Chapter 3

Determinants

除了標註※之簡報外，其餘採用李宏毅教授之投影片教材

Cofactor Expansion

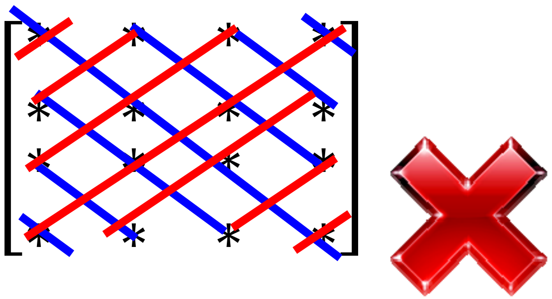
(Chapter 3.1)

Determinants in High School

- 2 X 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A) = ad - bc$$



- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$

$$det(A) =$$

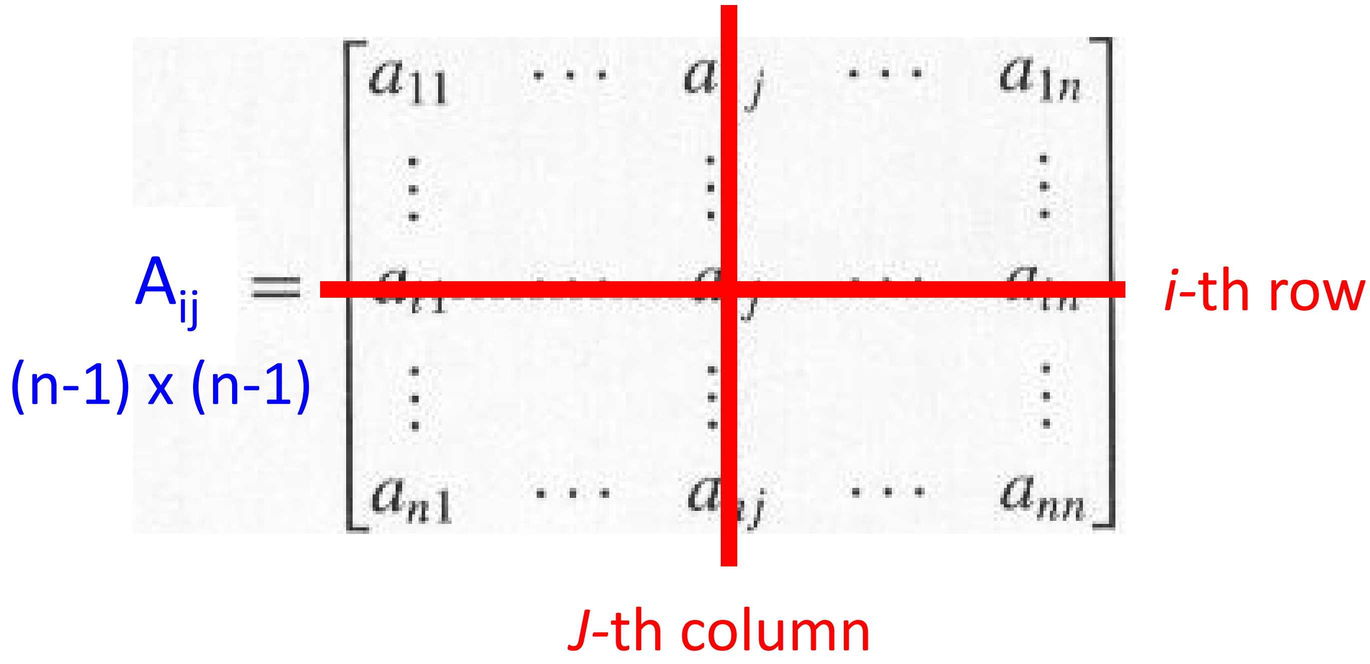
$$a_1a_5a_9 + a_2a_6a_7 + a_3a_4a_8$$

$$-a_3a_5a_7 - a_2a_4a_9 - a_1a_6a_8$$

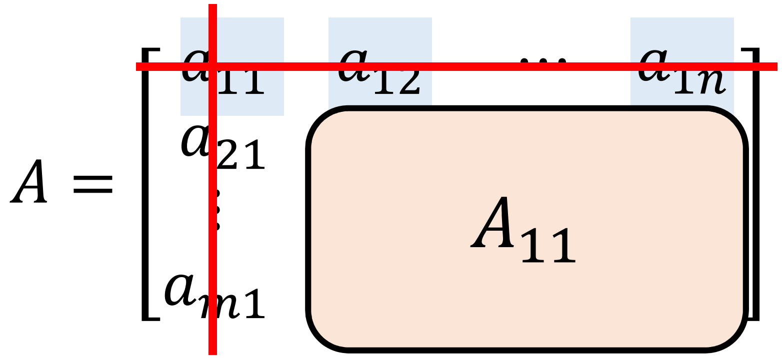
Cofactor Expansion
(Laplace Formula)

a_{ij}: scalar
A_{ij}: matrix

- Suppose A is an n x n matrix. A_{ij} is defined as the submatrix of A obtained by removing the i-th row and the j-th column.



Cofactor Expansion



- Pick row 1

$$detA = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

- Or pick row i

c_{ij}: (i,j)-cofactor

$$detA = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

- Or pick column j

$$detA = a_{1j}c_{1j} + a_{2j}c_{2j} + \cdots + a_{nj}c_{nj}$$

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$c_{11} = (-1)^{1+1} detA_{11}$$

Cofactor expansion again …

2 x 2 matrix

$$c_{ij} = (-1)^{i+j} det A_{ij}$$

- Define det([a]) = a

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad det(A) = ad - bc$$

Pick the first row

$$det(A) = ac_{11} + bc_{12} \quad = ad - bc$$

$$c_{11} = (-1)^{1+1} det([d]) \ = d$$

$$c_{12} = (-1)^{1+2} det([c]) \ = -c$$

3 x 3 matrix

$$c_{ij} = (-1)^{i+j} detA_{ij}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$detA = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23}$$

4   5   6

$$(-1)^{2+1} detA_{21} \qquad (-1)^{2+2} detA_{22} \qquad (-1)^{2+3} detA_{23}$$

$$A_{21} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

- Given tridiagonal n x n matrix A

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Find det A when n = 999

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$detA_4$$

$$= a_{11}c_{11} + a_{12}c_{12} + \cancel{a_{13}c_{13}} + \cancel{a_{14}c_{14}}$$

1 1 0 0

$$c_{11} = (-1)^2 det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= det(A_3)$$

$$c_{12} = (-1)^3 det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= -det(A_2)$$

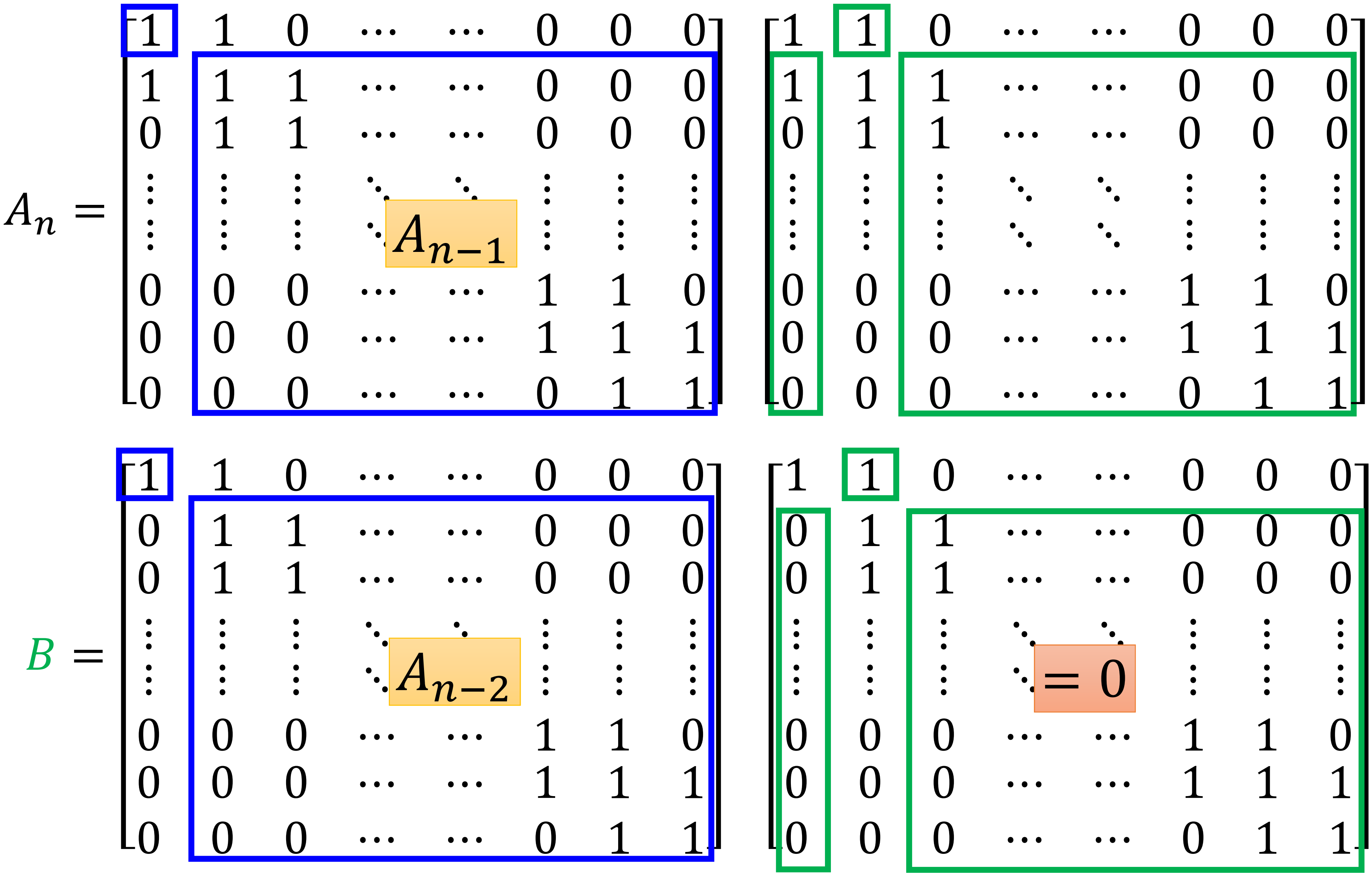
$$= a_{11}c_{11} + a_{12}c_{12} + \cancel{a_{13}c_{13}}$$

1 1 0

$$= det(A_2)$$

$$det(A_4) = det(A_3) - det(A_2)$$

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$



Example

$$det(A_n) = det(A_{n-1}) - det(A_{n-2})$$

$det(A_1) = 1$ $\quad$ $det(A_2) = 0$ $\quad$ $det(A_3) = -1$

$det(A_4) = -1$ $\quad$ $det(A_5) = 0$ $\quad$ $det(A_6) = 1$

$det(A_7) = 1$ $\quad$ $det(A_8) = 0$ $\quad$ … …

Another Example

Vandermonde Determinant

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

@TamasGorbe

See https://nl.wikipedia.org/wiki/Vandermonde-matrix

※ for how to prove the above

Basic Properties of Determinant

(Chapter 3.2)

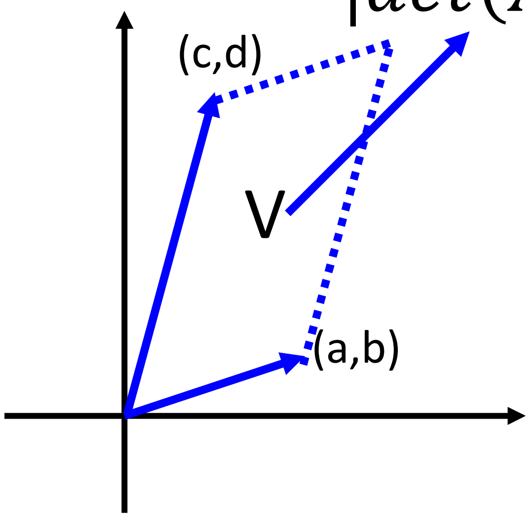
自行閱讀

Determinant in High School

- 2 X 2

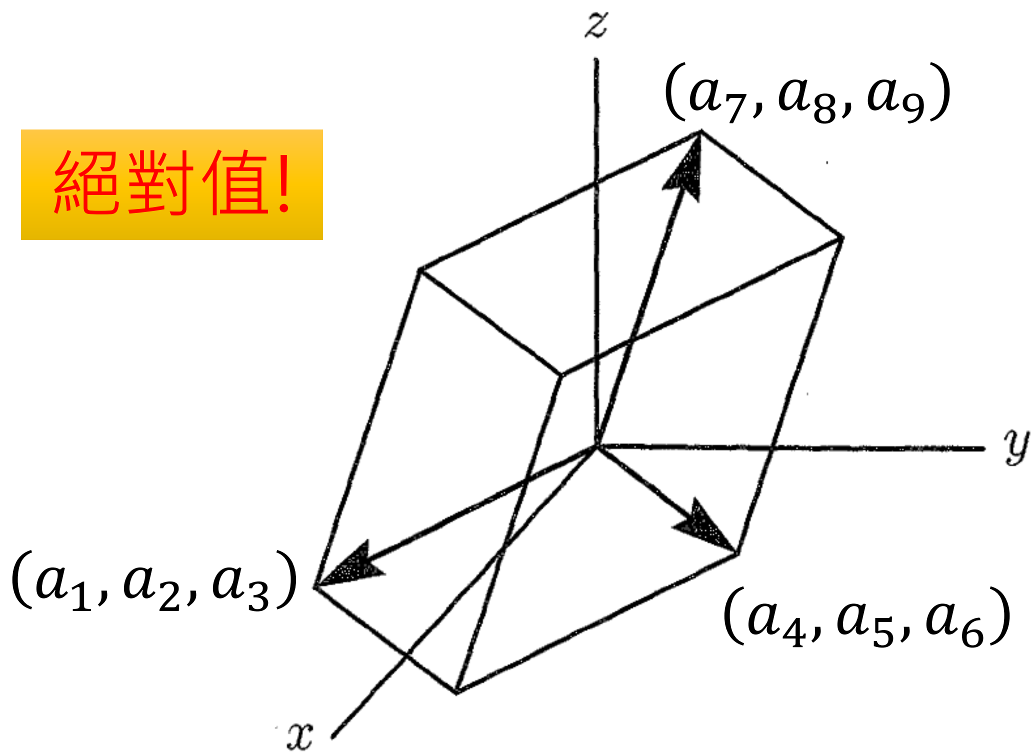
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|det(A)|$ 絕對值!



- 3 x 3

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$



Three Basic Properties

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows only *reverses the sign* of det (do not change absolute value)
- Basic Property 3: Determinant is “linear” for each row

Area in 2d and Volume in 3d have the above properties

So det is “Volume” in high dimension?

Three Basic Properties

- Basic Property 1:
  - $det(I) = 1$

正方形　面積為 1

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(I_2) = 1$$

正立方體　體積為 1

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(I_3) = 1$$

Three Basic Properties

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

$$det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 1$$

$$det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = -1$$

$$det\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1$$

Three Basic Properties

- Basic Property 2:
  - Exchanging rows only reverses the sign of det

If a matrix A has ***2 equal rows***

⇒ $det(A) = 0$

$$A \xrightarrow{\text{exchange the two rows}} A'$$

$det(A) = K$ = $det(A') = -K$

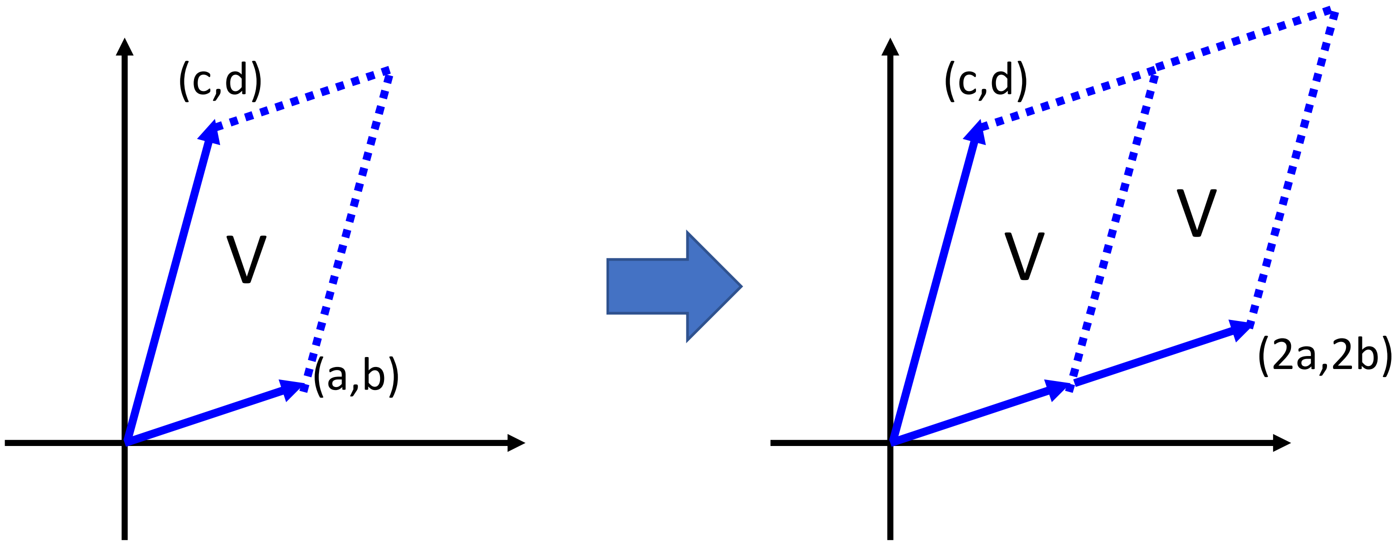
Exchanging the two equal rows yields the same matrix

Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$



Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Q: find $det(2A)$

$$det\left(2\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}\right)$$

If A is n x n ……

$$= 2det\left(\begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}\right) = 4det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A: $det(2A) = 2^n det(A)$

Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

3-a

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

A row of zeros → $det(A) = 0$

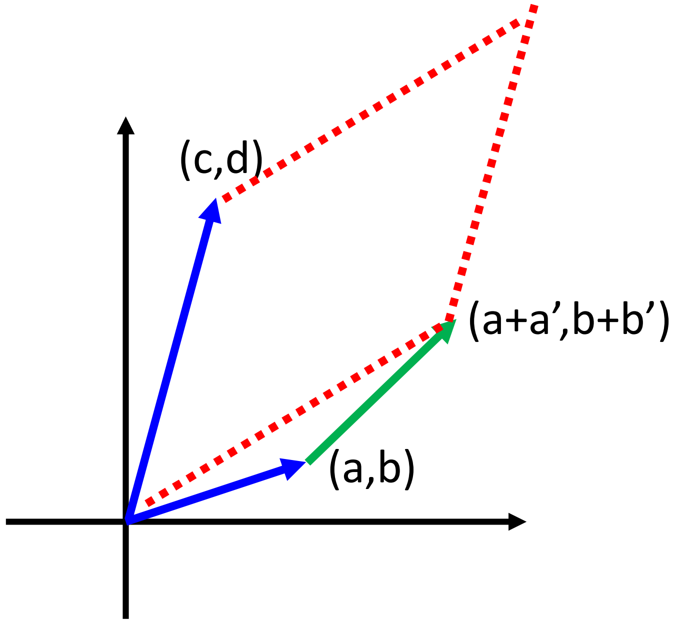
Set $t = 0$!

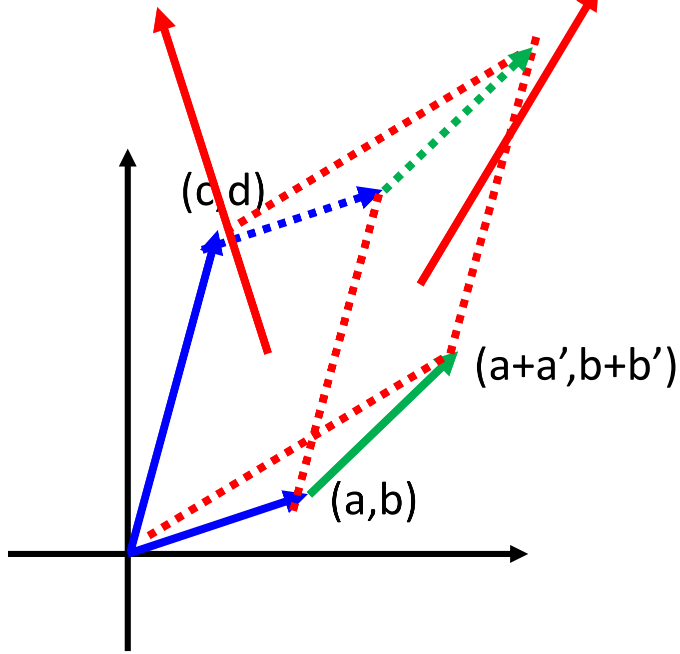
A row of zeros → "volume" is zero

Three Basic Properties

- Basic Property 3:
  - Determinant is “linear” for each row

3-b $$det\left(\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$





Three Basic Properties

- Basic Property 3:
  - Determinant is "linear" for each row

Subtract k x row i from row j (elementary row operation)

Determinant doesn't change

$$det\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right)$$

3-b $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a & b \\ -ka & -kb \end{bmatrix}\right)$$

3-a $$= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k det\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

Formulas Again*

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3! matrices have non-zero rows

$$= \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$a_{11}a_{22}a_{33} \qquad -a_{11}a_{23}a_{32} \qquad -a_{12}a_{21}a_{33}$$

$$+\det\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$a_{12}a_{23}a_{31} \qquad a_{13}a_{21}a_{32} \qquad -a_{13}a_{22}a_{31}$$

Pick an element at each row,
but they cannot be in the same column.

Formula from Three Properties
(Leibniz Formula)

- Given an n x n matrix A

$$det(A) = sum\ of\ n!\ terms$$

Format of each term: $a_{\underline{1}\underline{\alpha}} a_{\underline{2}\underline{\beta}} a_{\underline{3}\underline{\gamma}} \cdots a_{\underline{n}\underline{\omega}}$

Find an element in each row

permutation of 1,2, …, n

$$det(A) = \sum_{\sigma \in S_n} sign(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

sign(σ)= $(-1)^{S(\sigma)}$, S(σ) is the number of "reverse pairs".
1342 -> (3,2), (4,2)

Cramer’s Rule*

Chap. 3.2

Formula for A^{-1}

- $A^{-1} = \frac{1}{det(A)} C^T$
  - $det(A)$: scalar
  - C: cofactors of A (C has the same size as A, so does C^T)
  - C^T is adjugate of A (adj A, 伴隨矩陣)

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$det(A)$$

$$= ad - bc$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, A^{-1} = ?$

$$det(A) = aei + bfg + cdh - ceg - bdi - afh$$

$$C = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Formula for A^{-1}

$$A^{-1} = \frac{1}{det(A)} C^T$$

- Proof: $AC^T = det(A)I_n$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

transpose

Diagonal: By definition of determinants

Not Diagonal:

(Exercise 82, P221)

$$AC^T = det(A)I_n$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & det(A) \end{bmatrix}$$

$$det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$

$$det \begin{bmatrix} a_{n1} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{n1}c_{11} + a_{n2}c_{12} + \cdots + a_{nn}c_{1n}$$

$$= 0$$

Cramer's Rule

$$\begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} C^T \qquad Ax = b \qquad x = A^{-1}b = \frac{1}{det(A)} C^T b$$

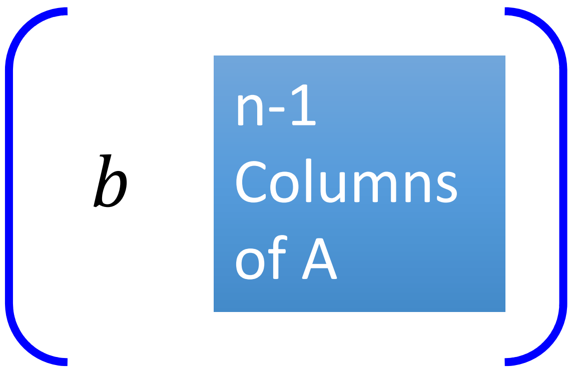
$$x_1 = \frac{1}{det(A)} (c_{11}b_1 + c_{21}b_2 + \cdots + c_{n1}b_n)$$

$$det(B_1)$$

$$x_1 = \frac{det(B_1)}{det(A)}$$

B_1 = with column 1 replaced by b

$$x_2 = \frac{det(B_2)}{det(A)}$$



$$\vdots$$

B_j = with column j replaced by b

$$x_j = \frac{det(B_j)}{det(A)}$$

Example

Cramer's Rule

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $D \neq 0$ then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D}$$

https://www.onlinemathlearning.com/cramers-rule.html

More Properties of Det

(Chapter 3.2)

A is invertible ⟷ $\det(A) \neq 0$

Properties of Determinants

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows reverse the sign of det
  - If a matrix A has 2 equal rows, det A = 0
- Basic Property 3: Determinant is "linear" for each row

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$det\left(\begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$

  - A row of zeros, det A = 0
  - "Subtract k x row i from row j" does not change det

Determinants for Upper Triangular Matrix

$$U = \begin{bmatrix} d_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

Killing everything above

Does not change the det

$$det(U) = det\left(\begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}\right)$$

Property 1

3-a $= d_1 d_2 \cdots d_n det\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right)$

= 1

$det(U) = d_1 d_2 \cdots d_n$ (Products of diagonal)

Determinant vs. Invertible

A is invertible ⟷ det(A) ≠ 0

A → R

Elementary row operation

$det(A)$

$$det(R) = \pm k_1 k_2 \cdots det(A)$$

Exchange: Change sign

Scaling: Multiply k

Add row: nothing

If A is invertible, R is identity

$det(R) = 1$ ⇒ $det(A) \neq 0$

If A is not invertible, R has zero row

$det(R) = 0$ ⇒ $det(A) = 0$

Example

A is invertible ⟷ det(A) ≠ 0

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix}$$

For what scalar c is the matrix not invertible?

det(A) = 0

$$detA = 1 \cdot 0 \cdot 7 \quad +(-1) \cdot c \cdot 2 \quad +2 \cdot (-1) \cdot 1$$

$$-2 \cdot 0 \cdot 2 \quad -(-1) \cdot (-1) \cdot 7 \quad -1 \cdot c \cdot 1$$

$$= 0 - 2c - 2 - 7 - c \quad = -3c - 9$$

not invertible ⇒ $-3c - 9 = 0$ ⇒ $c = -3$

More Properties of Determinants

$det(A + B) \neq det(A) + det(B)$

- $det(AB) = det(A)det(B)$

  Q: find $det(A^{-1})$

  $\because A^{-1}A = I \quad \therefore det(A^{-1})det(A) = det(I) = 1$

  $\therefore det(A^{-1}) = 1/det(A)$

  Q: find $det(A^2)$

  $$det(A^2) = det(A)det(A) = det(A)^2$$

- $det(A^T) = det(A)$
  - Zero row → zero column
  - Same row → same column ……

More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is not invertible:

A is not invertible ⇒ AB is not invertible.
(because $(AB)M = I_n$ implies $A(BM)= I_n$)

⇒ det AB = 0

A is not invertible ⇒ det A = 0

⇒ det A det B = 0

More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

  If A is invertible:

  $$A = E_k \cdots E_2 E_1$$

You have to prove that det EA = det E det A

(E is elementary matrix)

You have to prove that det EA = det E det A

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$detE_1A = -detA$$
$$= detE_1detA$$
$$detE_1 = -1$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$detE_2A = -4detA$$
$$= detE_2detA$$
$$detE_2 = -4$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$detE_3A = detA$$
$$= detE_3detA$$
$$detE_3 = 1$$

More Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is invertible:

$$A = E_k \cdots E_2 E_1$$

You have to prove that det EA = det E det A

(E is elementary matrix)

$$det(A) = det(E_k) \cdots det(E_2)det(E_1)$$

$$\begin{aligned} det(A)det(B) &= det(E_k) \cdots det(E_2)det(E_1)det(B) \\ &= det(E_k) \cdots det(E_2)det(E_1 B) \\ &= det(E_k \cdots E_2 E_1 B) = det(AB) \end{aligned}$$

More Properties of Determinants

- $\det A = \det A^T$
- Proof:

A is not invertible → $\det A = 0$

=

A^T is not invertible → $\det A^T = 0$

A is invertible $\det E = \det E^T$ ...... in the textbook

det E = det E^T …… in the textbook

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{E_1}^{\mathrm{T}} \qquad det\mathrm{E_1} = det\mathrm{E_1}^{\mathrm{T}}$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{E_2}^{\mathrm{T}} \qquad det\mathrm{E_2} = det\mathrm{E_2}^{\mathrm{T}}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \mathrm{E_3}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad det\mathrm{E_3} = det\mathrm{E_3}^{\mathrm{T}}$$

More Properties of Determinants

det E = det E^T ...... in the textbook

- det A = det A^T
- Proof:

A is invertible $\quad A = E_k \cdots E_2 E_1$

$$det(A) = det(E_k) \cdots det(E_2) det(E_1)$$

$$A^T = (E_k \cdots E_2 E_1)^T = {E_1}^T {E_2}^T \cdots {E_k}^T$$

$$det(A^T) = det\left({E_1}^T\right) det\left({E_2}^T\right) \cdots det\left({E_k}^T\right)$$

$$= det(E_1) det(E_2) \cdots det(E_k)$$

More Properties of Determinants

det E = det E^T ...... in the textbook

- det A = det A^T
- Proof:

$$det(A) = sum\ of\ n!\ terms$$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

Sorted by column indices

Find an element in each row

permutation of 1,2, ..., n

Format of each term: $a_{\alpha'1}a_{\beta'2}a_{\gamma'3}\cdots a_{\omega'n}$

Find an element in each column

permutation of 1,2, ..., n