Chapter 4
Subspaces and Their Properties

除了標註※之簡報外，其餘採用李宏毅教授之投影片教材

Subspaces

(Chapter 4.1)

Subspace

- A vector set V is called a subspace of a vector space W if it has the following three properties:
- 1. The zero vector **0** belongs to V
- 2. If **u** and **w** belong to V, then **u+w** belongs to V

Closed under (vector) addition

- 3. If **u** belongs to V, and c is a scalar, then c**u** belongs to V

Closed under scalar multiplication

2+3 is linear combination

Examples

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathcal{R}^3 : 6w_1 - 5w_2 + 4w_3 = 0 \right\}$$

Subspace of $\mathcal{R}^3$?

Property 1. $\mathbf{0} \in W$ $\quad 6(0) - 5(0) + 4(0) = 0$

Property 2. $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u}+\mathbf{v} \in W$

$\mathbf{u} = [\, u_1 \;\; u_2 \;\; u_3 \,]^T, \mathbf{v} = [\, v_1 \;\; v_2 \;\; v_3 \,]^T \quad \mathbf{u}+\mathbf{v} = [\, u_1+v_1 \;\; u_2+v_1 \;\; u_3+v_1 \,]^T$

$6(u_1+v_1) - 5(u_2+v_2) + 4(u_3+v_3)$

$= (6u_1 - 5u_2 + 4u_3) + (6v_1 - 5v_2 + 4v_3) = 0 + 0 = 0$

Property 3. $\mathbf{u} \in W \Rightarrow c\mathbf{u} \in W$

$6(cu_1) - 5(cu_2) + 4(cu_3) = c(6u_1 - 5u_2 + 4u_3) = c0 = 0$

Examples

$V = \{c\mathbf{w} \mid c \in \mathcal{R}\}$ Subspace?

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1 \geq 0 \text{ and } w_2 \geq 0 \right\}$$

Subspace? $\mathbf{u} \in \mathcal{S}_1, \mathbf{u} \neq \mathbf{0} \Rightarrow -\mathbf{u} \notin \mathcal{S}_1$

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1^2 = w_2^2 \right\}$$

Subspace? $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in S_2$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin S_2$

$\mathcal{R}^n$ Subspace? $\{\mathbf{0}\}$ Subspace? zero subspace

Subspace vs. Span

- The span of a vector set is a subspace

Let $S = \{w_1, w_2, \cdots, w_k\}$ $\quad V = Span\ S$

Property 1. $\mathbf{0} \in V$

Property 2. $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} + \boldsymbol{v} \in V$

Property 3. $\boldsymbol{u} \in V$, $c\boldsymbol{u} \in V$

Column Space and Row Space

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

- Column space of an $m \times n$ matrix A is the span of its columns. It is denoted as Col A.

$$A \in \mathcal{R}^{m \times n} \Rightarrow \text{Col } A = \{A\mathbf{v} : \mathbf{v} \in \mathcal{R}^n\}$$

If matrix A represents a function A: $\mathcal{R}^n \rightarrow \mathcal{R}^m$

Col A is the range of the function (a subspace in $\mathcal{R}^m$)

- Row space of an $m \times n$ matrix A is the span of its rows. It is denoted as Row A.

$$\text{Row } A = \text{Col } A^T$$

(Row A is a subspace in $\mathcal{R}^n$)

Column Space = Range

- The range of a linear transformation is the same as the column space of its matrix.

Linear Transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$

Standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$ $\Rightarrow$ Range of *T* =

$$\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix}\right\}$$

RREF

- Original Matrix A vs. its RREF R
  - Columns:
    - The relations between the columns are the same.
    - The span of the columns are different.

$$Col\ \mathrm{A} \neq Col\ R$$

  - Rows:
    - The relations between the rows are changed.
    - The span of the rows are the same.

$$Row\ \mathrm{A} = Row\ R$$

Consistent

Ax = b have a solution (consistent)

b is the linear combination of columns of A

b is in the span of the columns of A

b is in Col A

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \text{Col } A? \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \text{Col } A?$$

Solving Ax = u

RREF([A u]) = $\begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solving Ax = v

RREF([A v]) = $\begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Null Space

- The null space of an $m \times n$ matrix A is the solution set of Ax=0. It is denoted as Null A.

$$\text{Null } A = \{ \mathbf{v} \in \mathcal{R}^n : A\mathbf{v} = \mathbf{0} \}$$

The solution set of the homogeneous linear equations $A\mathbf{v} = \mathbf{0}$.

- Null A is a subspace in $\mathcal{R}^n$

A linear function is one-to-one ⇄ Null space only contain **0**

Null Space - Example

$$T : \mathcal{R}^3 \to \mathcal{R}^2 \text{ with } T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$$

Find a generating set for the null space of *T*.

The null space of *T* is the set of solutions to *A***x** = **0**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -3 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = x_2$$

~~$x_1 - x_2 = 0$~~

$$x_3 = 0$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

a generating set for the null space

Basis

(Chapter 4.2)

Basis

Why nonzero?

- Let V be a nonzero subspace of R^n. A basis B for V is a linearly independent generating set of V.

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathcal{R}^n$.

1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is independent
2. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ generates $\mathcal{R}^n$.

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for $\mathcal{R}^2$

$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$ …… any two independent vectors form a basis for $\mathcal{R}^2$

Basis

- The pivot columns of a matrix form a basis for its column space.

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

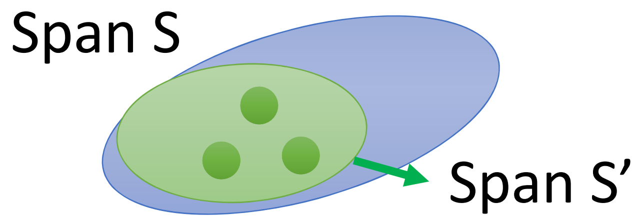
$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Property

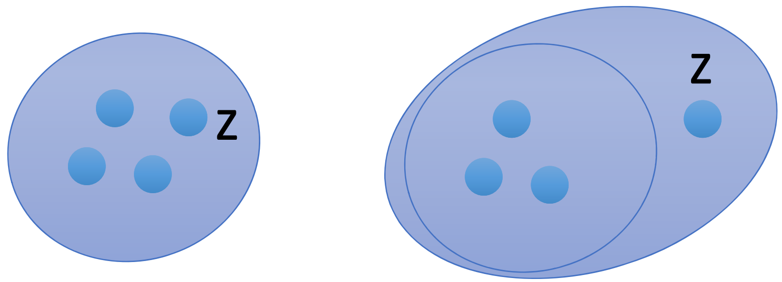
- (a) S is contained in Span S

Basis is always in its subspace

- (b) If a finite set S' is contained in Span S, then Span S' is also contained in Span S
  - Because Span S is a subspace



- (c) For any vector z, Span S = Span SU{z} if and only if z belongs to the Span S



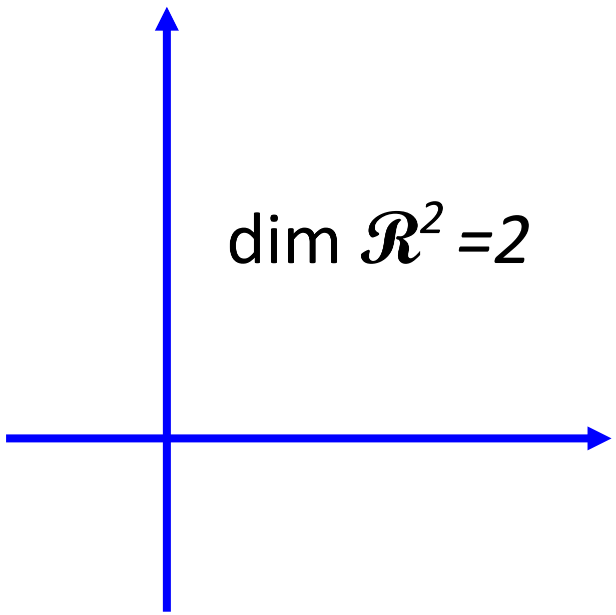
Theorem

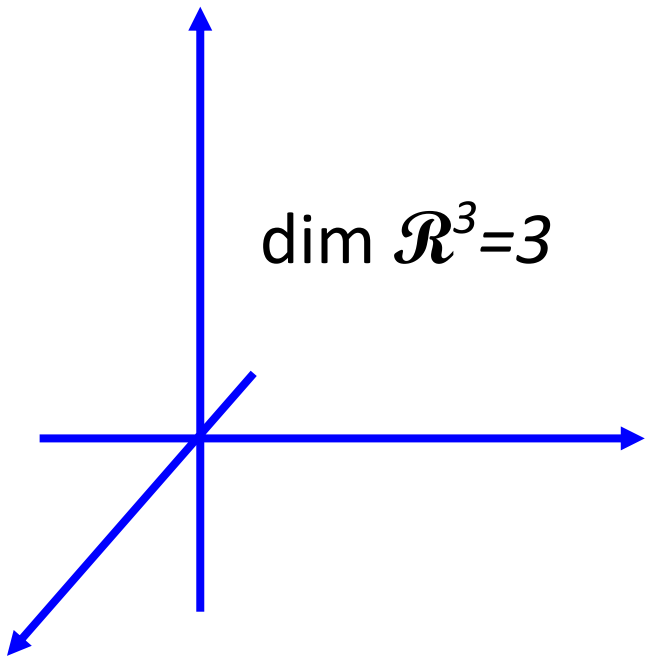
- 1. A basis is the smallest generating set.
- 2. A basis is the largest independent vector set in the subspace.
- 3. Any two bases for a subspace contain the same number of vectors.
  - The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).

Theorem 3

Every basis of $\mathcal{R}^n$ has n vectors.

- The number of vectors in a basis for a subspace V is called the dimension of V, and is denoted dim V
  - The dimension of zero subspace is 0





Example

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 : \sout{x_1 - 3x_2 + 5x_3 - 6x_4 = 0} \quad x_1 = 3x_2 - 5x_3 + 6x_4 \right\}$$

Find dim V

dim V = 3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Basis? Independent vector set that generates V

More from Theorems

Any two bases for a subspace contain the same number of vectors.

$\mathcal{R}^m$ have a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ All bases have m vectors

$\boldsymbol{dim}\ \mathcal{R}^m = m$

A basis is the smallest generating set.

A vector set generates $\mathcal{R}^m$ must contain at least m vectors.

Because a basis is the smallest generating set

Any other generating set has at least m vectors.

A basis is the largest independent set in the subspace.

Any independent vector set in $\mathcal{R}^m$ contain at most m vectors.

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linea independent

RREF ↓

Cannot be a pivot column

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Rank

Matrix A is ***full rank***
if Rank A = min(m,n)

Matrix A is ***rank deficient***
if Rank A < min(m,n)

- Given a mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because “the columns of A are independent” is equivalent to “rank A = n”
    - If m < n, the columns of A is dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors belonging to R^3 is dependent

In R^m, you cannot find more than m vectors that are independent.

Consistent or not

$A: m \times n$

$Span\{a_1, \cdots, a_n\} = R^m$ = Rank A = no. of rows

m independent vectors can span R^m

More than m vectors in R^m must be dependent.

Theorem 1

A basis is the smallest generating set.

If there is a generating set S for subspace V,

The size of basis for V is smaller than or equal to S.

Reduction Theorem

There is a basis containing in any generating set S.

S can be reduced to a basis for V by removing some vectors.

Theorem 1 – Reduction Theorem

所有的 generating set 心中都有一個 basis

S can be reduced to a basis for V by removing some vectors.

Suppose $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a generating set of subspace V

Subspace $V = Span\ S$ $= Col\ A$

Let $A = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_k\ ]$.

The basis of Col A is the pivot columns of A   Subset of $\mathcal{S}$

Theorem 1 – Reduction Theorem

所有的 generating set 心中都有一個 basis

Subspace $V = Span\ S = Col\ A$ = Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

Smallest generating set

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 3 \\ 9 \end{bmatrix} \right\}$$

Generation set

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2

A basis is the largest independent set in the subspace.

If the size of basis is k, then you cannot find more than k ***independent*** vectors in the subspace.

Extension Theorem

Given an independent vector set S in the space

S can be extended to a basis by adding more vectors

Theorem 2 – Extension Theorem

Independent set:
我不是一個 basis 就是正在成為一個 basis

There is a subspace V

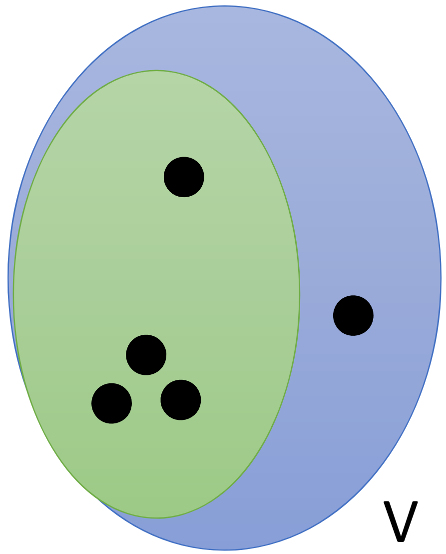
Given a independent vector set S (elements of S are in V)

- If Span S = V, then S is a basis
- If Span S ≠ V, find v_1 in V, but not in Span S

  S = S ∪ {v_1} is still an independent set

- If Span S = V, then S is a basis
- If Span S ≠ V, find v_2 in V, but not in Span S

  S = S ∪ {v_2} is still an independent set



…… You will find the basis in the end.

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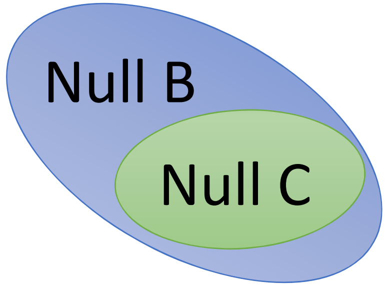
Theorem 3



- Any two bases of a subspace V contain the same number of vectors

Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ are two bases of V.

Let $A = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_k]$ and $B = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p]$.

Since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ spans V, $\exists\ \mathbf{c}_i \in \mathcal{R}^k$ s.t. $A\mathbf{c}_i = \mathbf{w}_i$ for all i

$\Rightarrow A[\mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p] = [\mathbf{w}_1\ \mathbf{w}_2 \cdots \mathbf{w}_p] \Rightarrow AC = B$

Suppose $C\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \in \mathcal{R}^p \Rightarrow AC\mathbf{x} = B\mathbf{x} = \mathbf{0}$

B is independent vector set $\Rightarrow x = \mathbf{0} \Rightarrow \mathbf{c}_1\ \mathbf{c}_2 \cdots \mathbf{c}_p$ are independent

$\mathbf{c}_i \in \mathcal{R}^k \quad \Rightarrow p \leq k$

Reversing the roles of the two bases one has $k \leq p \Rightarrow p = k$.

Theorem 4.9

- If V and W are subspaces of R^n with V contained in W, then dim V $\leq$ dim W
- If dim V = dim W, V=W
- Proof:

B_V is a basis of V, V in W, B_V in W

⇒ B_V is an independent set in W

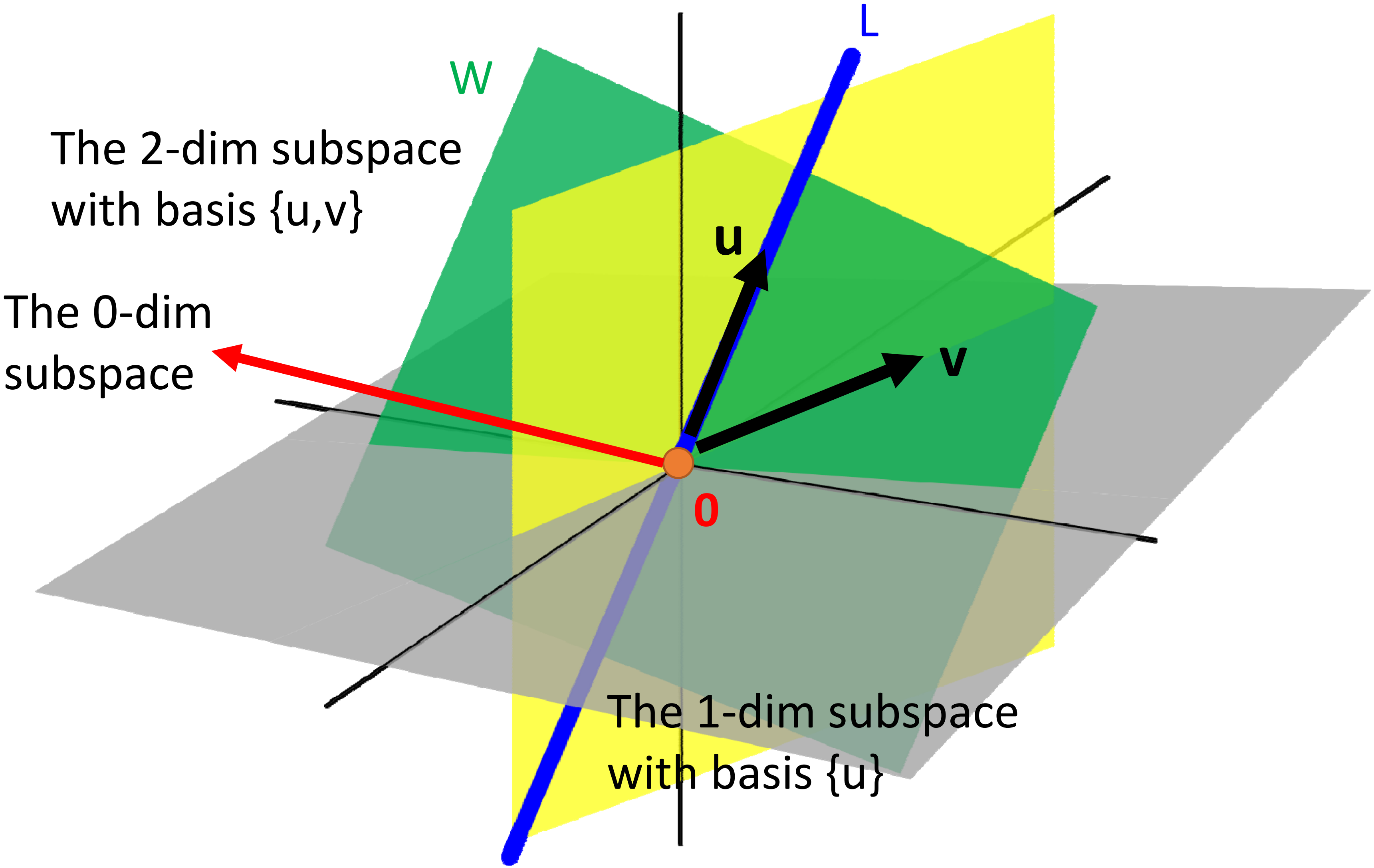
By extension theorem, B_V is in the basis of W ⇒ dim V $\leq$ dim W

If dim V = dim W =k

B_V is a linear independent set in W, with k elements

⇒ It is also the span of W

R^3 is the only 3-dim subspace of itself



Concluding Remarks

- 1. A basis is the smallest generation set.
- 2. A basis is the largest independent vector set in the subspace.
- 3. Any two bases for a subspace contain the same number of vectors.
  - The number of vectors in a basis for a nonzero subspace V is called dimension of V (dim V).

Confirming that a set is a Basis

(Chapter 4.2)

Intuitive Way

- Definition: A basis B for V is an independent generating set of V.

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Is $\mathcal{C}$ a basis of V?

Independent? yes

Generating set? difficult

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$ generates V

Another way

Find a basis for V

- Given a subspace V, assume that we already know that dim V = k. Suppose S is a subset of V with k vectors

If S is independent → S is basis

If S is a generating set → S is basis

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 : v_1 - v_2 + 2v_3 = 0 \right\} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Dim V = 2 (parametric representation)

Is $\mathcal{C}$ a basis of V?

$\mathcal{C}$ is a subset of V with 2 vectors

Independent? yes

→ $\mathcal{C}$ is a basis of V

Another way

Assume that dim V = k. <u>Suppose S is a subset of V with k vectors</u>

If S is independent → S is basis

By the extension theorem, we can add more vector into S to form a basis.

However, S already have k vectors, so it is already a basis.

If S is a generating set → S is basis

By the reduction theorem, we can remove some vector from S to form a basis.

However, S already have k vectors, so it is already a basis.

Example

- Is $\mathscr{B}$ a basis of V?

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Independent set in V? yes

Dim V = ? 3 → $\mathscr{B}$ is a basis of V.

Example

- Is $\mathscr{B}$ a basis of V = Span $\mathcal{S}$?

$\mathscr{B}$ is a subset of V with 3 vectors

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \Rightarrow R_A = \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad dimA = 3$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow R_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Independent

⇒ $\mathscr{B}$ is a basis of V.

Dimension of Basis

(Chapter 4.3)

Col A = Range

- Basis: The pivot columns of A form a basis for Col A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow \text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

pivot columns

pivot columns

- Dimension:

Dim (Col A) = number of pivot columns

= rank A

Rank A (revisit)

- Maximum number of Independent Columns
- Number of Pivot Columns
- Number of Non-zero rows
- Number of Basic Variables
- Dim (Col A): dimension of column space
- Dimension of the range of A

Row A

- Basis: Nonzero rows of RREF(A)

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \xrightarrow{\text{RREF}} R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row A = Row R

(The elementary row operations do not change the row space.)

a basis of Row *R* = a basis of Row *A*

- Dimension: Dim (Row A) = Number of Nonzero rows

= Rank A

Rank A (revisit)

- Maximum number of Independent Columns
- Number of Pivot Column
- Number of Non-zero rows
- Number of Basic Variables
- Dim (Col A): dimension of column space = Dim (Row A)
- Dimension of the range of A = Dim (Col A^T)

Rank A = Rank A^T

- Proof

Rank A = Dim (Col A)

Rank A = Dim (Row A)

= Dim (Col A^T) = Rank A^T

Example 2, P256

Null A

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Basis:
  - Solving Ax = 0
  - Each free variable in the parametric representation of the general solution is multiplied by a vector.
  - The vectors form the basis.

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 5x_3 + 4x_5 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 5x_3 - 4x_5 \\ x_3 &= x_3 \text{ (free)} \\ x_4 &= 2x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Basis

Null A

- Basis:
  - Solving Ax = 0
  - Each free variable in the parametric representation of the general solution is multiplied by a vector.
  - The vectors form the basis.
- Dimension:

Dim (Null A) = number of free variables

= Nullity A

= n - Rank A

Dimension Theorem

Dim (Col A)

=Rank A

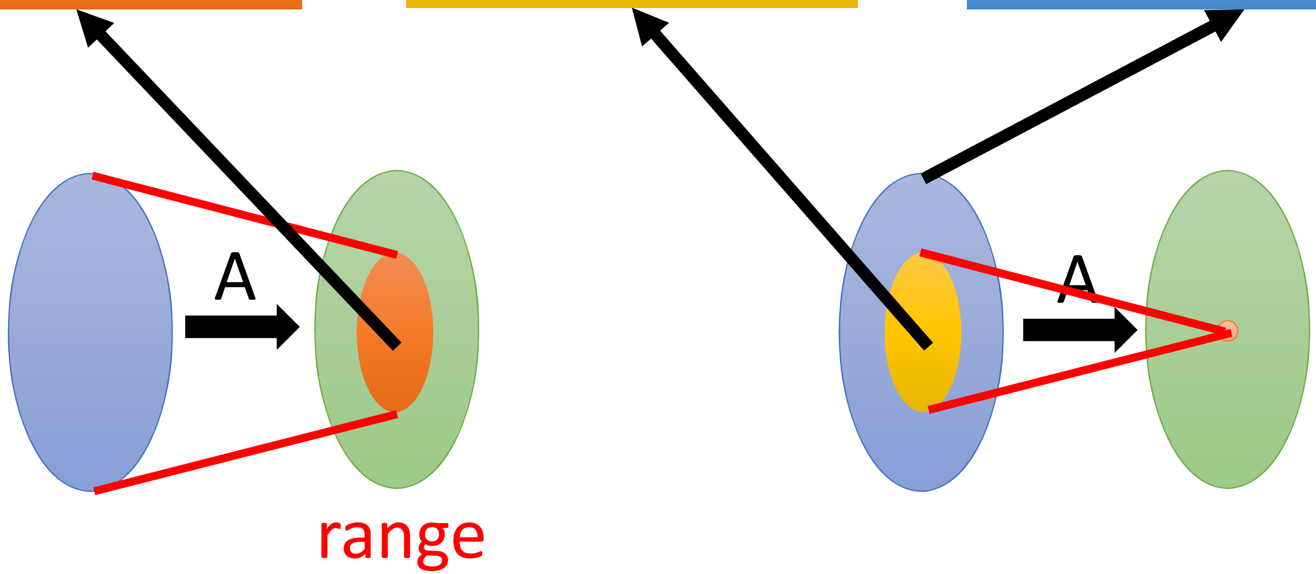
Dim (Null A)

=n - Rank A

If A is mxn

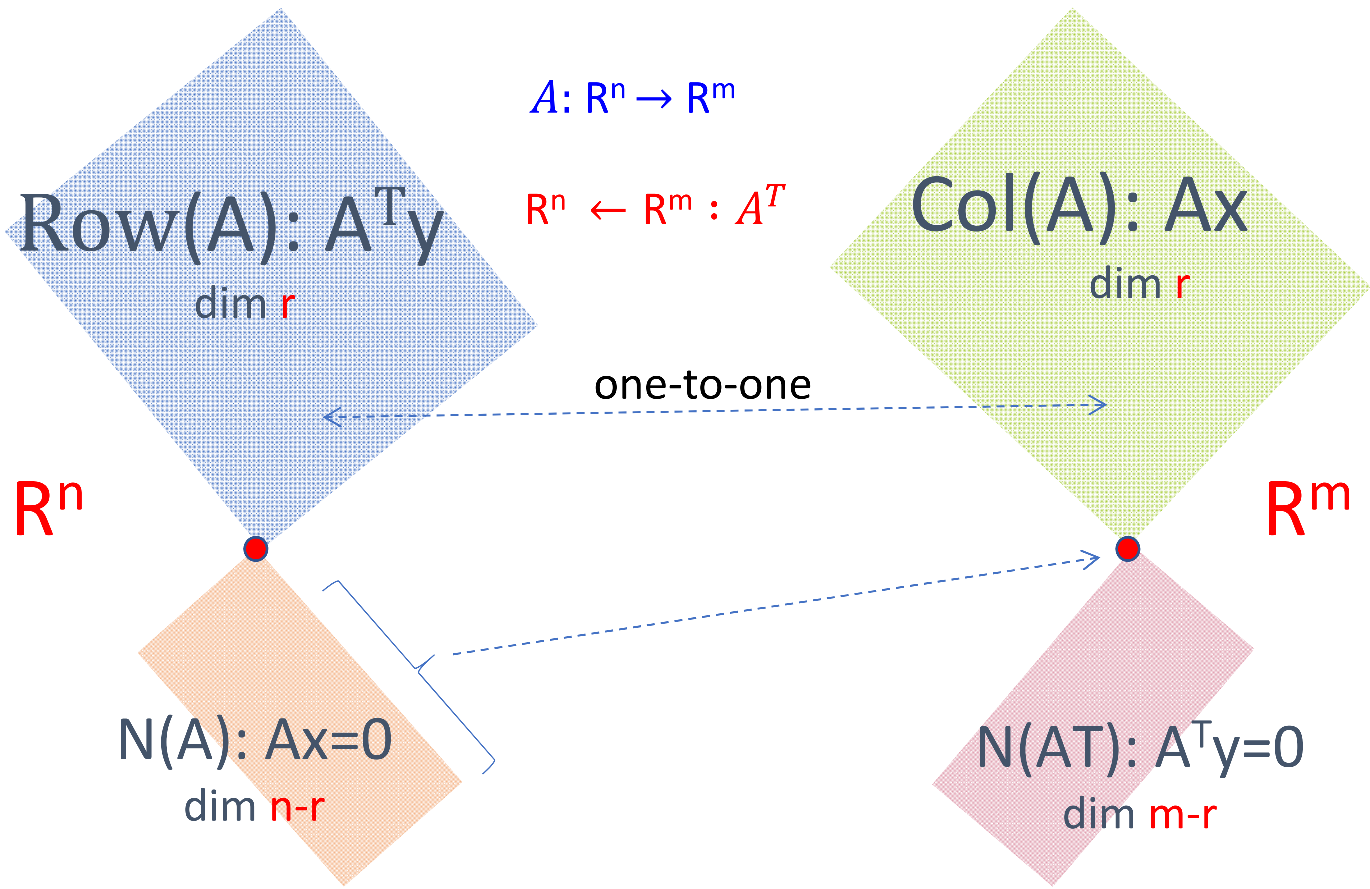
Dim (R^n) =n

Dim of Range + Dim of Null = Dim of Domain



Four fundamental subspaces of $A: \mathrm{R}^n \rightarrow \mathrm{R}^m$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

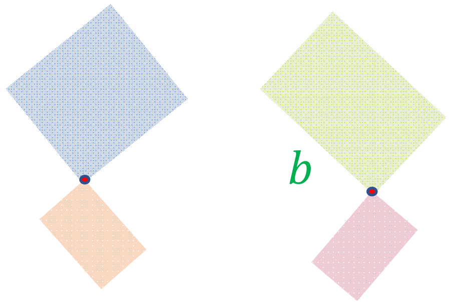


※

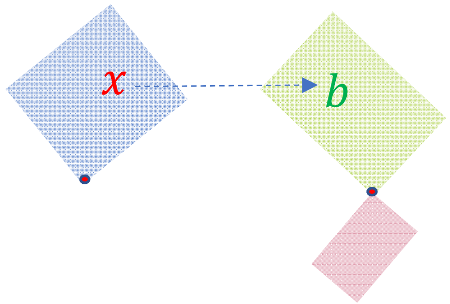
Solutions of $Ax = b$

$Zero, One, Infinity$ ...

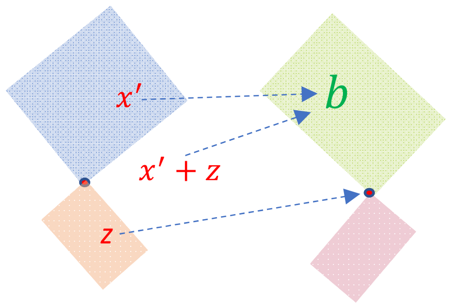
No Solution



One Solution



Infinite Solutions



The Meaning of Matrix Transpose

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A: $R^n \rightarrow R^m$ $\qquad$ A^T: $R^m \rightarrow R^n$

$$x \longrightarrow Ax$$

$$A^Ty \longleftarrow y$$

$$Ax \cdot y = x \cdot A^Ty$$

Preservation of dot product in R^n and R^m

※

Finite vs. Infinite-dimension Vector Space

- Care has to be taken when dealing with infinite-dimension vector spaces.
- E.g. Consider the "vector space" containing all polynomial functions with basis P={1, x , x^2, x^3, …}

Is it really a vector space?

No!

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

which does not converge to a polynomial function.

Coordinate System

(Chapter 4.4)

Coordinate System

- Each coordinate system is a "*viewpoint*" for vector representation.
  - The same vector is represented differently in different coordinate systems.
  - Different vectors can have the same representation in different coordinate systems.
- A vector set $\mathcal{B}$ can be considered as a coordinate system for R^n if:
  - 1. The vector set $\mathcal{B}$ spans the R^n
  - 2. The vector set $\mathcal{B}$ is independent

$\mathcal{B}$ is a basis of R^n

Coordinate System

- Let vector set $\mathscr{B}$=$\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace R^n

$\mathscr{B}$ is a coordinate system

- For any v in R^n, there are unique scalars $c_1, c_2, \cdots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$\mathscr{B}$ -coordinate vector of v:

$$[v]_{\mathscr{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathcal{R}^n$$

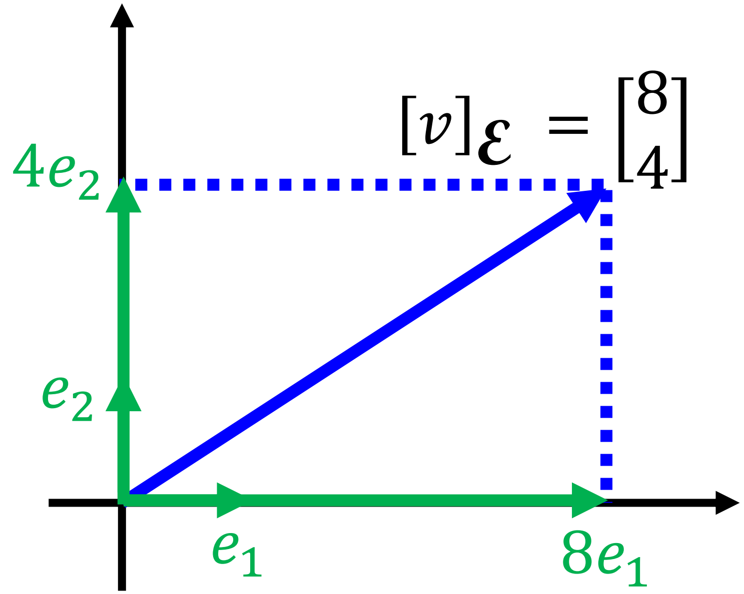
(用 $\mathscr{B}$ 的觀點來看 v)

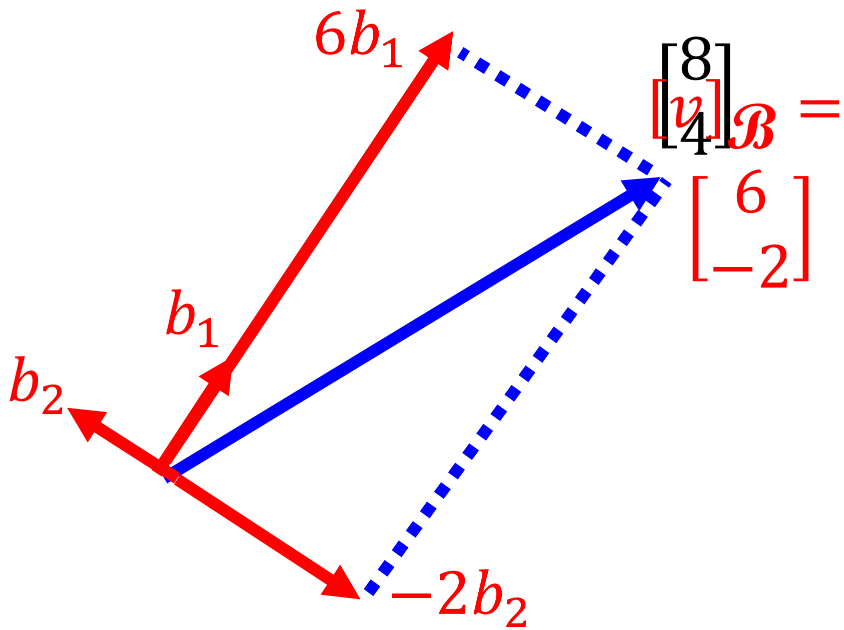
New Coordinate System $\mathcal{B}$

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\mathcal{E}$

$\{e_1, e_2\}$ is a coordinate system





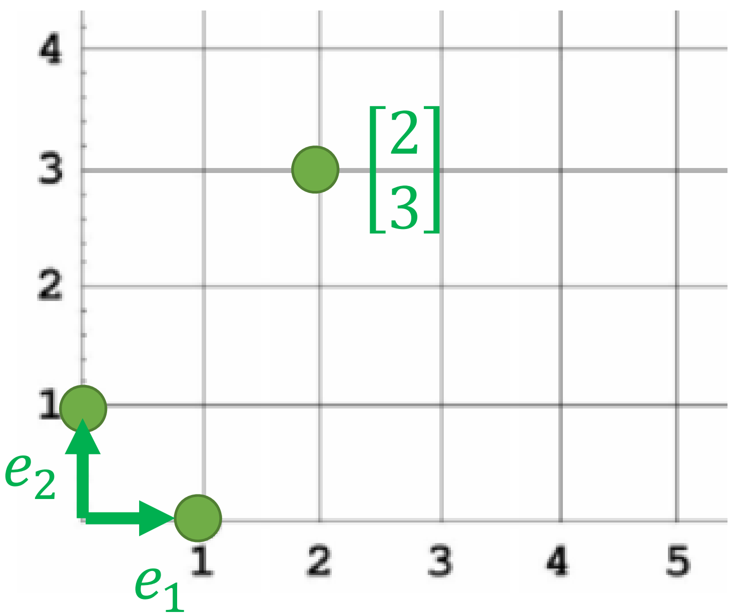
$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 8e_1 + 4e_2$

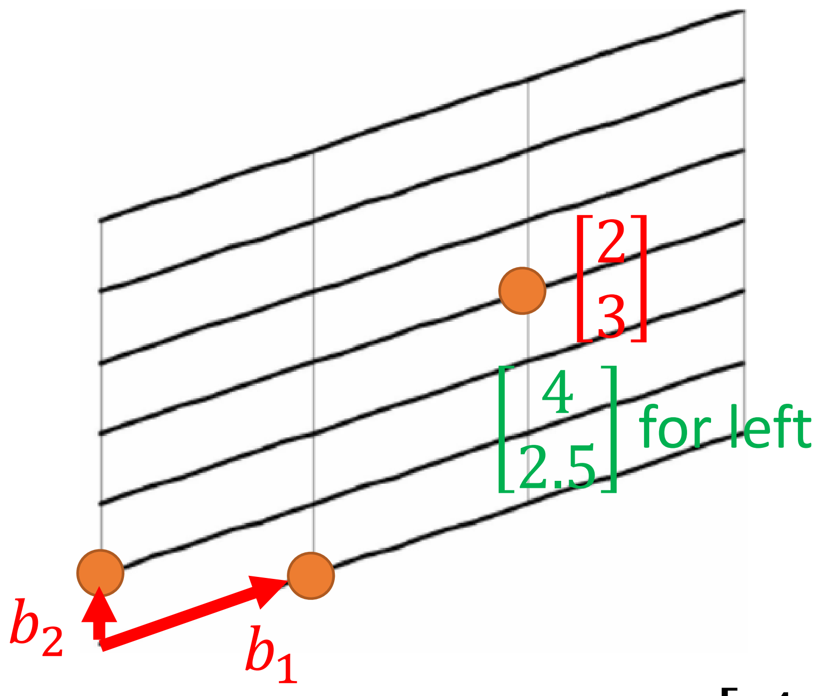
$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 6b_1 + (-2)b_2$

Vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

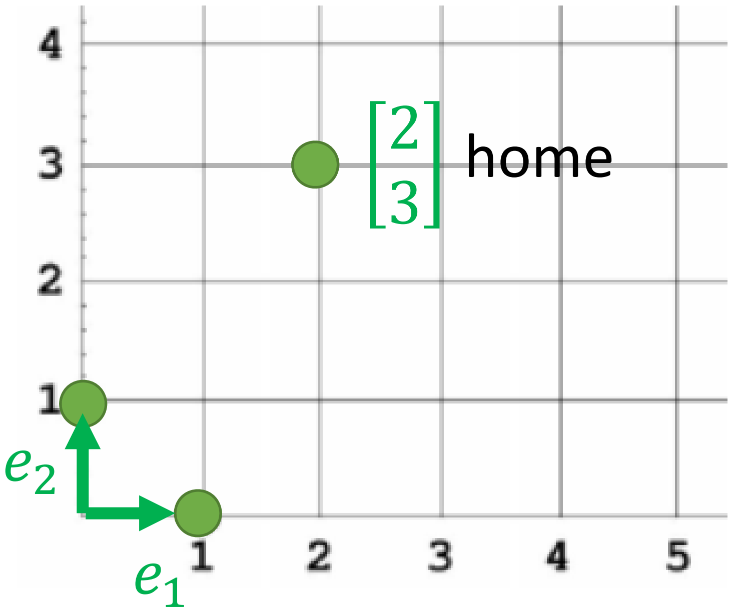




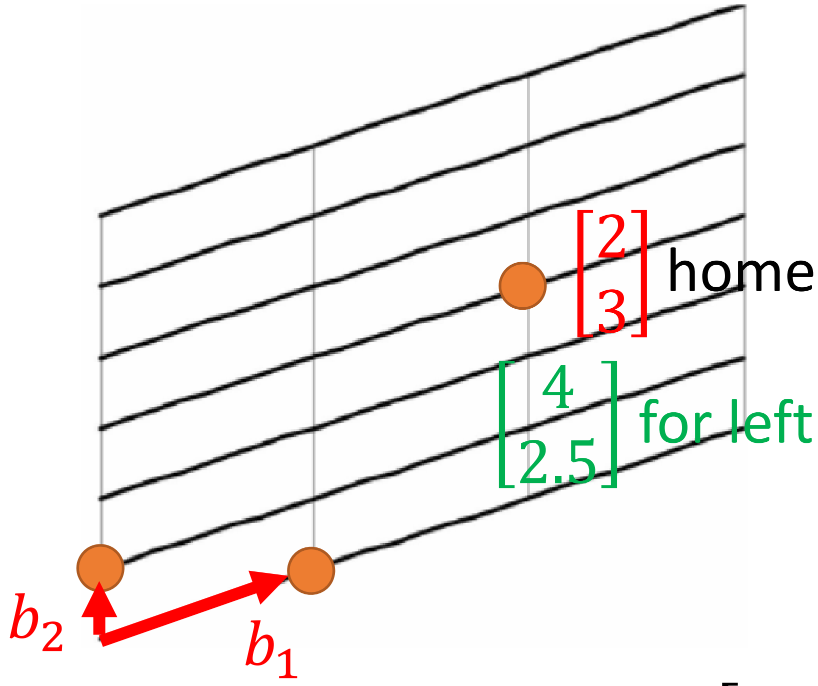
$$2b_1 + 3b_2 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

Vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$b_1 = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$



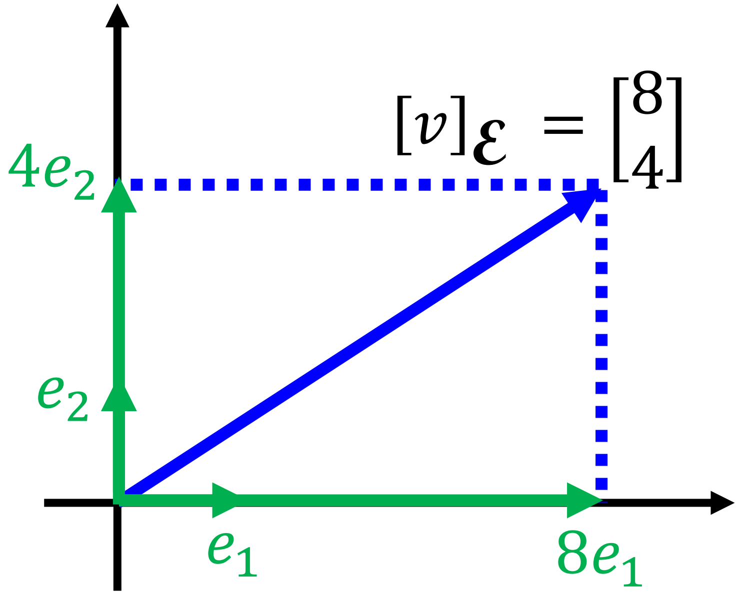
$$2b_1 + 3b_2 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

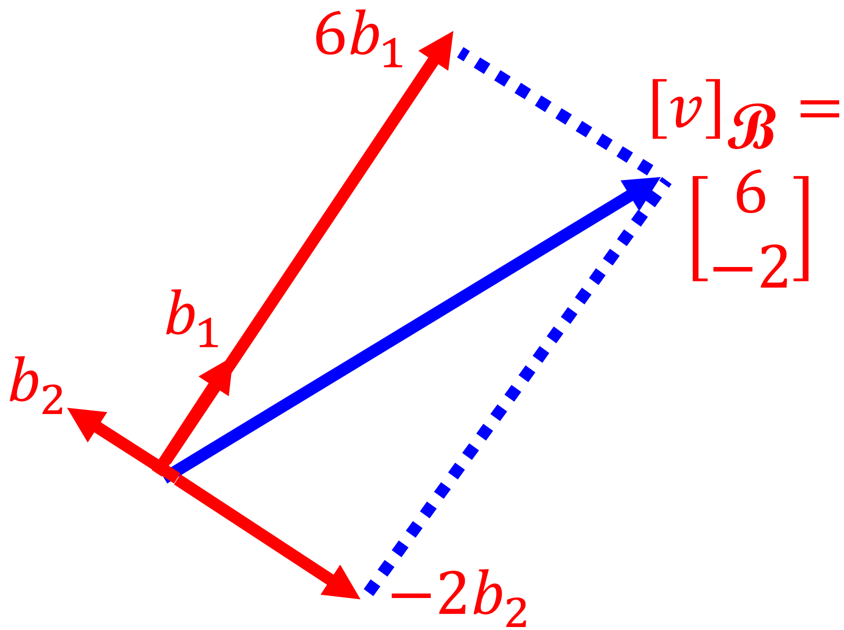
$\mathcal{E}$

{e_1, e_2} is a coordinate system

New Coordinate System $\mathcal{B}$

$b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$





$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 8e_1 + 4e_2$

$\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 6b_1 + (-2)b_2$

$\mathcal{E}=\{e_1, e_2, \cdots, e_n\}$ (standard vectors) $\quad v = [v]_{\mathcal{E}}$

$\mathcal{E}$ is Cartesian coordinate system (直角坐標系)

Coordinate System

- A vector set $\mathcal{B}$ can be considered as a coordinate system for R^n if:
- 1. The vector set $\mathcal{B}$ spans the R^n

→ Every vector should have a representation

- 2. The vector set $\mathcal{B}$ is independent

→ Unique representation

$\mathcal{B}$ is a basis of R^n

Why Basis?

- Let vector set $\mathcal{B}$=$\{u_1, u_2, \cdots, u_k\}$ be independent.
- Any vector v in Span $\mathcal{B}$ can be uniquely represented as a linear combination of the vectors in $\mathcal{B}$.
- That is, there are unique scalars $a_1, a_2, \cdots, a_k$ such that $v = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k$
- Proof:

Unique? $v = a_1 u_1 + a_2 u_2 + \cdots + a_k u_k$

$$v = b_1 u_1 + b_2 u_2 + \cdots + b_k u_k$$

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_k - b_k)u_k = 0$$

$\mathcal{B}$ is independent ⇨ $a_1 - b_1 = a_2 - b_2 = \cdots = a_k - b_k = 0$

Change Coordinate

(Chapter 4.4)

Coordinate System

- Let vector set $\mathcal{B}=\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace R^n

⇨ $\mathcal{B}$ is a coordinate system

- For any v in R^n, there are unique scalars $c_1, c_2, \cdots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$

$\mathcal{B}$ -coordinate vector of v:

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathcal{R}^n$$

(用 $\mathcal{B}$ 的觀點來看 v)

Other System → Cartesian

$$\mathcal{B}=\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \qquad [v]_{\mathcal{B}}=\begin{bmatrix}3\\6\\-2\end{bmatrix}$$

$$v=3\begin{bmatrix}1\\1\\1\end{bmatrix}+6\begin{bmatrix}1\\-1\\1\end{bmatrix}-2\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}7\\-7\\5\end{bmatrix}$$

$$\mathcal{C}=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\} \qquad [u]_{\mathcal{C}}=\begin{bmatrix}3\\6\\-2\end{bmatrix}$$

$$u=3\begin{bmatrix}1\\2\\3\end{bmatrix}+6\begin{bmatrix}4\\5\\6\end{bmatrix}-2\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}13\\20\\27\end{bmatrix}$$

Other System → Cartesian

- Let vector set $\mathcal{B}=\{u_1, u_2, \cdots, u_n\}$ be a basis for a subspace R^n
- Matrix $\mathrm{B} = [u_1 \quad u_2 \quad \cdots \quad u_n]$

Given $[v]_{\mathcal{B}}$, how to find v?

$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

$$= B[v]_{\mathcal{B}} \quad \text{(matrix-vector product)}$$

Cartesian → Other System

$$v = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

find $[\mathbf{v}]_{\mathcal{B}}$

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \qquad [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

B is invertible (?)

independent

$$B[v]_{\mathcal{B}} = v \Rightarrow [v]_{\mathcal{B}} = B^{-1}v = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

Cartesian ↔ Other System

- Let $\mathscr{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$

$$[v]_{\mathscr{B}} = B^{-1}v$$

v $\qquad$ $[v]_{\mathscr{B}}$

$$v = B[v]_{\mathscr{B}}$$

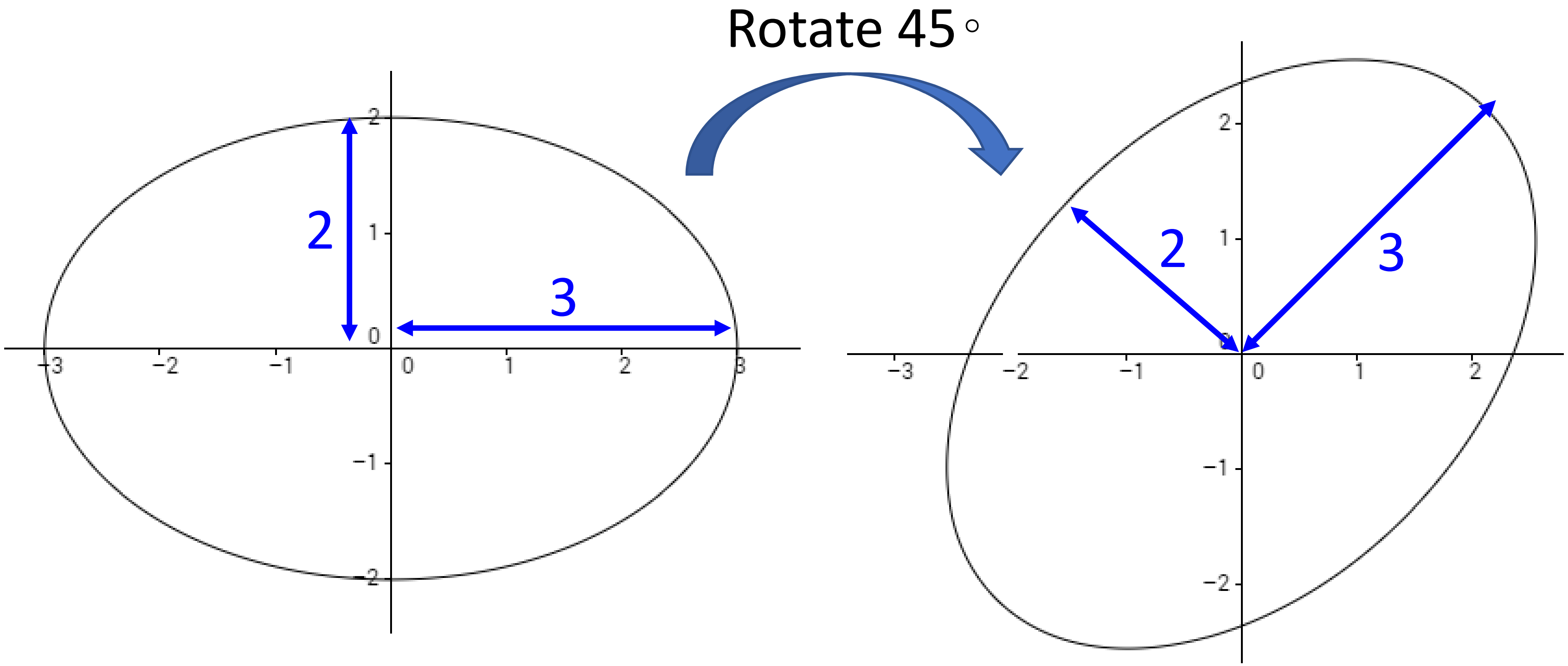
$$= c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

$$= \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Let $\mathscr{B}=\{b_1, b_2, \cdots, b_n\}$ be a basis of R^n. $[b_i]_{\mathscr{B}} = ?e_i$

(Standard vector)

Equation of ellipse

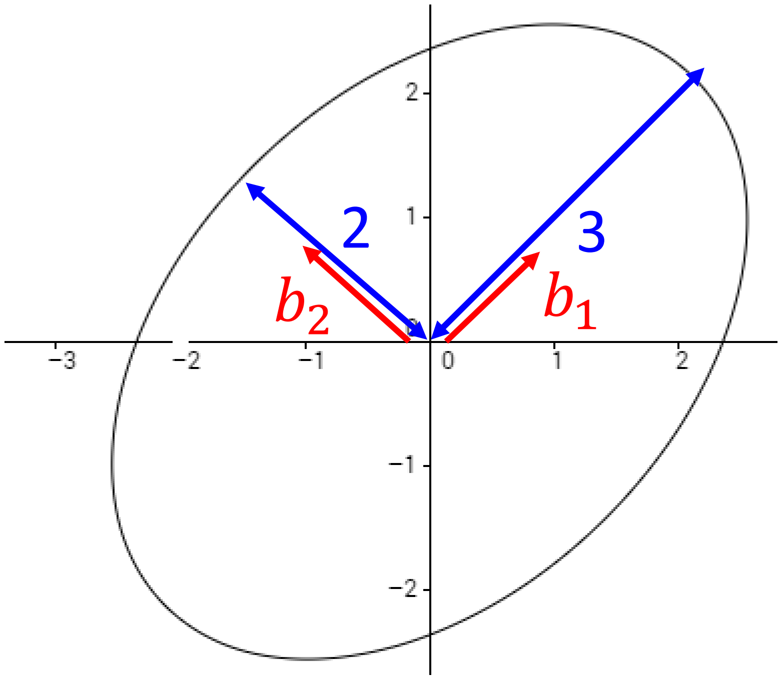


$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

?

Equation of ellipse

Use another coordinate system



$$\mathcal{B} = \left\{ \overset{b_1}{\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}}, \overset{b_2}{\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}} \right\}$$

What is the equation of the ellipse in the new coordinate system?

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

Equation of ellipse

$$\mathcal{B} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \ [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1 \Rightarrow \frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{3^2} + \frac{\left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{2^2} = 1$$

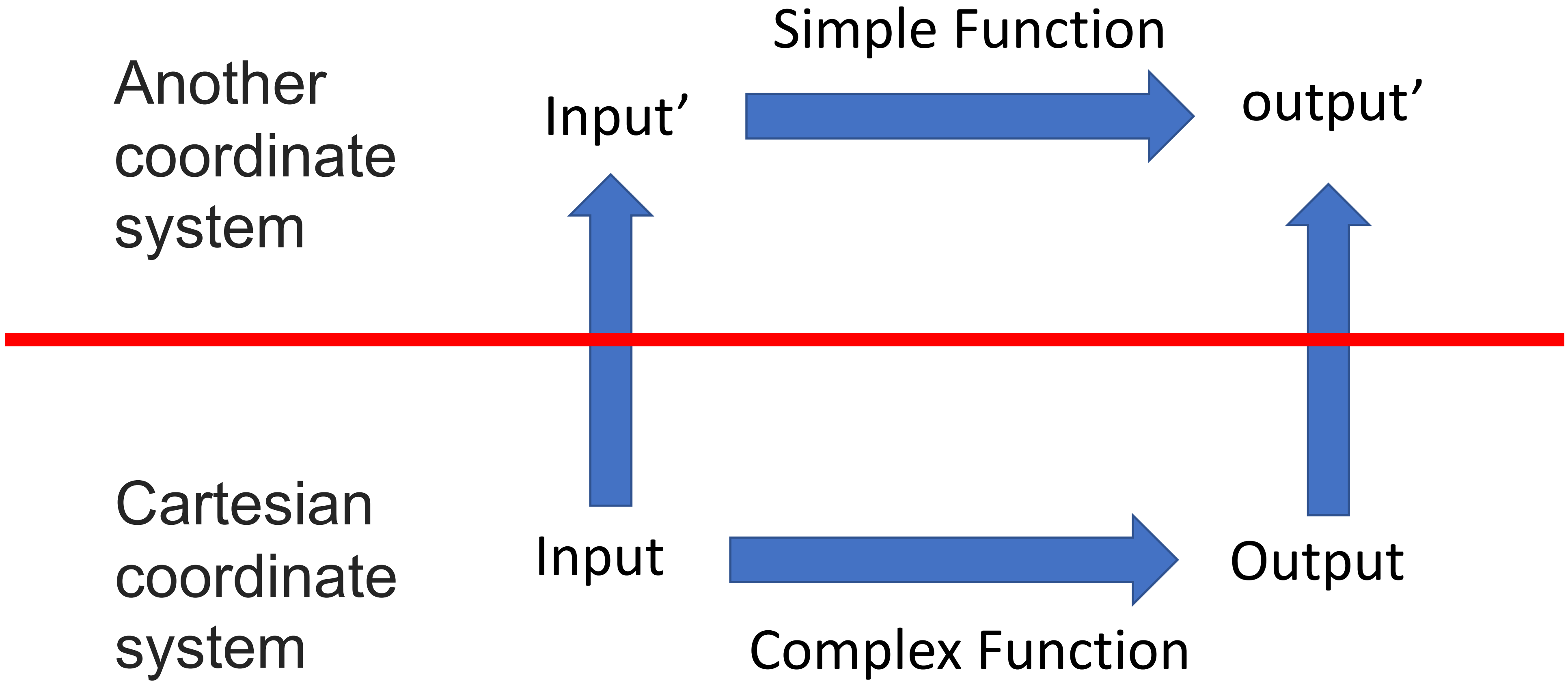
$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = B^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

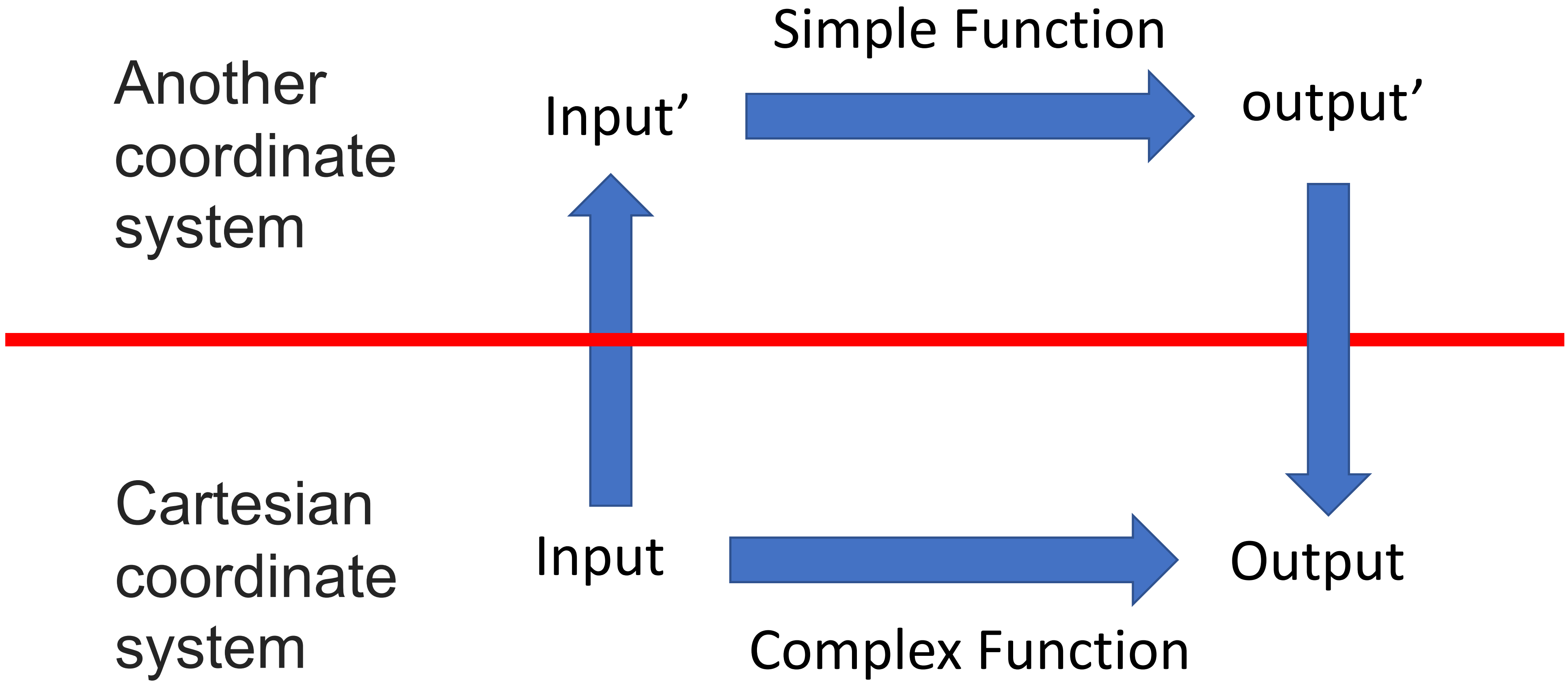
Linear Function in Coordinate System

(Chapter 4.5)

Basic Idea


Another coordinate system
Cartesian coordinate system
Simple Function
Input’
output’
Input
Output
Complex Function

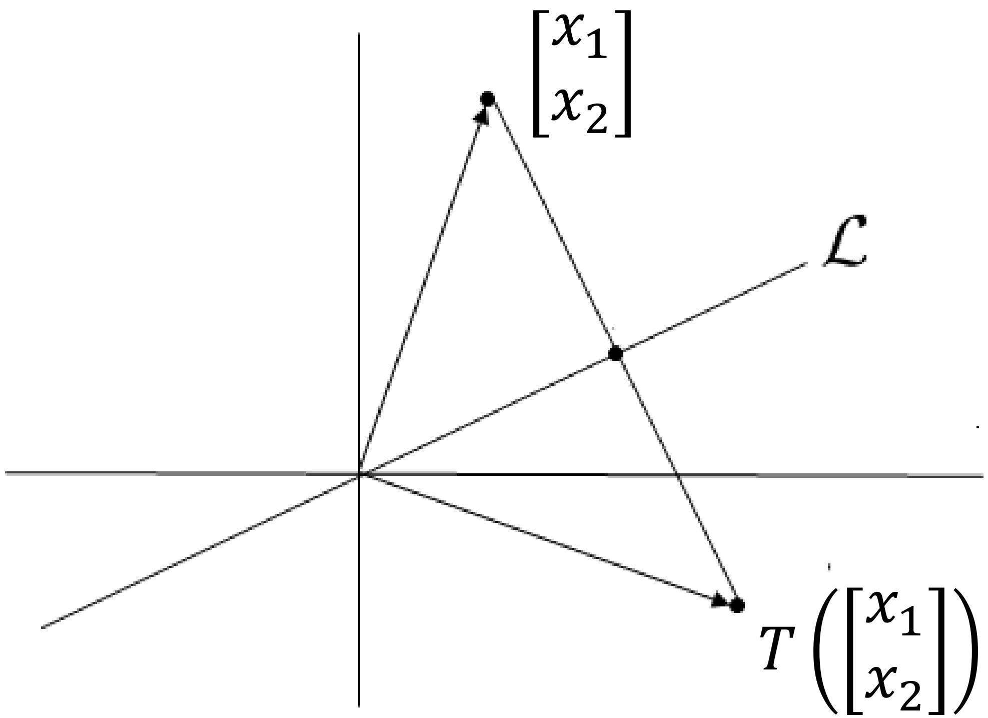
Basic Idea



Another coordinate system
Cartesian coordinate system
Simple Function
Input'
output'
Input
Output
Complex Function

Sometimes a function can be complex ……

- T: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$



$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =?$$

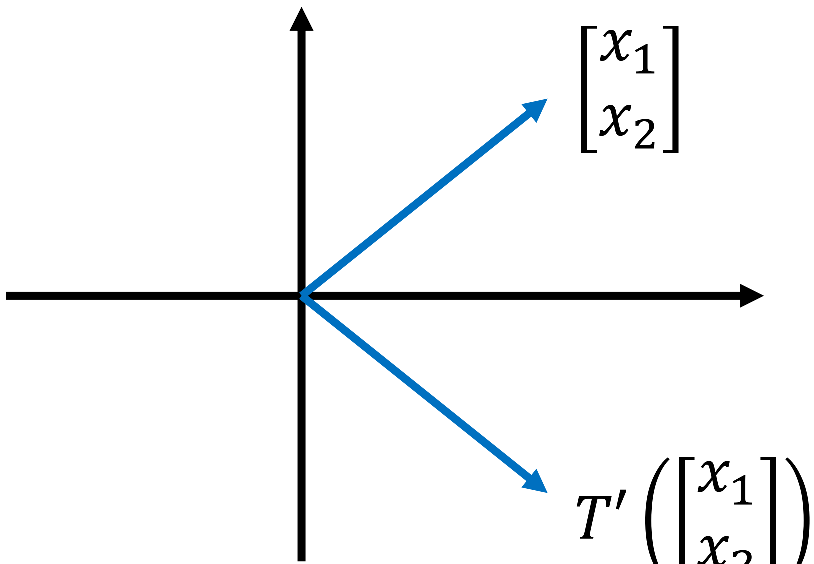
$$[\mathrm{T}] = [T(e_1) \quad T(e_2)\ ]$$

$$=?$$

Sometimes a function can be complex ……

- T: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

special case: $\mathcal{L}$ is the *horizontal axis*



$$T'\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =? \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

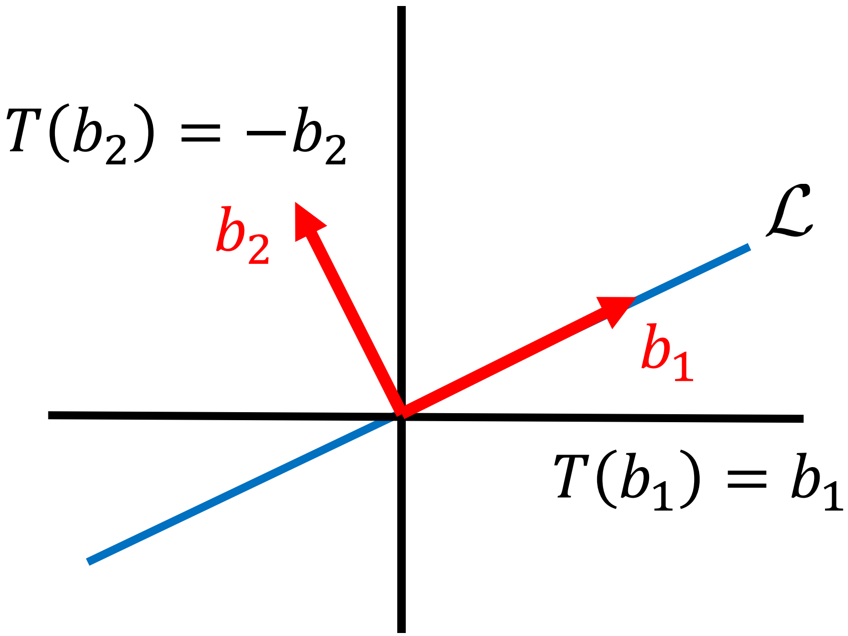
$$[T'] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$T'(e_1) = e_1$ $\quad$ $T'(e_2) = -e_2$

Describing the function in another coordinate system

- T: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

In another coordinate system $\mathcal{B}$ ....



$\mathcal{B} = \{b_1, b_2\}$

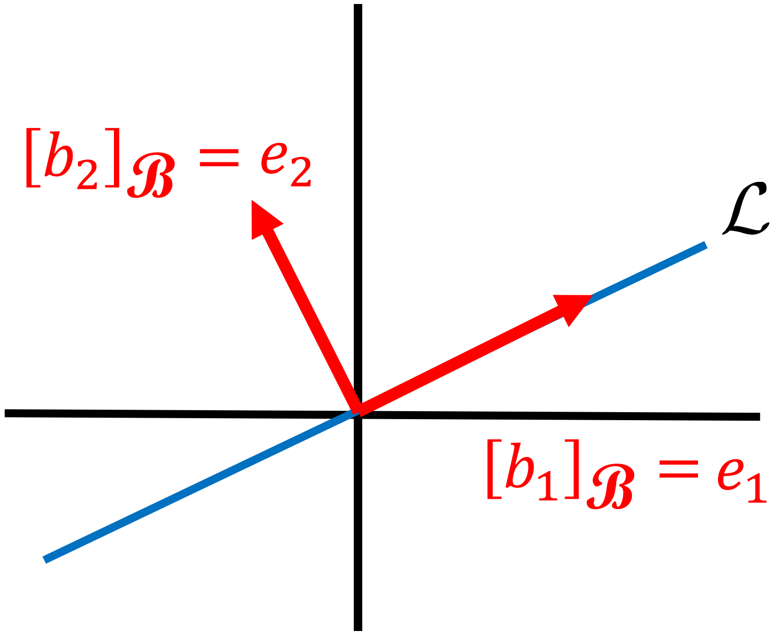
Describing the function in another coordinate system

- T: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

In another coordinate system $\mathcal{B}$ ....

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\mathcal{B}$ matrix of T: Input and output are both in $\mathcal{B}$



$[T]b_1 = b_1$

⇒ $[T]_{\mathcal{B}}([b_1]_{\mathcal{B}}) = [b_1]_{\mathcal{B}}$

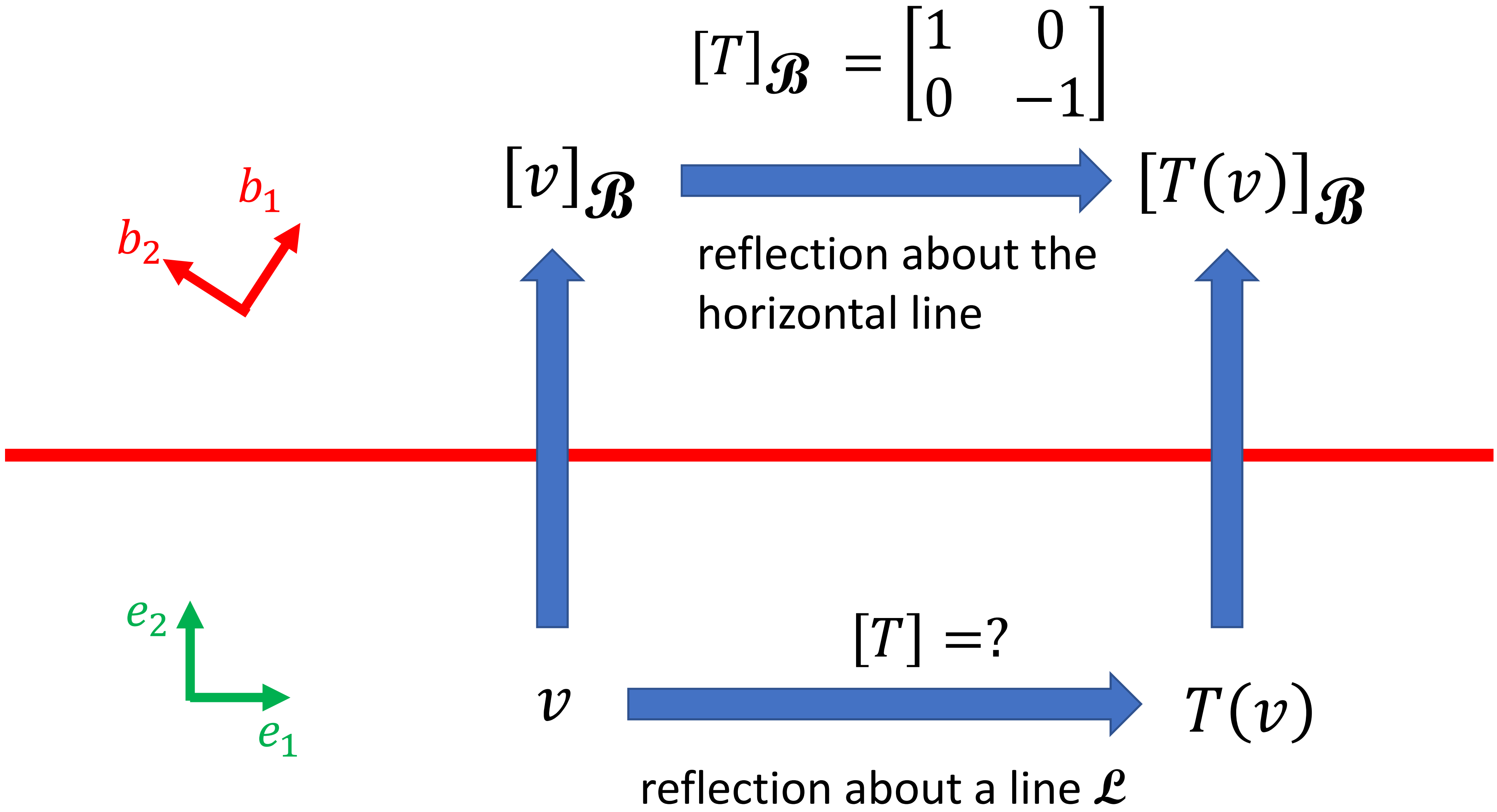
⇒ $[T]_{\mathcal{B}}(e_1) = e_1$

$[T]b_2 = -b_2$

⇒ $[T]_{\mathcal{B}}([b_2]_{\mathcal{B}}) = [-b_2]_{\mathcal{B}}$

⇒ $[T]_{\mathcal{B}}(e_2) = -e_2$

Flowchart



$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
b_1
b_2
$[v]_{\mathcal{B}}$
$[T(v)]_{\mathcal{B}}$
reflection about the horizontal line
e_2
e_1
$[T] = ?$
v
$T(v)$
reflection about a line $\mathcal{L}$

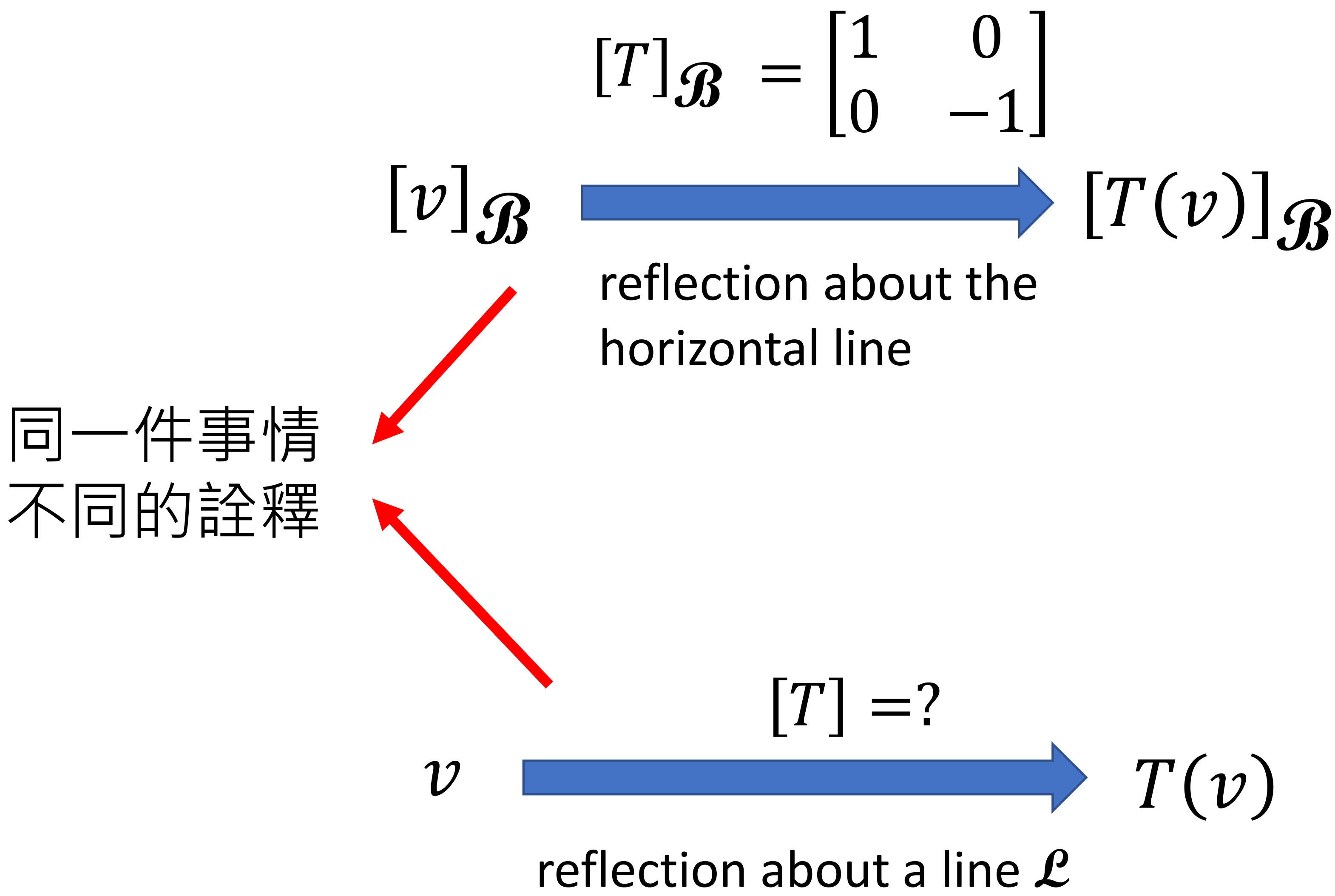
Linear Operator vs. Matrix

$$v \longrightarrow T(v)$$

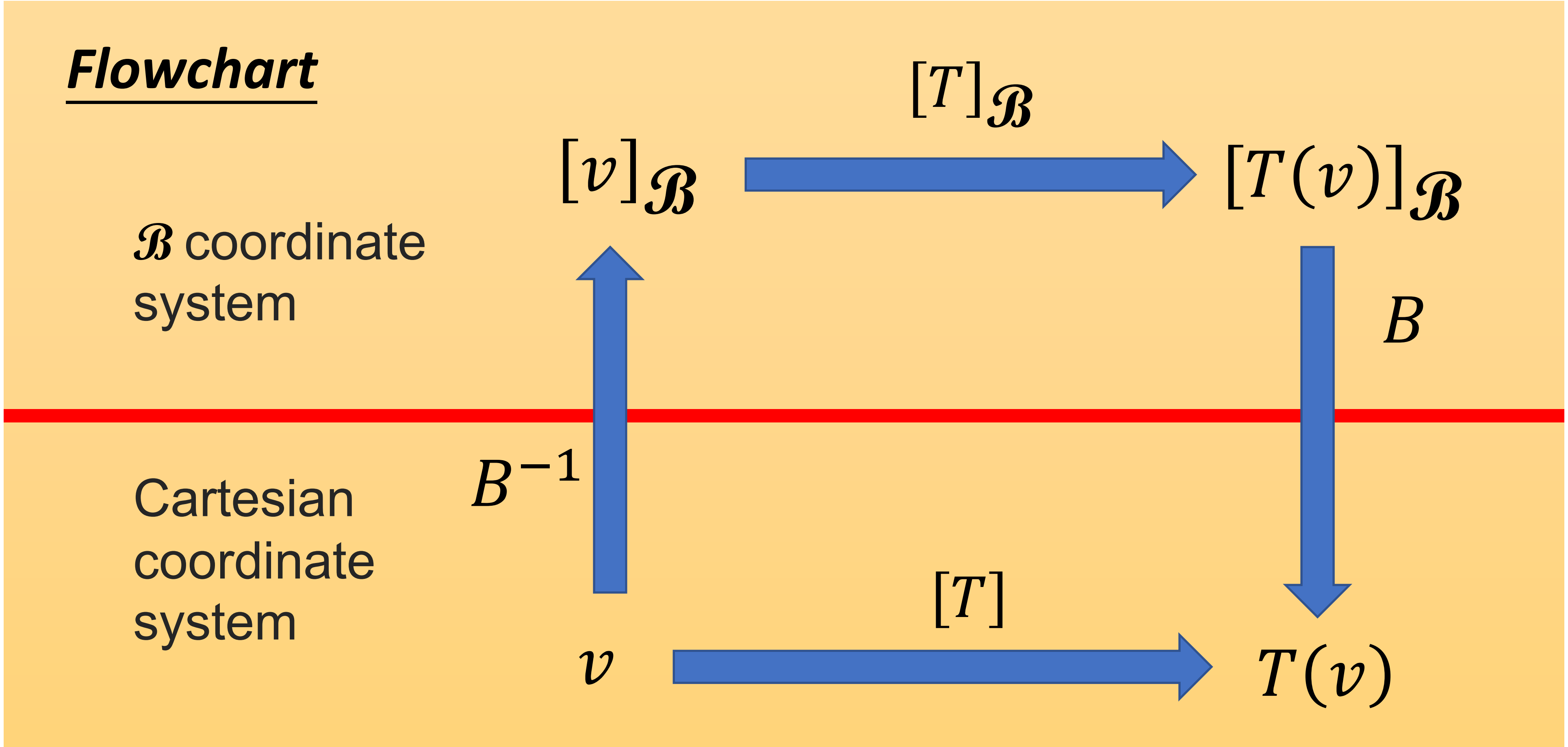
$$[v]_{\mathcal{B}} \xrightarrow{\text{Matrix } D} [T(v)]_{\mathcal{B}}$$

$$[v]_{\mathcal{C}} \xrightarrow{\text{Matrix } E} [T(v)]_{\mathcal{C}}$$

Corresponding matrix of operator T depends on the coordinate system



$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
$[v]_{\mathcal{B}}$
$[T(v)]_{\mathcal{B}}$
reflection about the horizontal line
同一件事情
不同的詮釋
$[T] = ?$
v
$T(v)$
reflection about a line $\mathcal{L}$



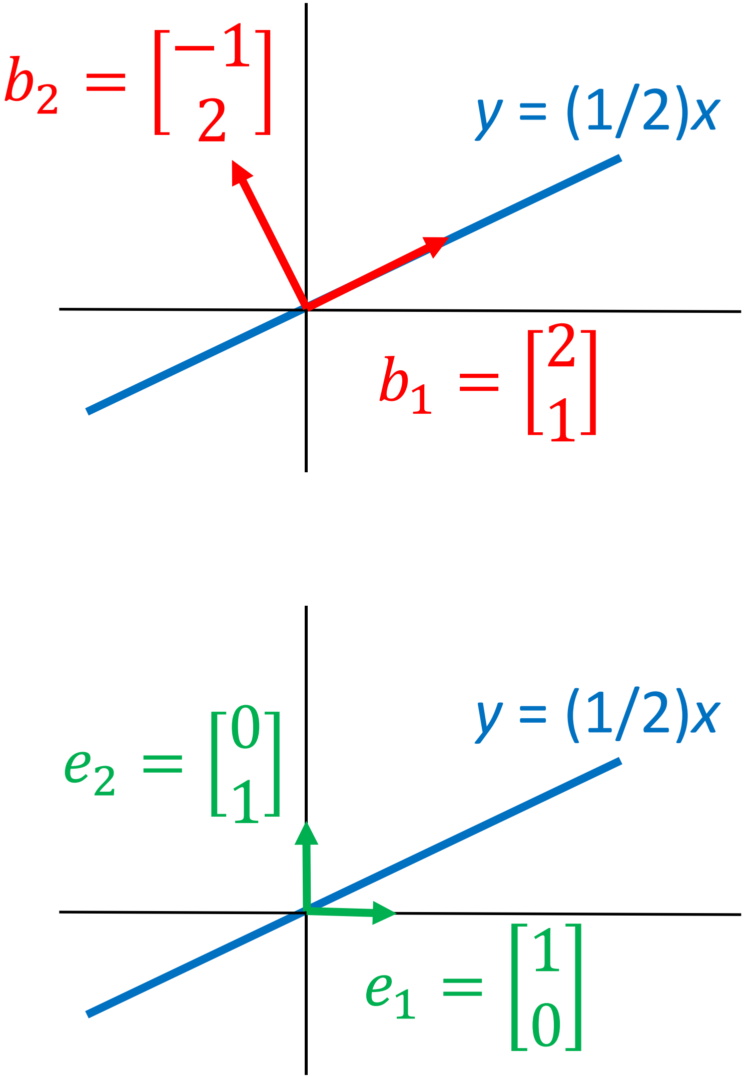
$$[T] = B[T]_{\mathcal{B}}B^{-1}$$

similar

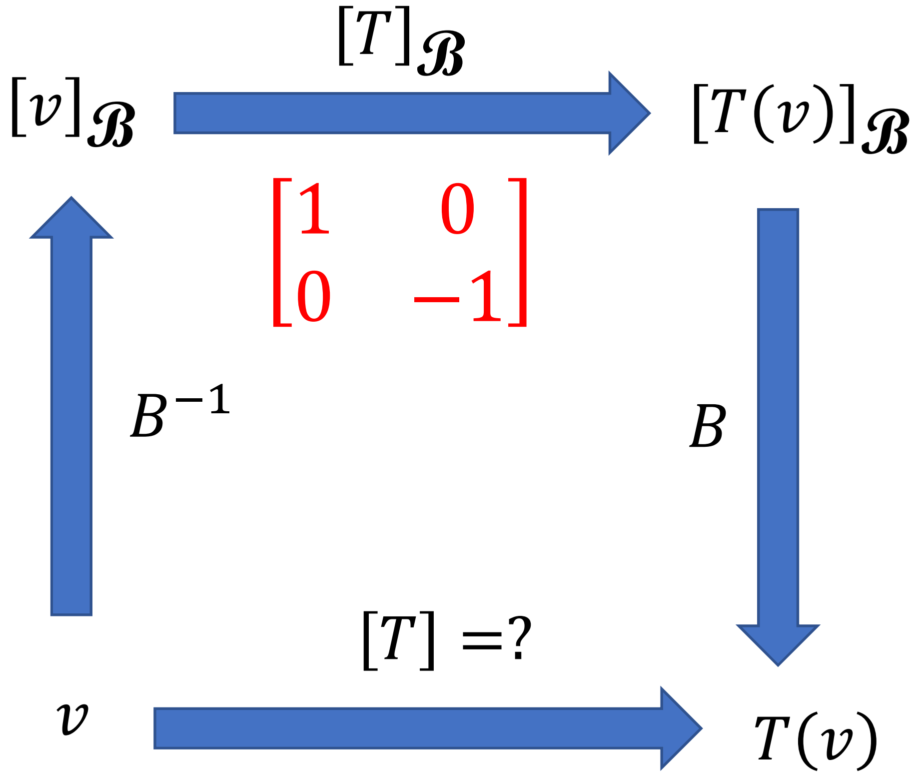
$$[T]_{\mathcal{B}} = B^{-1}[T]B$$

similar

- Example: reflection operator T about the line $y = (1/2)x$



$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

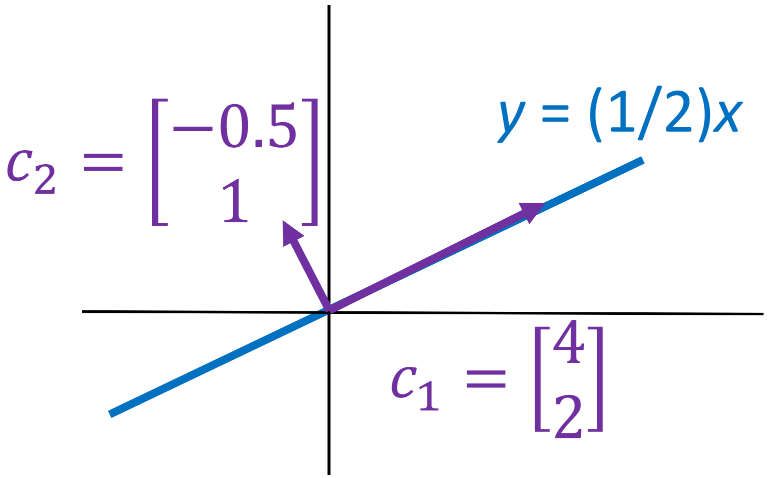


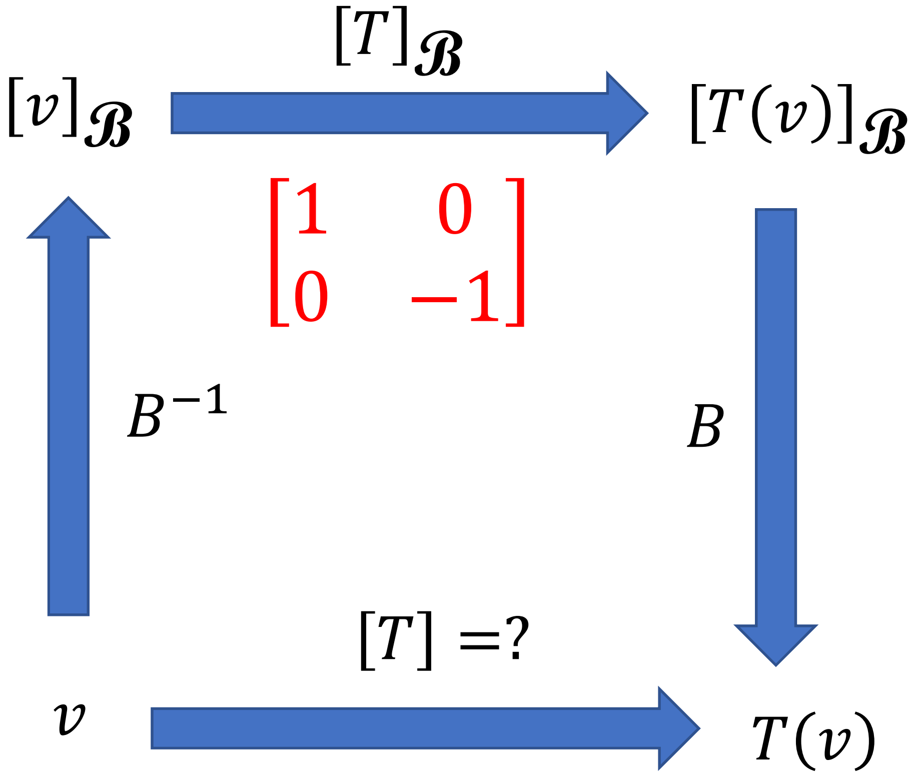
$$[T] = B[T]_{\mathcal{B}}B^{-1}$$

- Example: reflection operator *T* about the line *y* = (1/2)*x*

$$[T] = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$





$$[T] = C[T]_C C^{-1}$$

$$= \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

$$[T] = B[T]_{\mathcal{B}} B^{-1}$$

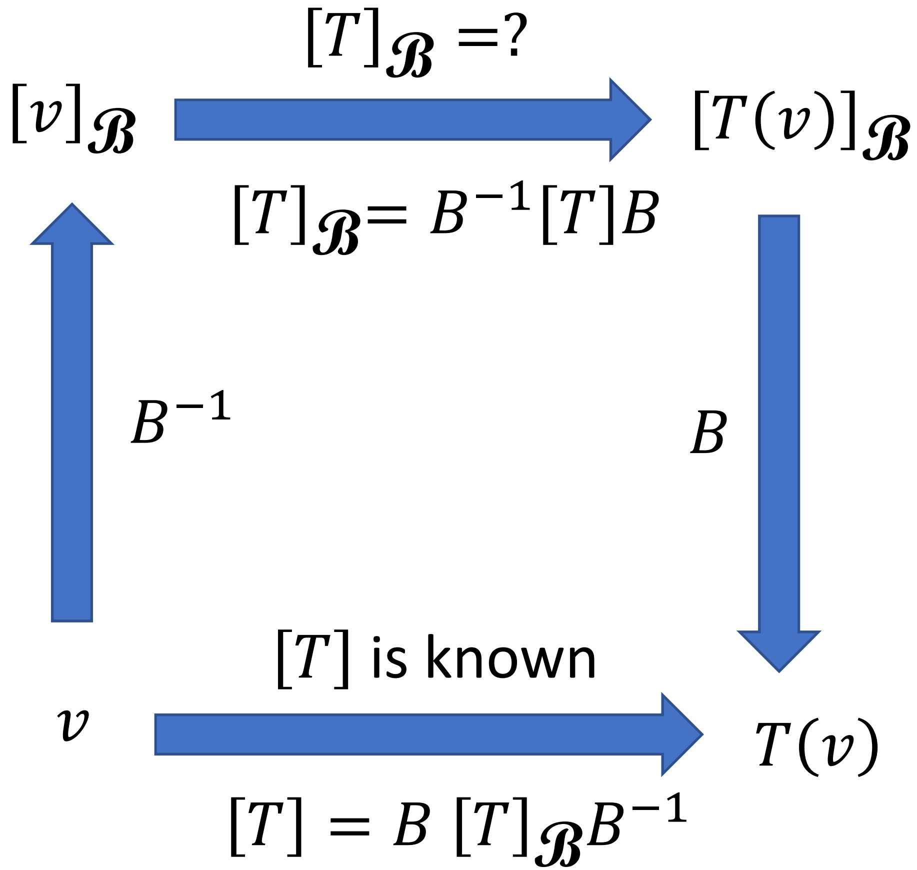
Example (P279)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$[T] = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}$$



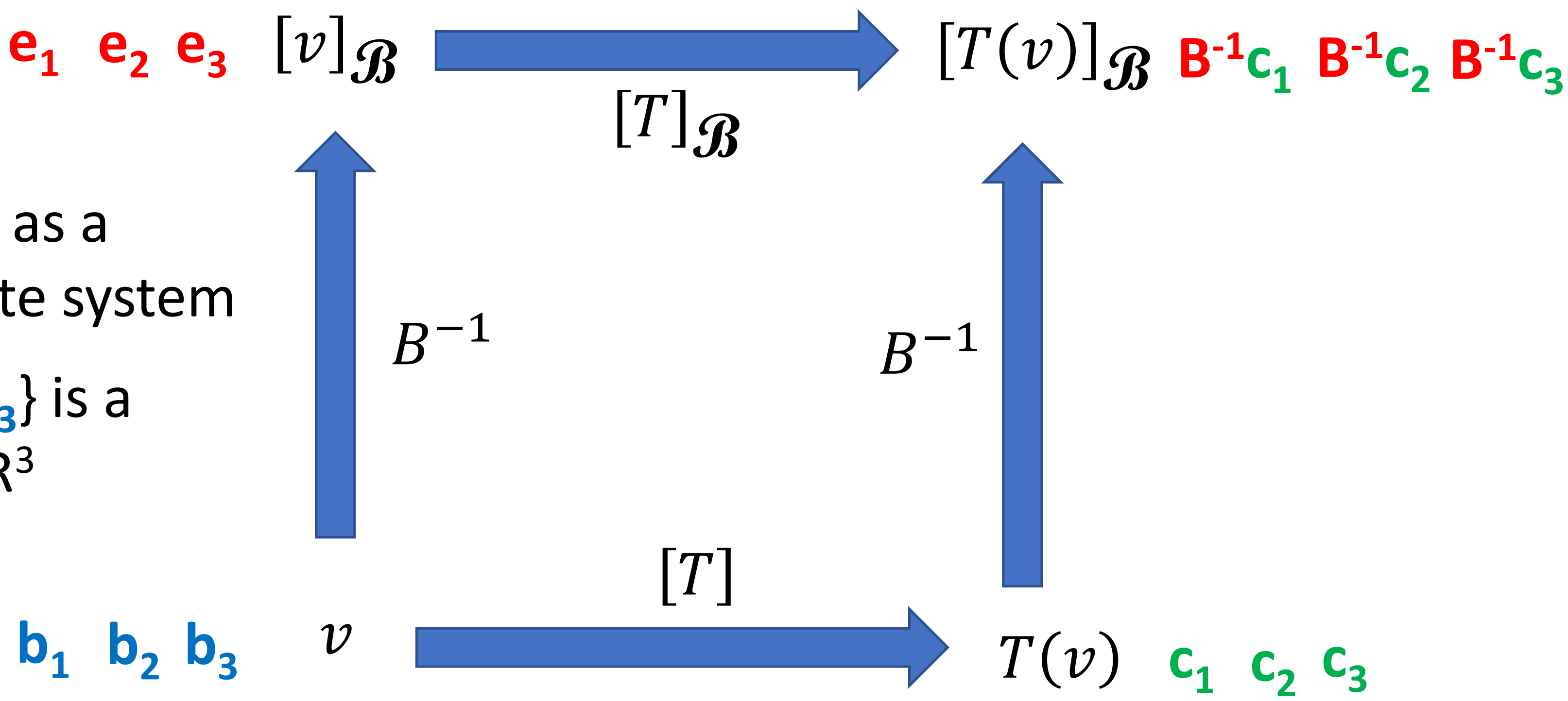
Example (P279)   Determine T

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\-1\\1\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$\mathbf{b}_1$ $\mathbf{c}_1$ $\mathbf{b}_2$ $\mathbf{c}_2$ $\mathbf{b}_3$ $\mathbf{c}_3$

$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ as a coordinate system

$\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of R^3

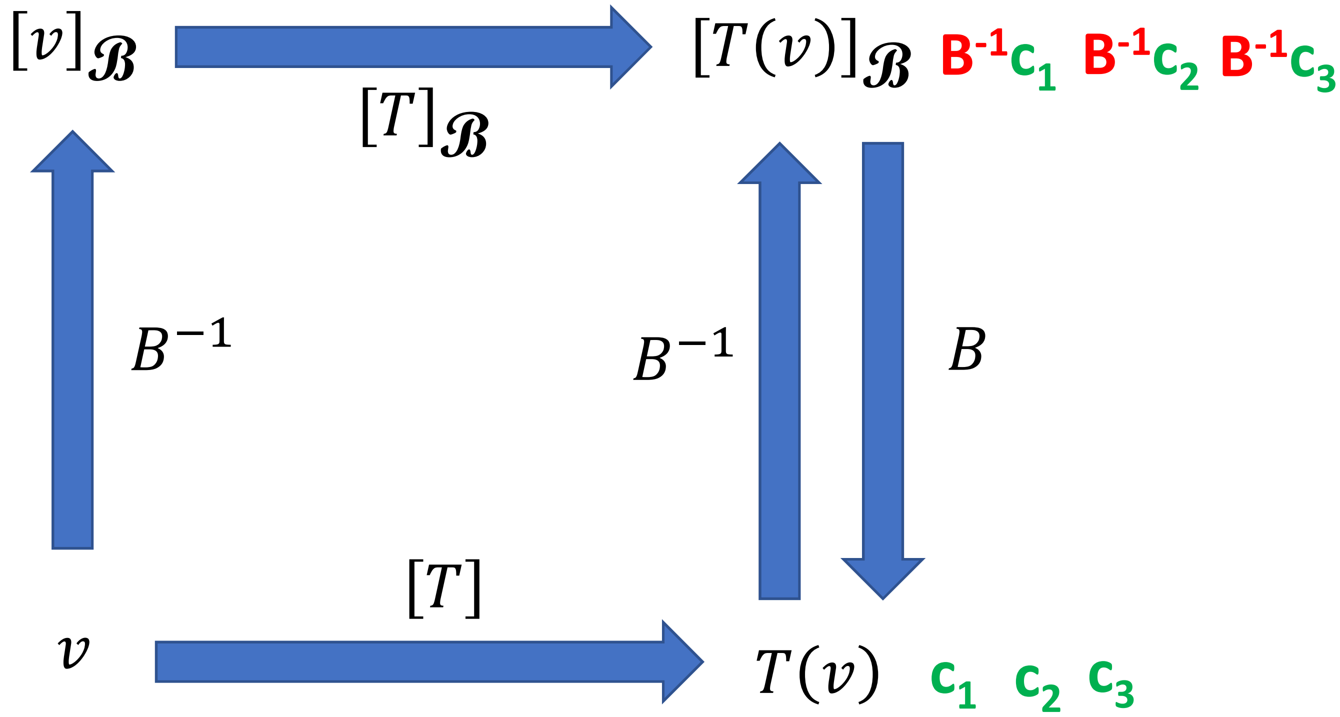


Example (P279) Determine T

$$[T]_{\mathcal{B}} = [B^{-1}c_1 \quad B^{-1}c_2 \quad B^{-1}c_3] = B^{-1}C$$

$$[T] = B[T]_{\mathcal{B}}B^{-1} = BB^{-1}CB^{-1} = CB^{-1}$$

$\mathbf{e_1}$ $\mathbf{e_2}$ $\mathbf{e_3}$

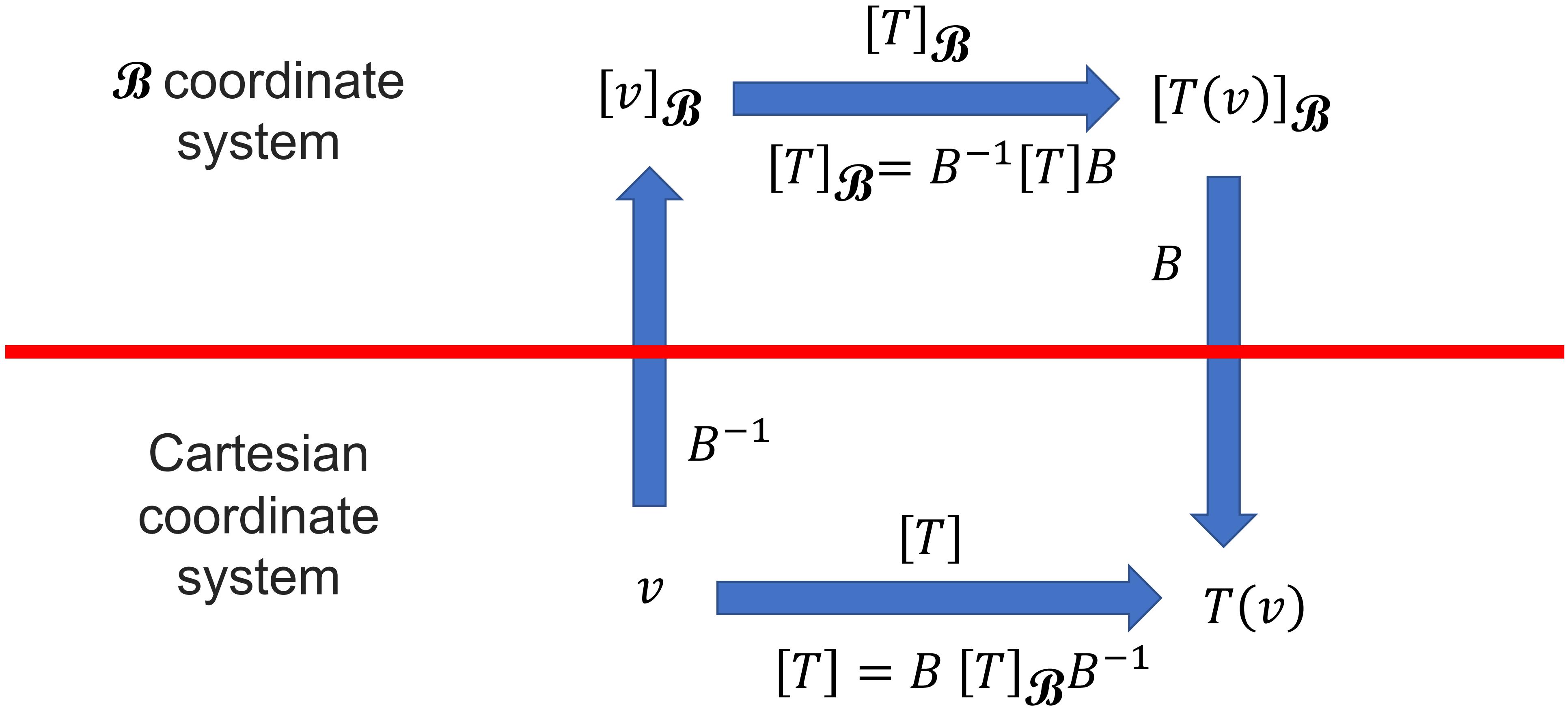


$\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ as a coordinate system

{$\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$} is a basis of R^3

$\mathbf{b_1}$ $\mathbf{b_2}$ $\mathbf{b_3}$

Conclusion



$\mathcal{B}$ coordinate system
$[T]_{\mathcal{B}}$
$[v]_{\mathcal{B}}$
$[T(v)]_{\mathcal{B}}$
$[T]_{\mathcal{B}} = B^{-1}[T]B$
B
B^{-1}
Cartesian coordinate system
$[T]$
v
$T(v)$
$[T] = B\ [T]_{\mathcal{B}}B^{-1}$