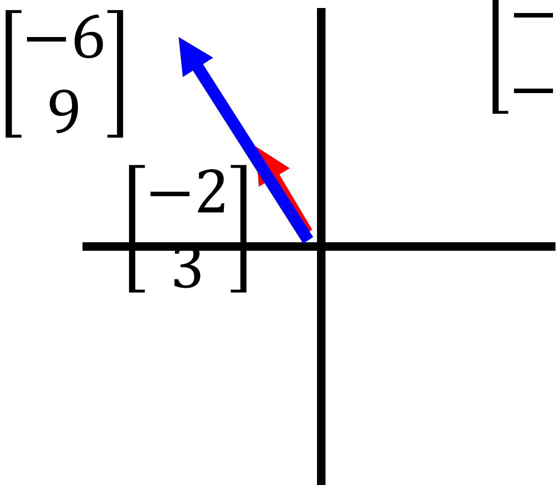
Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

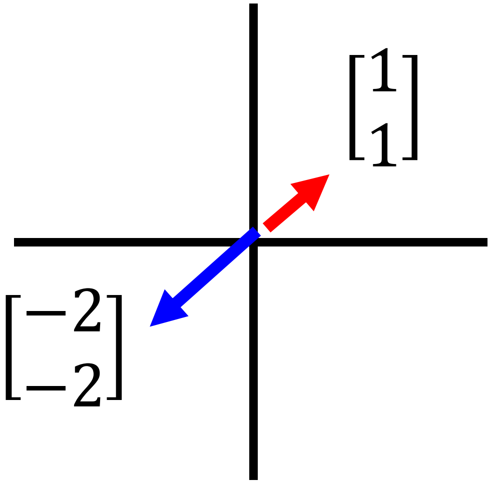
- Example: to check 3 and -2 are eigenvalues of the linear operator T

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$Null(A - 3I_n)$=? $Null(A + 2I_n)$=?



$$\begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

Check Eigenvalues

$Null(A - \lambda I_n)$:
eigenspace of λ

- Example: check that 3 is an eigenvalue of B and find a basis for the corresponding eigenspace

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

find the solution set of $(B - 3I_3)\mathbf{x} = \mathbf{0}$

find the RREF of

$B - 3I_3$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Looking for eigenvalues

(Chapter 5.2)

Looking for Eigenvalues

A scalar t is an eigenvalue of A

⇔ Existing $v \neq 0$ such that $Av = tv$

⇔ Existing $v \neq 0$ such that $Av - tv = 0$

⇔ Existing $v \neq 0$ such that $(A - tI_n)v = 0$

⇔ $(A - tI_n)v = 0$ has multiple solution

⇔ The columns of $(A - tI_n)$ are Dependent

⇔ $(A - tI_n)$ is not invertible

⇔ $det(A - tI_n) = 0$

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

A scalar t is an eigenvalue of A ⟺ $det(A - tI_n) = 0$

$$A - tI_2 = \begin{bmatrix} -4-t & -3 \\ 3 & 6-t \end{bmatrix}$$

$\det(A - tI_2)$ =0

➡ t = -3 or 5

The eigenvalues of *A* are -3 or 5.

Looking for Eigenvalues

- Example 1: Find the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$

The eigenvalues of *A* are -3 or 5.

<u>Eigenspace of -3</u>

$Ax = -3x$ ➡ $(A + 3I)x = 0$

find the solution

<u>Eigenspace of 5</u>

$Ax = 5x$ ➡ $(A - 5I)x = 0$

find the solution

Looking for Eigenvalues

- Example 2: find the eigenvalues of linear operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix} \xrightarrow[\text{standard matrix}]{} A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

A scalar t is an eigenvalue of A $\Leftrightarrow$ $det(A - tI_n) = 0$

$$A - tI_n = \begin{bmatrix} -1-t & 0 & 0 \\ 2 & -1-t & -1 \\ 0 & 0 & -1-t \end{bmatrix}$$

$\Rightarrow$ $det(A - tI_n) = (-1-t)^3$

Looking for Eigenvalues

- Example 3: linear operator on $\mathcal{R}^2$ that rotates a vector by 90°

A scalar t is an eigenvalue of A ⟺ $det(A - tI_n) = 0$

standard matrix of the 90°-rotation: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - tI_2\right)$$

No eigenvalues, no eigenvectors

Characteristic Polynomial

A scalar t is an eigenvalue of A $\Leftrightarrow$ $det(A - tI_n) = 0$

A is the standard matrix of linear operator T

$det(A - tI_n)$: Characteristic polynomial of A

linear operator T

$det(A - tI_n) = 0$: Characteristic equation of A

linear operator T

Eigenvalues are the roots of characteristic polynomial or solutions of characteristic equation.

Characteristic Polynomial

- In general, a matrix A and RREF of A have different characteristic polynomials. ➡ Different Eigenvalues
- Similar matrices have the same characteristic polynomials ➡ The same Eigenvalues

$B = P^{-1}AP$

$$det(B - tI) = det(P^{-1}AP - P^{-1}(tI)P)$$

$$= det(P^{-1}(A - tI)P)$$

$$= det(P^{-1})det(A - tI)det(P)$$

$$= \left(\frac{1}{det(P)}\right)det(A - tI)det(P) = det(A - tI)$$

Characteristic Polynomial

- Question: What is the order of the characteristic polynomial of an $n \times n$ matrix A?
  - The characteristic polynomial of an $n \times n$ matrix is indeed a polynomial with degree n
  - Consider $\det(A - tI_n)$

$$det \begin{bmatrix} ?-t & \cdots & ? \\ \vdots & \ddots & \vdots \\ ? & \cdots & ?-t \end{bmatrix}$$

- Question: What is the number of eigenvalues of an $n \times n$ matrix A?
  - Fact: An n x n matrix A have less than or equal to n eigenvalues
  - Consider complex roots and multiple roots

Characteristic Polynomial vs. Eigenspace

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	$d_1 \leq m_1$	$d_2 \leq m_2$	$d_k \leq m_k$

Characteristic Polynomial

- The eigenvalues of an upper triangular matrix are its diagonal entries.

$$\begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}$$

Characteristic Polynomial:

$$det\begin{bmatrix} a-t & * & * \\ 0 & b-t & * \\ 0 & 0 & c-t \end{bmatrix}$$

$$= (a-t)(b-t)(c-t)$$

The determinant of an upper triangular matrix is the product of its diagonal entries.

Diagonalization

(Chapter 5.3)

Review

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= Null(A - \lambda I_n) - \{\mathbf{0}\}$$

($Null(A - \lambda I_n)$: eigenspace)

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

- A scalar t is an eigenvalue of A $\Leftrightarrow$ $det(A - tI_n) = 0$

Review

- Characteristic polynomial of A is

$det(A - tI_n)$ **Factorization**

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\ldots(t-\lambda_k)^{m_k}(\ldots\ldots)$$

Algebraic multiplicity

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace: (dimension)	$d_1 \le m_1$	$d_2 \le m_2$	$d_k \le m_k$

Outline

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Is a matrix A diagonalizable?
  - If yes, find D and P

Diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

➡ $A^2 = 0$ (?)

- Not all matrices are diagonalizable

If $A = PDP^{-1}$ for some invertible P and diagonal D

➡ $A^2 = PD^2P^{-1} = 0$ ➡ $D^2 = 0$ ➡ $D = 0$

D is diagonal

➡ A= 0 ✗

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} \qquad D^2 = \begin{bmatrix} (d_1)^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (d_n)^2 \end{bmatrix}$$

$d_1 = 0$, $d_n = 0$; $(d_1)^2 = 0$, $(d_n)^2 = 0$

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix} = [d_1 e_1 \quad \cdots \quad d_n e_n]$$

- If A is diagonalizable

$$A = PDP^{-1} \Rightarrow AP = PD$$

$$\Rightarrow AP = [\underline{Ap_1} \quad \cdots \quad \underline{Ap_n}]$$

$$\Rightarrow PD = P[d_1 e_1 \quad \cdots \quad d_n e_n]$$

$$= [Pd_1 e_1 \quad \cdots \quad Pd_n e_n]$$

$$= [d_1 Pe_1 \quad \cdots \quad d_n Pe_n]$$

$$= [\underline{d_1 p_1} \quad \cdots \quad \underline{d_n p_n}] \Rightarrow Ap_i = d_i p_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

=

There are n eigenvectors that form an invertible matrix

=

There are n independent eigenvectors

=

The eigenvectors of A can form a basis for R^n.

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

How to diagonalize a matrix A?

Step 1: Find *n* independent eigenvectors corresponding if possible, and form an invertible *P*

Step 2: The eigenvalues corresponding to the eigenvectors in P form the diagonal matrix *D.*

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linearly independent.

$det(A - tI_n)$ **Factorization**

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \dots (t-\lambda_k)^{m_k}(\dots\dots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: (dimension) $d_1 \le m_1$ $d_2 \le m_2$ …… $d_k \le m_k$

Independent

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linearly independent.

Eigenvalue: λ_1 λ_2 ...... λ_m

Eigenvector: v_1 v_2 ...... v_m

Assume dependent → a contradiction

$$\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{k-1}\mathbf{v}_{k-1}$$

$$A\mathbf{v}_k = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_{k-1} A\mathbf{v}_{k-1}$$

$$\lambda_k\mathbf{v}_k = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}$$

(λ_k)

$$- \quad \lambda_k\mathbf{v}_k = c_1\lambda_k\mathbf{v}_1 + c_2\lambda_k\mathbf{v}_2 + \cdots + c_{k-1}\lambda_k\mathbf{v}_{k-1}$$

$$\mathbf{0} = c_1(\lambda_1-\lambda_k)\mathbf{v}_1 + c_2(\lambda_2-\lambda_k)\mathbf{v}_2 + \cdots + c_{k-1}(\lambda_{k-1}-\lambda_k)\mathbf{v}_{k-1}$$

Not $c_1 = c_2 = \cdots = c_{k-1} = 0$ → *Same eigenvalue* → a contradiction

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

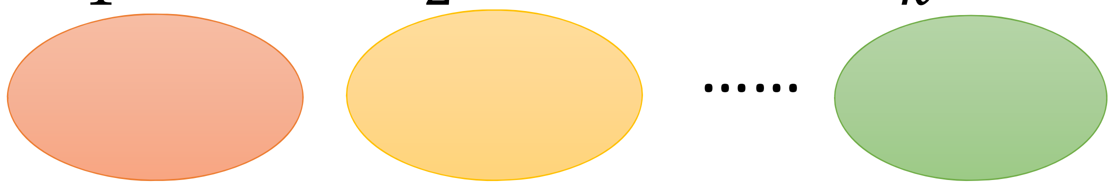
$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$det(A - tI_n)$$
$$= (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}(\ldots \ldots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: ……



{ Basis for λ_1 Basis for λ_2 Basis for λ_3 }

Independent Eigenvectors

You can't find more!

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t+1)^2(t-3)$ ⇨ eigenvalues: 3, –1

eigenvalue 3

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue –1

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Test for a Diagonalizable Matrix

- An *n* x *n* matrix *A* is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of *A* factors into a product of linear factors.

$det(A - tI_n)$ Factorization

$= (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \ldots (t - \lambda_k)^{m_k}(\ldots\ldots)$

For each eigenvalue λ of *A*, the multiplicity of λ (algebraic multiplicity) equals the dimension of the corresponding eigenspace (geometric multiplicity).

Independent Eigenvectors

An n x n matrix A is diagonalizable

$\Vert$

The eigenvectors of A can form a basis for R^n.

$\Vert$

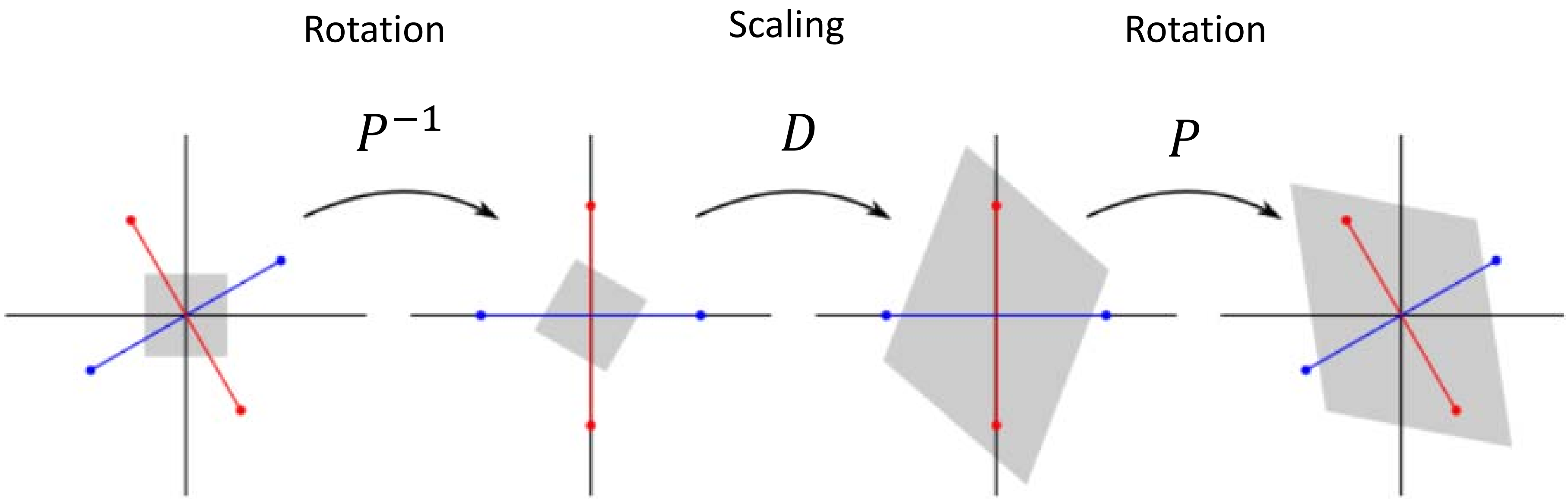
$$det(A - tI_n)$$

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots\ldots)$$

Eigenvalue:	λ_1	λ_2	......	λ_k
Eigenspace: (dimension)	$d_1 = m_1$	$d_2 = m_2$	......	$d_k = m_k$

$$d_1 + d_2 + \cdots + d_k = n$$

Geometric Meaning of Diagonalization $A = PDP^{-1}$



Red and blue axes correspond to the directions of two eigenvectors

※

How to Cope with Non-diagonalizable Matrices

Question: If A is not diagonizable (i.e., $A \neq PDP^{-1}$) , can we write $\mathrm{A} = UTU^{-1}$, where T is "near diagonal"?

Schur Decomposition: Any square matrix A can be written as $\mathrm{A} = UTU^{-1}$, where U is an "orthonormal matrix" and T is "upper triangular" with eigenvalues on the diagonal.

$$T = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Jordan Normal Form

Theorem: For every n x n matrix A, there exists an invertible matrix Q such that $Q^{-1}AQ = J$ where J is in Jordan Normal Form.

Jordan Block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$$

Jordan Normal Form

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigenspace of 2 has dim = 1

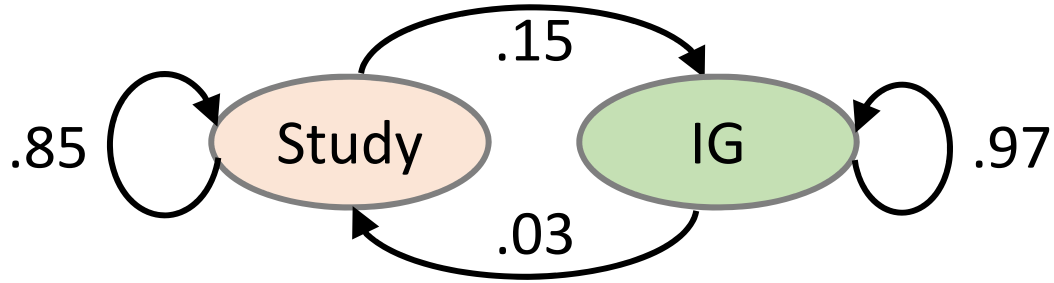
Eigenspace of 3 has dim = 2

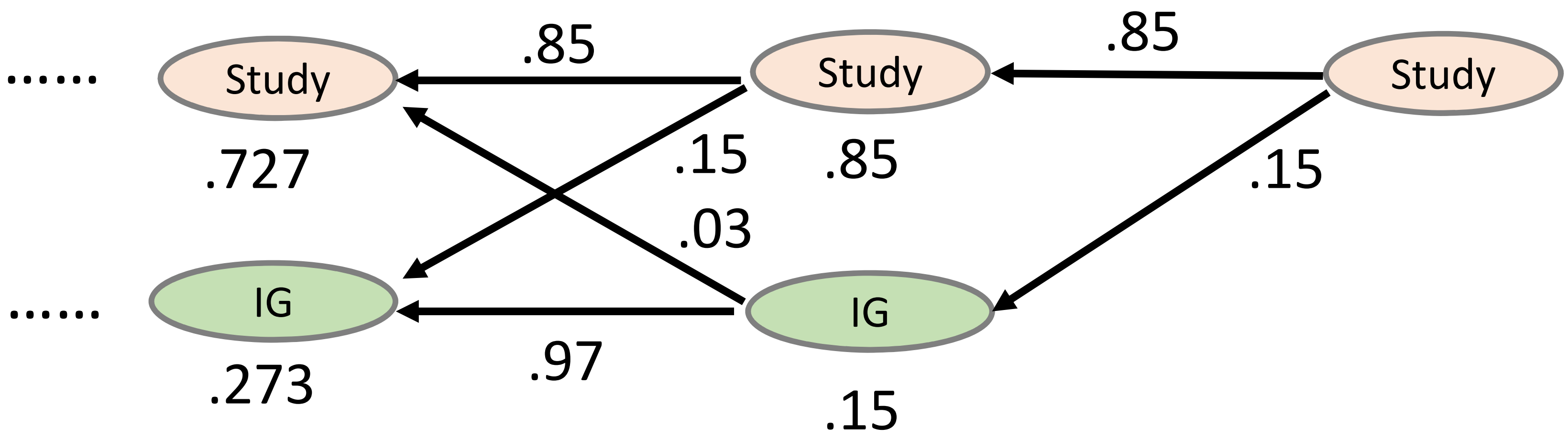
Application of Diagonalization

- If A is diagonalizable,

$$A = PDP^{-1} \Rightarrow A^m = PD^mP^{-1}$$

- Example:

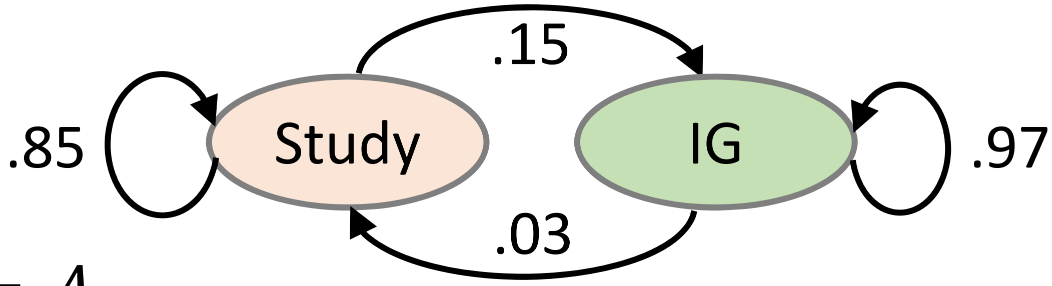


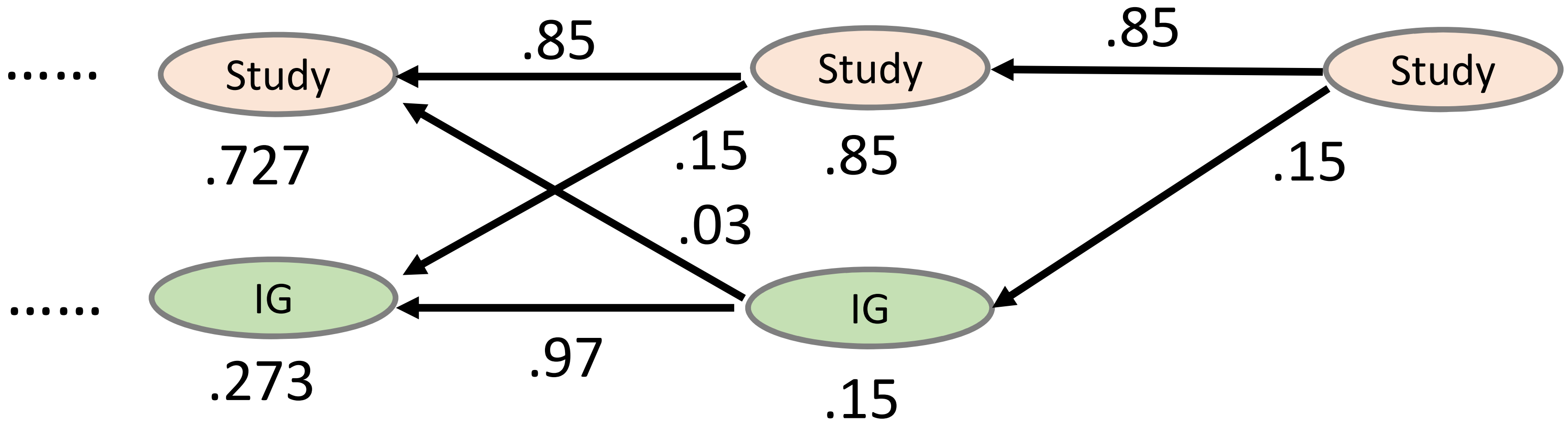


From

		Study	IG
To	Study	.85	.03
	IG	.15	.97

$$\begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} = A$$





$$\begin{bmatrix} .727 \\ .273 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} .85 \\ .15 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det\ (A - tI_2)$$

$$A - .82I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A - I_2 \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$$

(invertible)

$$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1+5(.82)^m & 1-(.82)^m \\ 5-5(.82)^m & 5+(.82)^m \end{bmatrix} \end{aligned}$$

When $m \to \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

$$\begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} \quad \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$$

Diagonalization of linear operators*

(Chapter 5.4)

Diagonalization of Linear Operator

- Example 1: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

det

The standard matrix is $A = \begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}$

-t -t -t

$\Rightarrow$ the characteristic polynomial is $-(t+1)^2(t-2)$

Eigenvalue -1:

$\mathcal{B}_1 = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Eigenvalue 2:

$\mathcal{B}_2 = \left\{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right\}$

$\Rightarrow \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathcal{R}^3$

Diagonalization of Linear Operator

- Example 2: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

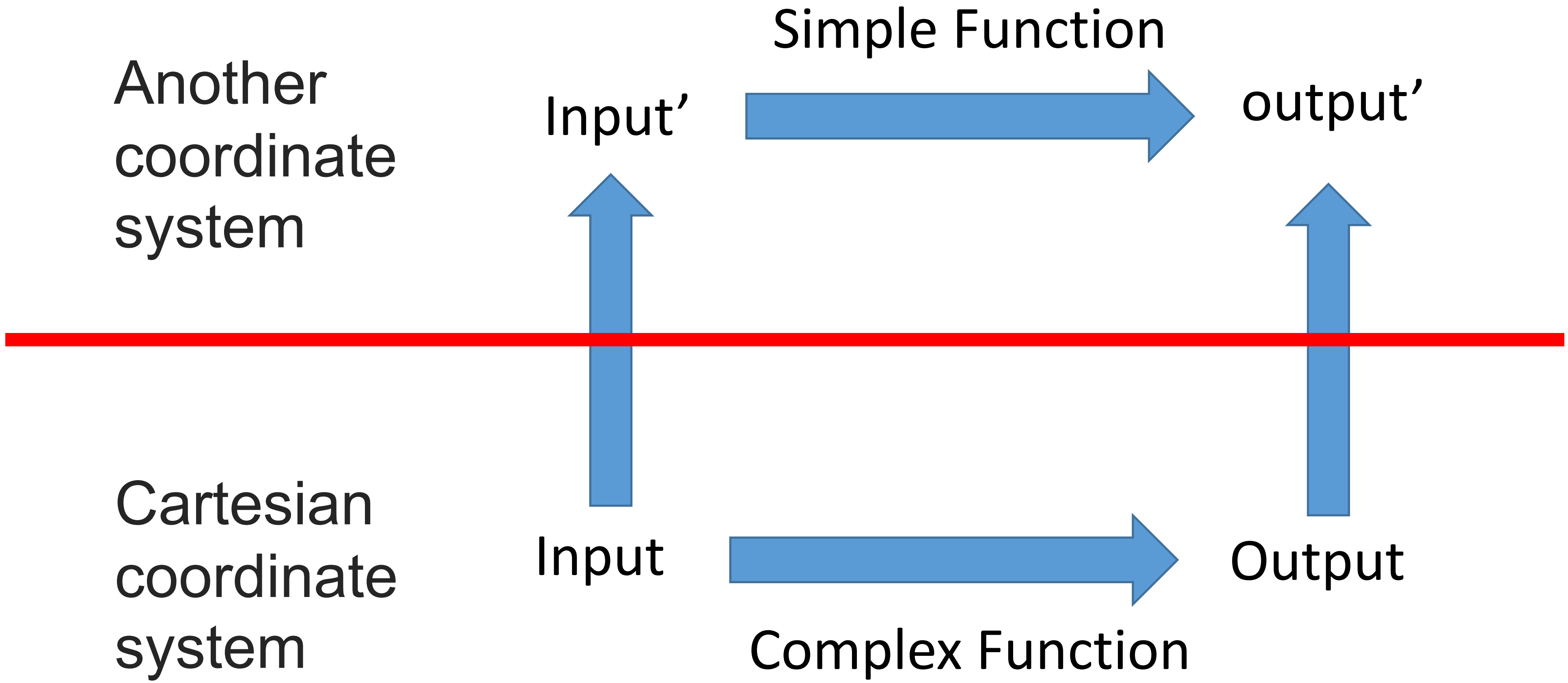
det -t -t -t

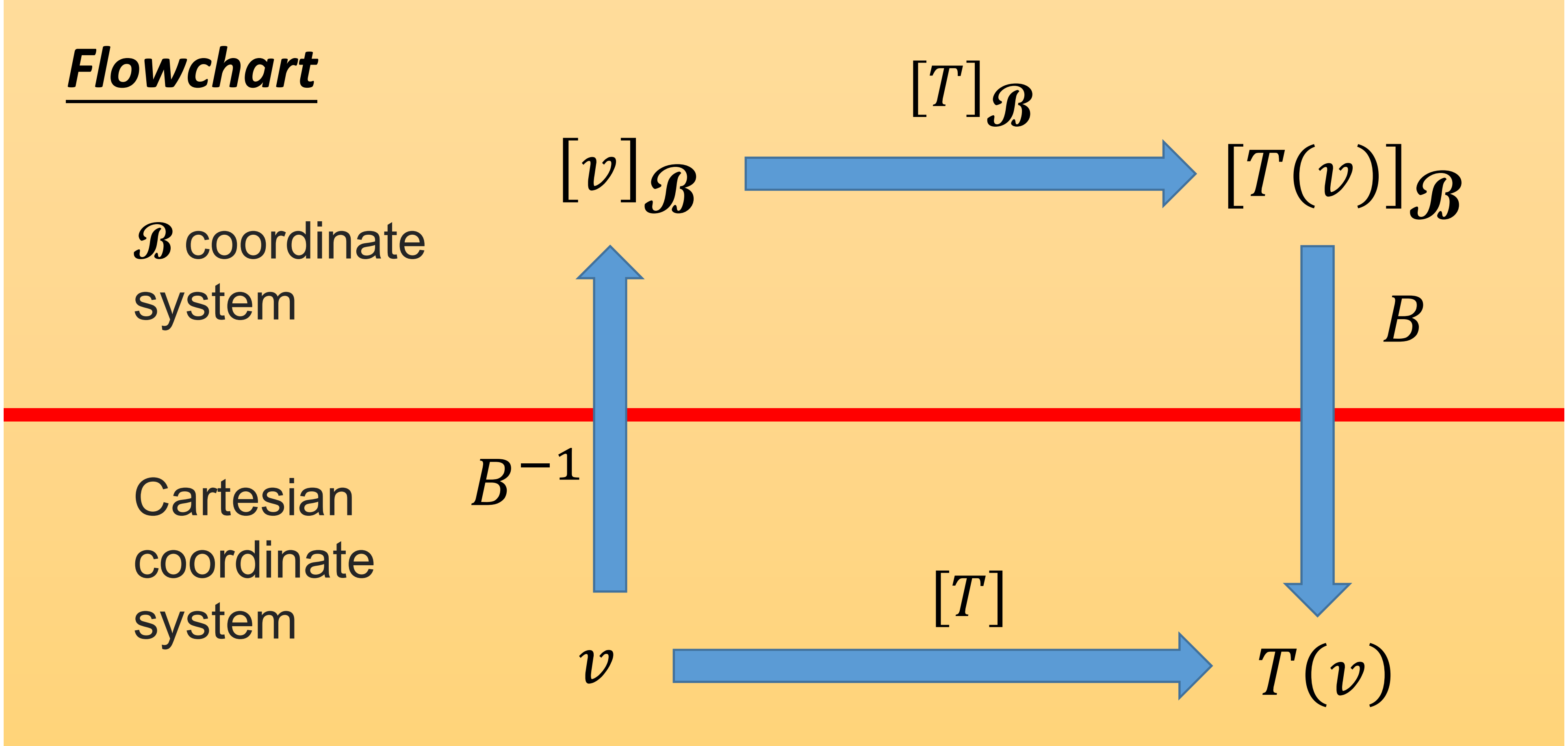
The standard matrix is $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Review


Another coordinate system
Simple Function
Input'
output'
Cartesian coordinate system
Input
Output
Complex Function



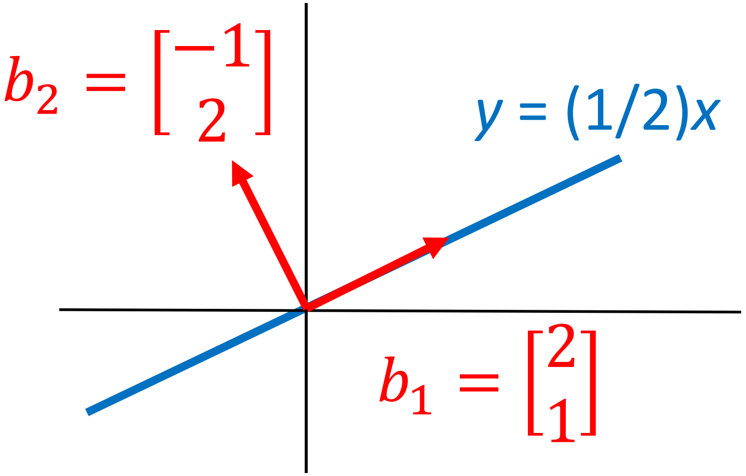
$$[T] = B[T]_{\mathscr{B}}B^{-1}$$

similar

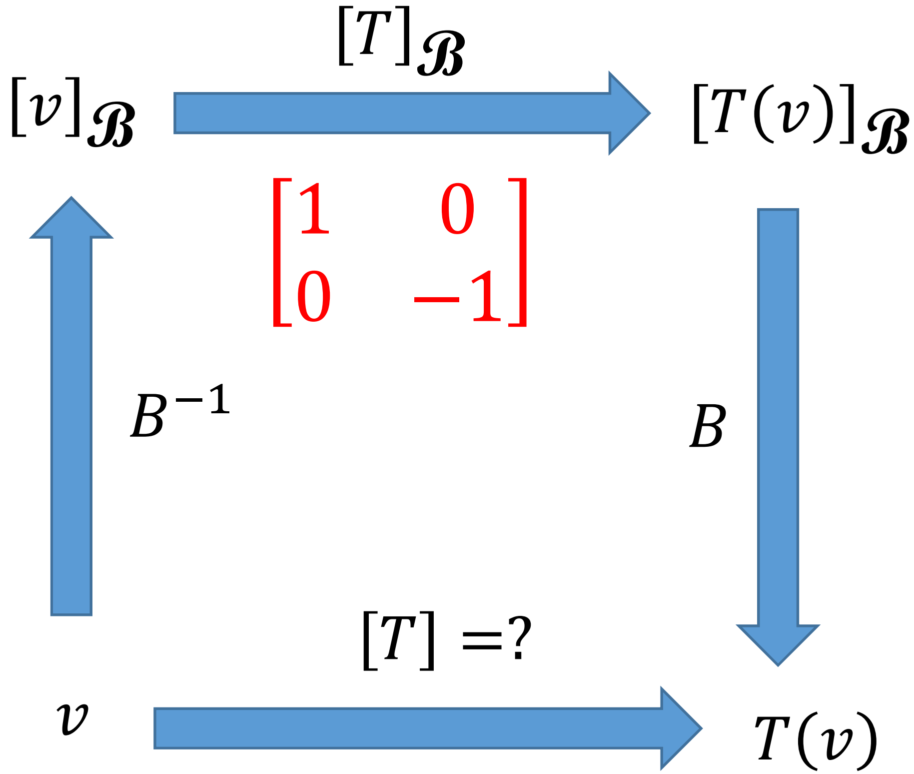
$$[T]_{\mathscr{B}} = B^{-1}[T]B$$

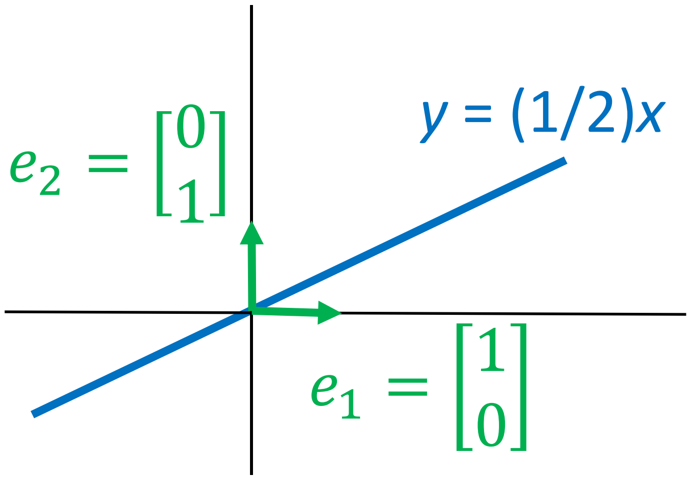
similar

- Example: reflection operator T about the line $y = (1/2)x$



$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

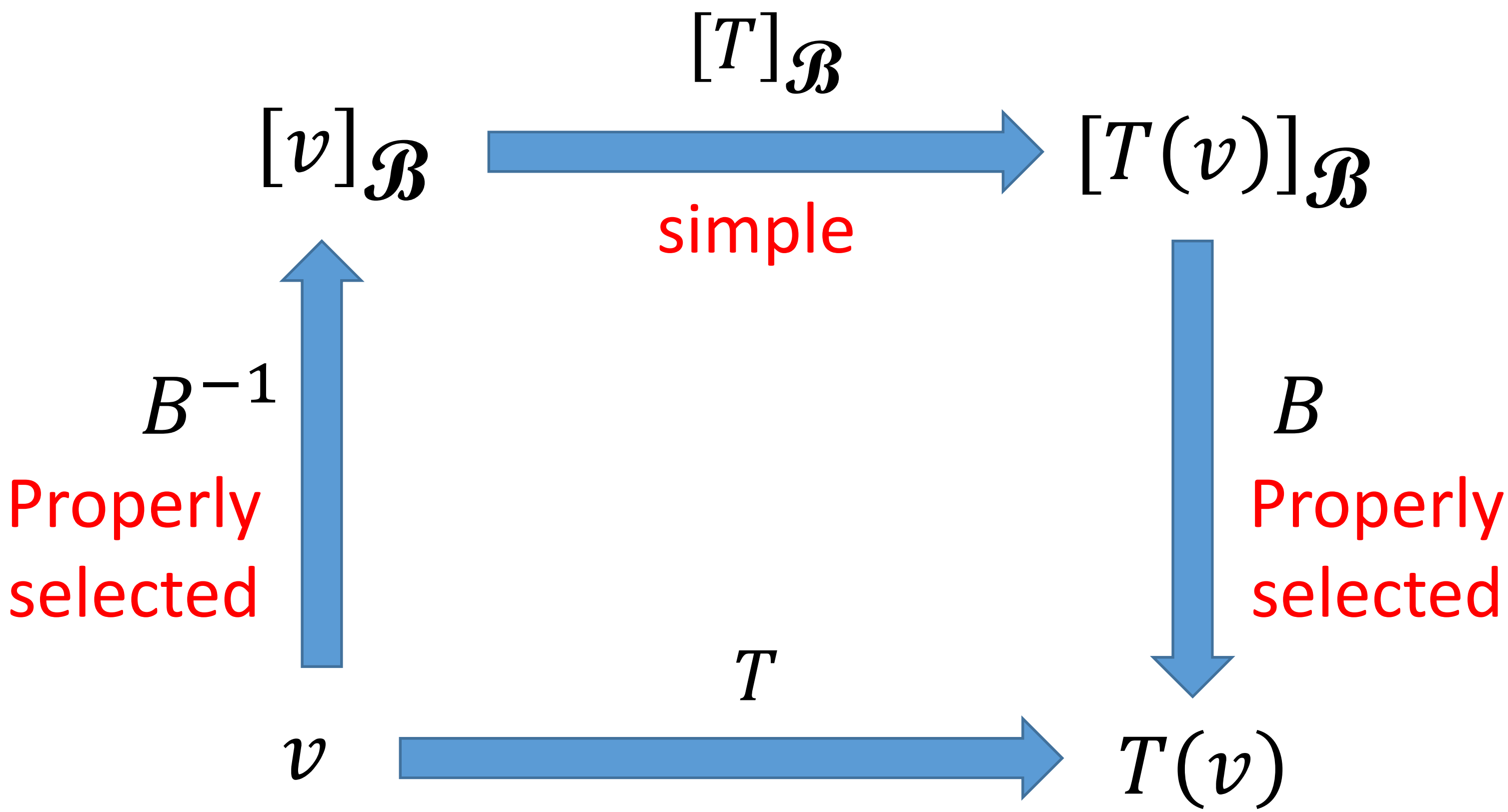




$$[T] = B[T]_{\mathcal{B}}B^{-1}$$

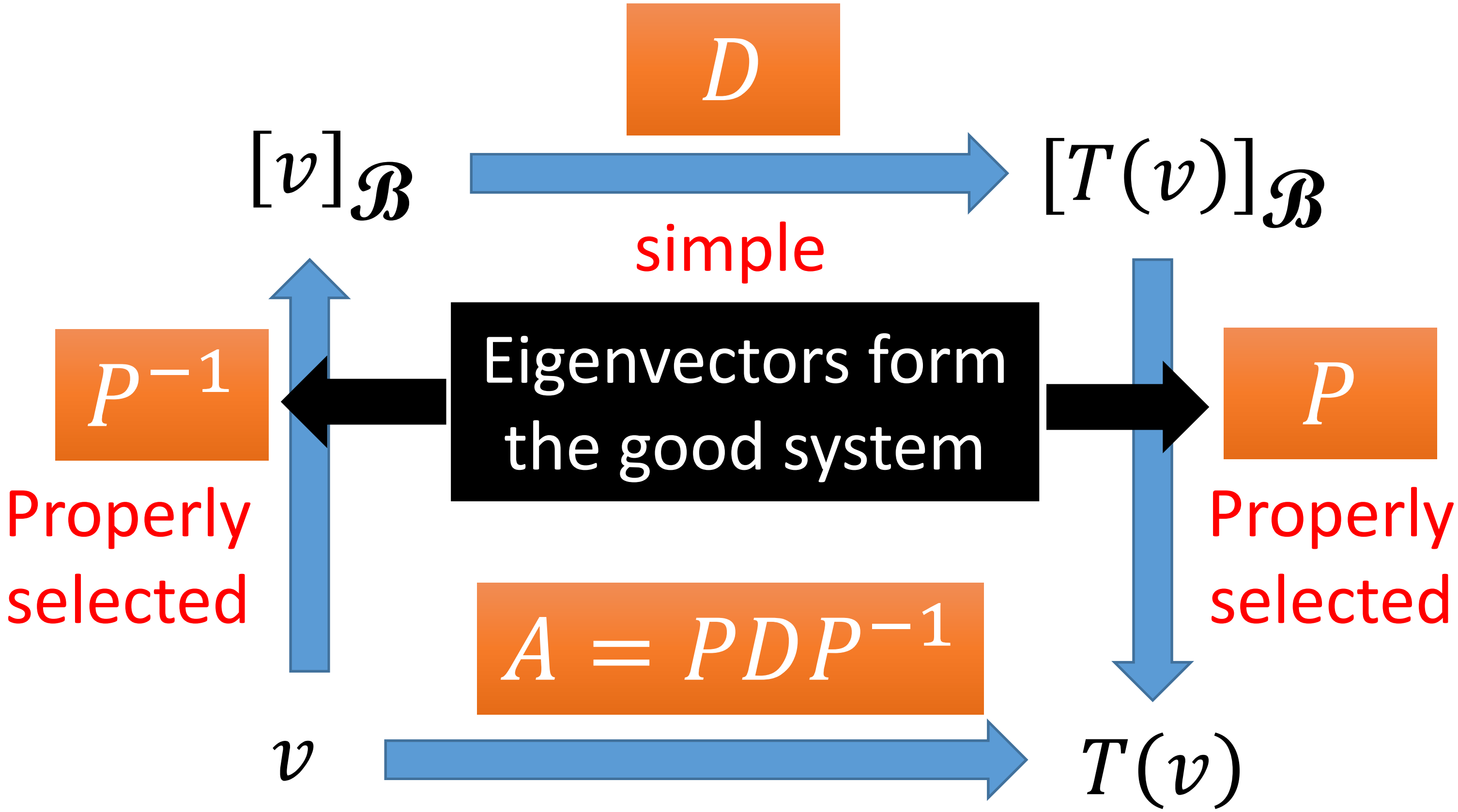
Diagonalization of Linear Operator

- Reference: Chapter 5.4



Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$$

-1: $\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2: $\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$

$$[v]_{\mathcal{B}} \xrightarrow{[T]_{\mathcal{B}}} [T(v)]_{\mathcal{B}}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$v \to [v]_{\mathcal{B}}$: $\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$

$[T(v)]_{\mathcal{B}} \to T(v)$: $\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

$$v \xrightarrow{\begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}} T(v)$$