Chapter 2
Matrices and Linear Transformations

除了標註※之簡報外，其餘採用李宏毅教授之投影片教材

Matrix Multiplication

(Chapter 2.1)

Four aspects for matrix multiplication

1. Dot Product

(What you have learned in high school)

Dot Product
(special case of Inner Product)

- **Dot product**: dot product of u and v is

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \boxed{= u^T v}$$

- Three properties of Dot Product V × V → R
  - $u \cdot v = v \cdot u$ (commutative)
  - $u \cdot (cv + w) = c(u \cdot v) + u \cdot w$ (Linear)
  - $u \cdot u \geqq 0$, and =0 only when u = **0**

Dot Product

- Given two matrices A and B, the (*i*, *j*)-entry of AB is the dot product of row i of A and column j of B

$$C = AB \qquad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$\begin{array}{rc} & \\ & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \end{bmatrix} \\ \text{row } i \text{ of } A \longrightarrow & \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \\ & \begin{bmatrix} \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \qquad \begin{array}{c} \text{column } j \text{ of } B \\ \downarrow \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} \end{array}$$

A B

Dot Product

- Given two matrices A and B, the (*i*, *j*)-entry of AB is the dot product of row i of A and column j of B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\boxed{C = AB} = \begin{bmatrix} (-1) \times 1 + 3 \times 2 & 1 \times 1 + 2 \times 2 \\ (-1) \times 3 + 3 \times 4 & 1 \times 3 + 2 \times 4 \\ (-1) \times 5 + 3 \times 6 & 1 \times 5 + 2 \times 6 \end{bmatrix}$$

Dot Product

Given two matrices A and B, the (*i*, *j*)-entry of AB is the dot product of row i of A and column j of B

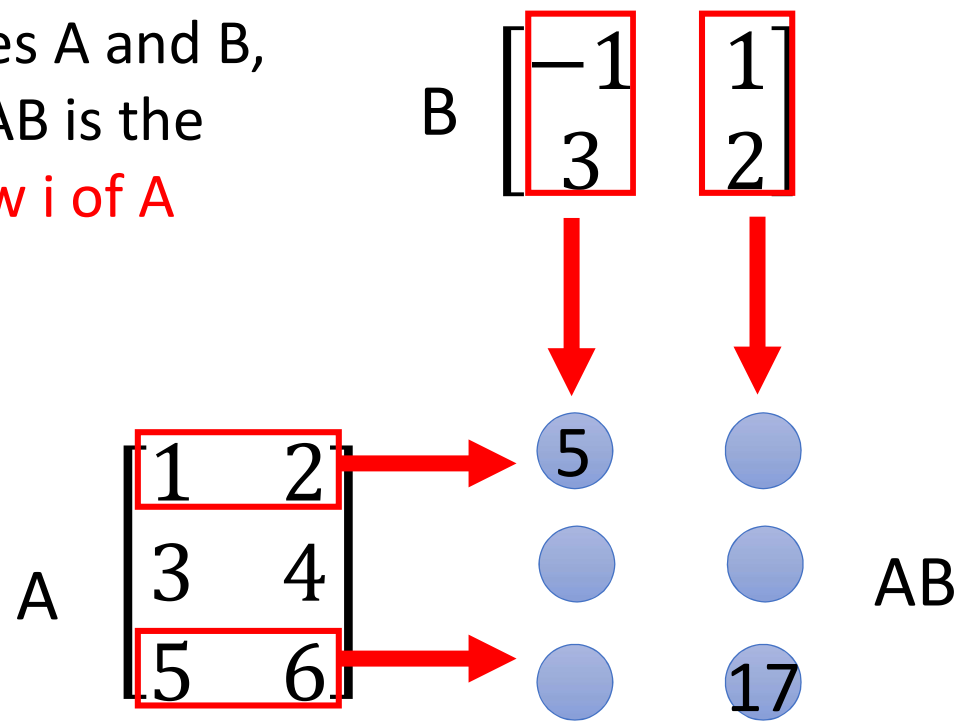
$$B \quad \begin{bmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1p} \\ b_{21} & \cdots & \boxed{b_{2j}} & \cdots & b_{2p} \\ \vdots & & \boxed{\vdots} & & \vdots \\ b_{n1} & \cdots & \boxed{b_{nj}} & \cdots & b_{np} \end{bmatrix}$$

$$A \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow \begin{bmatrix} & & \\ & \text{(i,j)-entry} & \\ & & \end{bmatrix}$$

AB

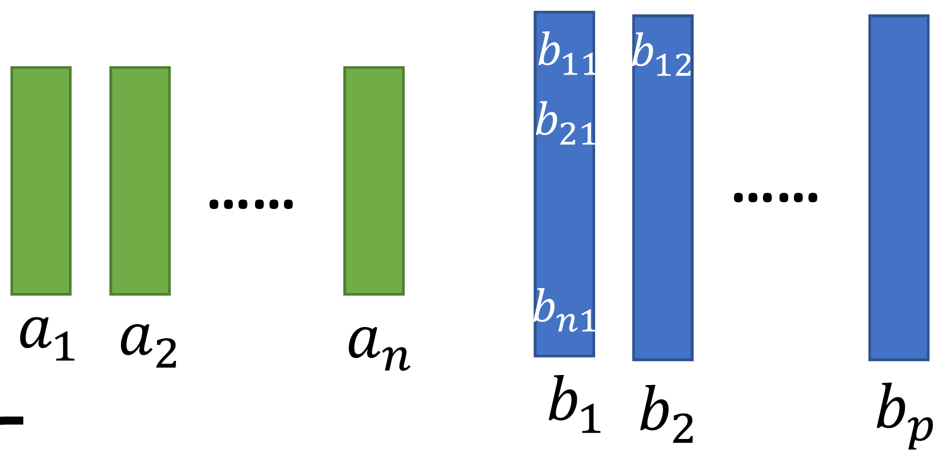
Dot Product

Given two matrices A and B, the (i, j)-entry of AB is the dot product of row i of A and column j of B

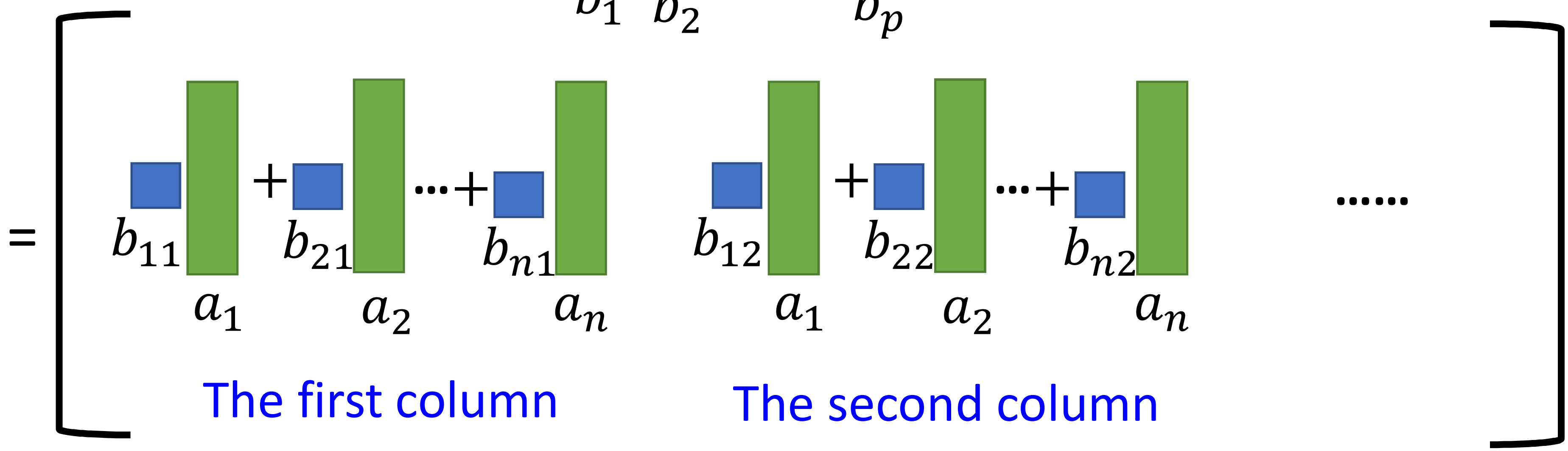


2. Combination of Columns

Combination of Columns



$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$
$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$



Combination of Columns

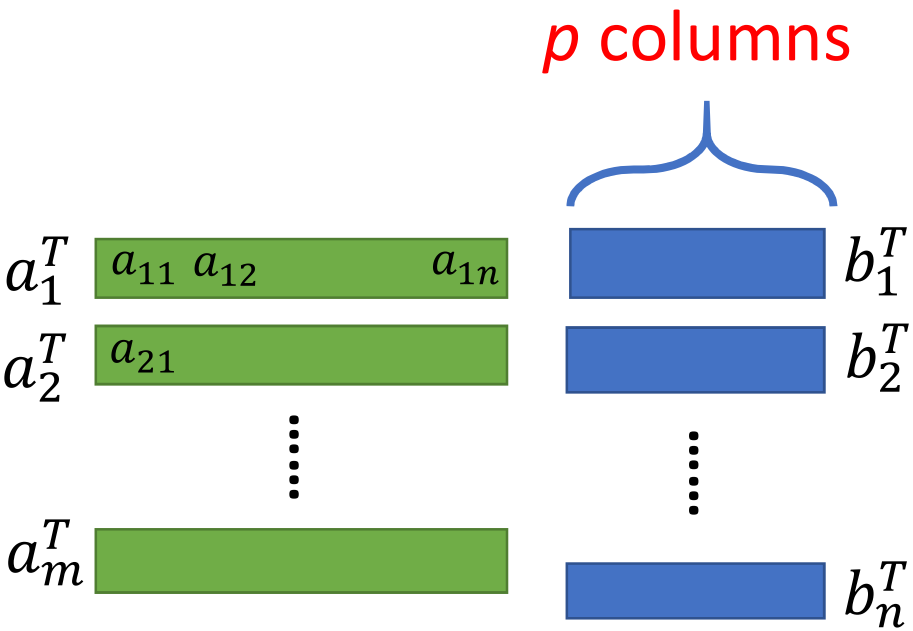
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$
$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

$$= \left[\begin{array}{cc} -1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} & 1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \\ \text{The first column} & \text{The second column} \end{array} \right]$$

3. Combination of Rows

Combination of Rows



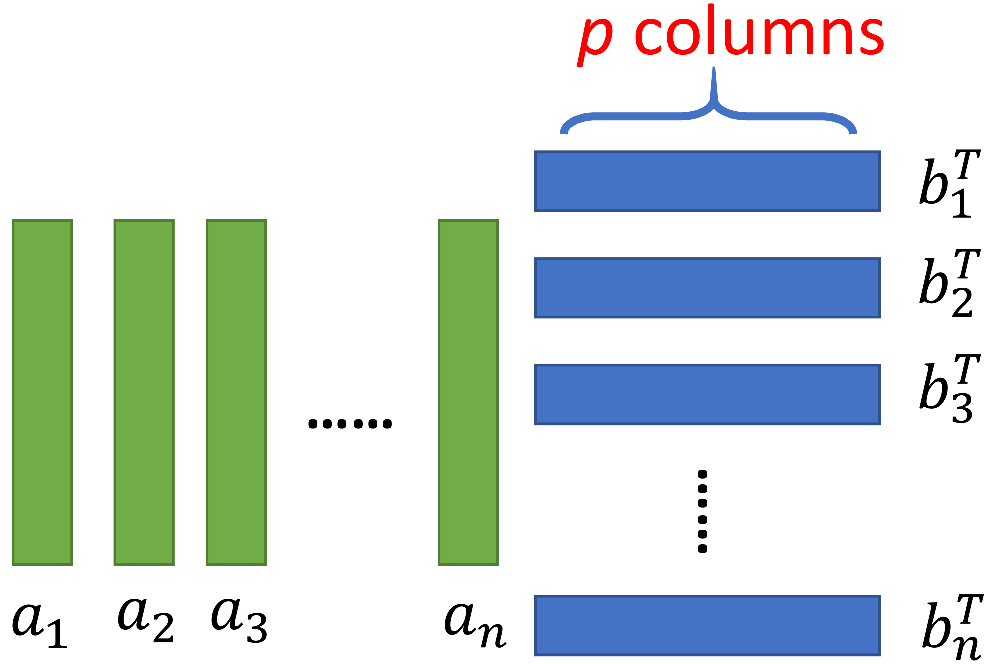
$$= \begin{bmatrix} a_{11}b_1^T + a_{12}b_2^T \cdots + a_{1n}b_n^T \\ a_{21}b_1^T + a_{22}b_2^T \cdots + a_{2n}b_n^T \\ \vdots \\ a_{m1}b_1^T + a_{m2}b_2^T \cdots + a_{mn}b_n^T \end{bmatrix}$$

Combination of Rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1[-1 \quad 1] + 2[3 \quad 2] \\ \text{The first row} \\ 3[-1 \quad 1] + 4[3 \quad 2] \\ \text{The second row} \\ 5[-1 \quad 1] + 6[3 \quad 2] \\ \text{The third row} \end{bmatrix}$$

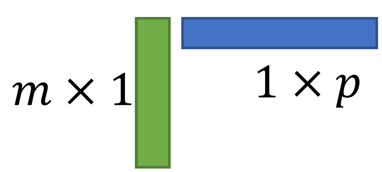
4. Summation of Matrices

Summation of Matrices



$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

matrices



Summation of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} \right]$$

"1 x 2" "2 x 1"

"1 x 1"

$$= \left[\underbrace{\begin{bmatrix} -1 & 1 \\ -3 & 3 \\ -5 & 5 \end{bmatrix}}_{\text{Rank = ?}} + \underbrace{\begin{bmatrix} 6 & 4 \\ 12 & 8 \\ 18 & 12 \end{bmatrix}}_{\text{Rank = ?}} \right]$$

Block Multiplication

Augmentation and Partition

- Augment: the augment of A and B is [A B]
- Partition:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right] \qquad A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right] \qquad A = \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right]$$

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Multiply as the small matrices are scalar

Don't switch the order

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix}$$

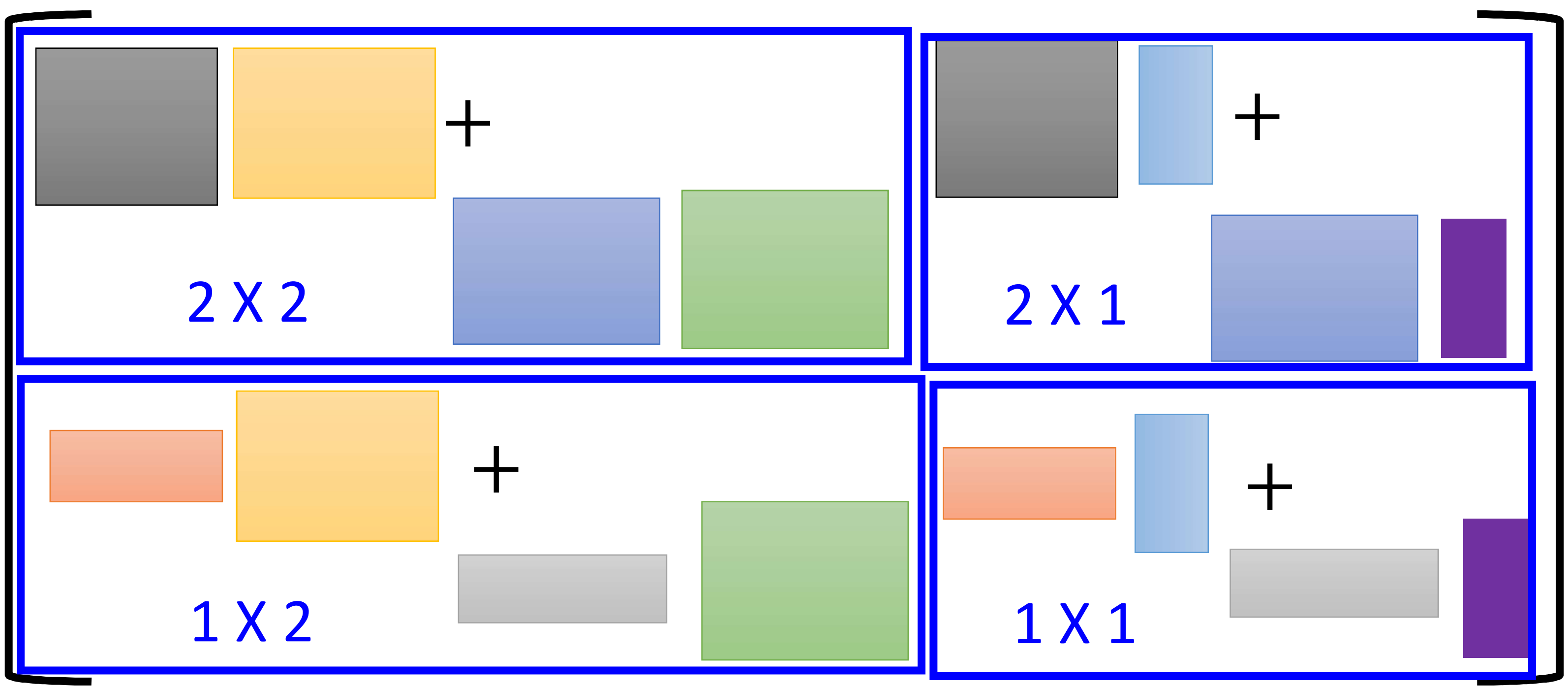
"2 x 2"

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

"2 x 2"

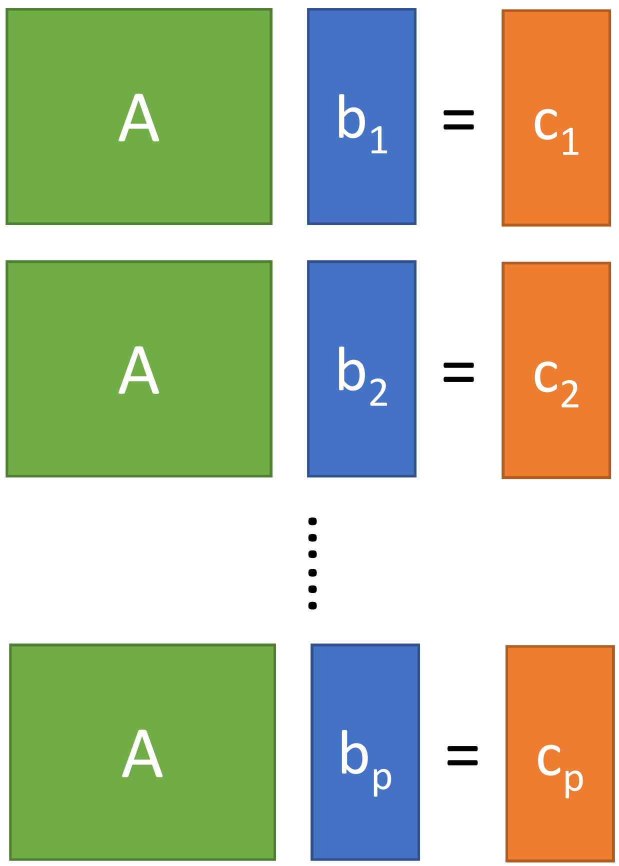
$AB =$

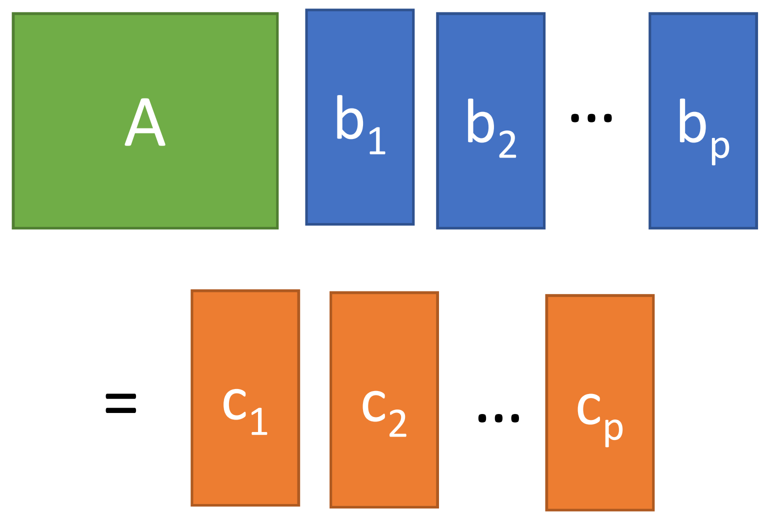
"2 x 2"



Matrix Multiplication - Multiple Input

- ***Multiple Input*** $C = AB$





$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$
$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

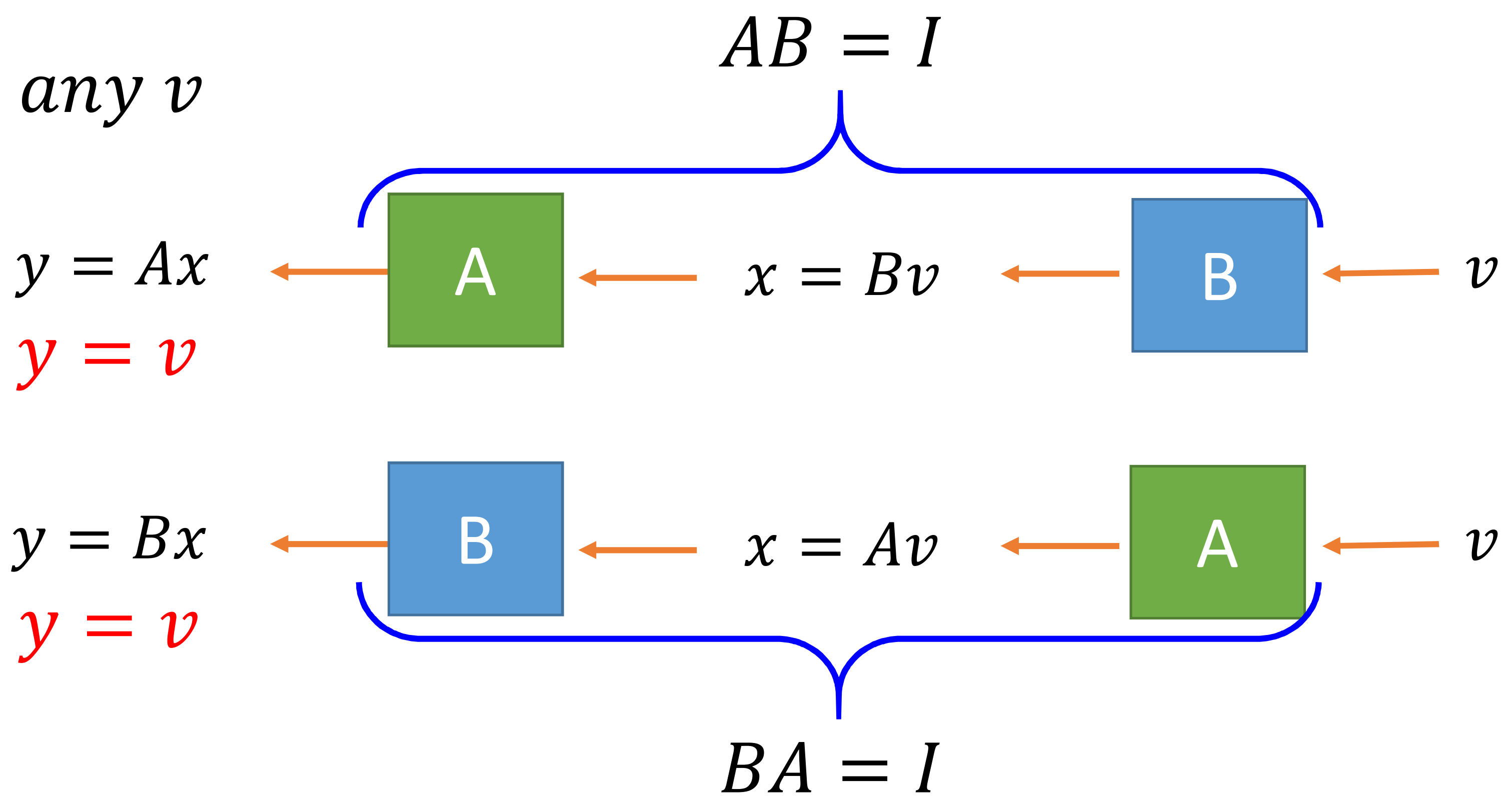
Matrix Inverse

(Chapter 2.3-2.4)

Inverse of Matrix

- A and B are inverses to each other

For $any\ v$



Inverse of Matrix

Invertible = Non-singular
Not Invertible = Singular

A is called invertible if there is a matrix B such that $AB = I$ and $BA = I$

B is an inverse of A

$$B = A^{-1} \qquad A = B^{-1}$$

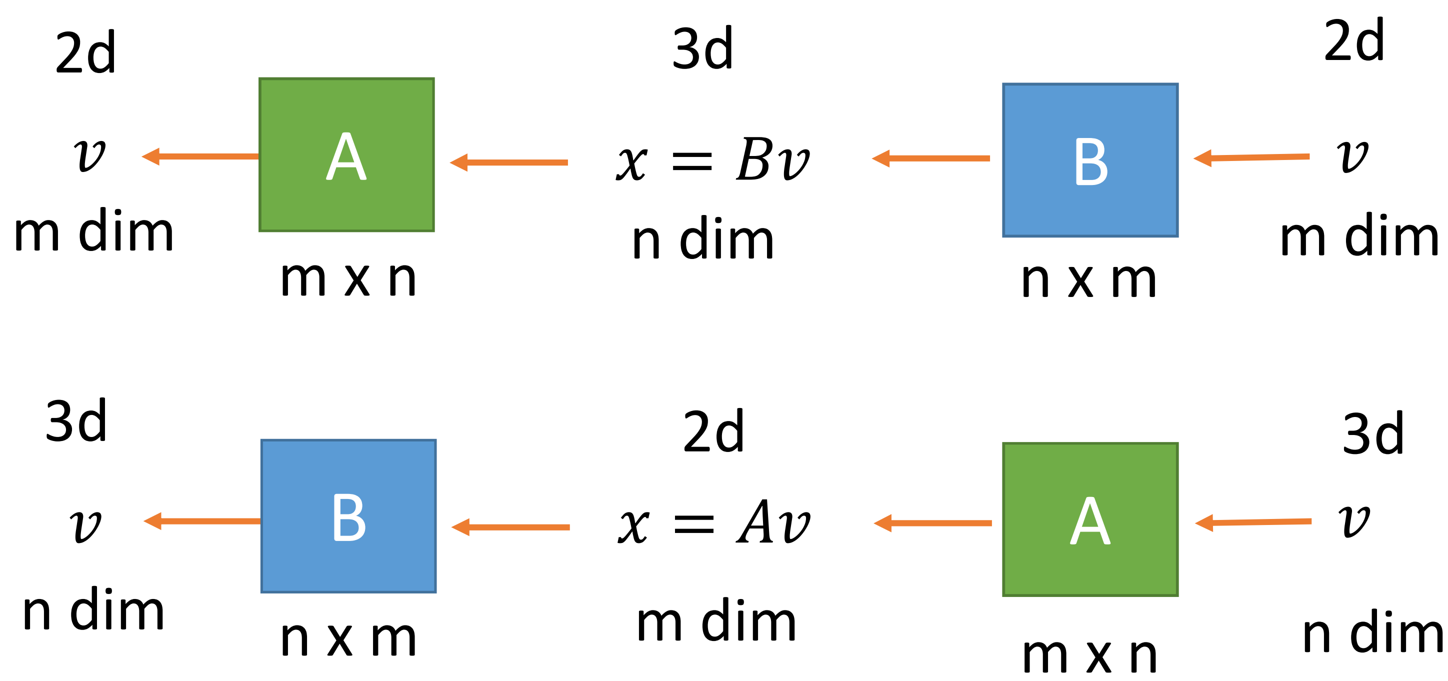
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Non-square matrix cannot be invertible

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}$$

Inverse of Matrix

- Non-square matrix cannot be invertible?



Is BA (dim 3 x 3) invertible?

Inverse of Matrix

- Not all the square matrices are invertible

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Unique

$$AB = I \qquad BA = I \qquad AC = I \qquad CA = I$$

$$B = BI = B(AC) = (BA)C = IC = C$$

Inverse for matrix product

- A and B are invertible nxn matrices, is AB invertible? yes

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$B^{-1}A^{-1}(AB) \; = B^{-1}\,(A^{-1}A)B \; = B^{-1}\,B \; = I$$

$$(AB)B^{-1}A^{-1} \; = A(BB^{-1})A^{-1} \; = A\,A^{-1} \; = I$$

- Let $A_1, A_2, \cdots, A_k$ be nxn invertible matrices. The product $A_1A_2\cdots A_k$ is invertible, and

$$(A_1A_2\cdots A_k)^{-1} = (A_k)^{-1}(A_{k-1})^{-1}\cdots(A_1)^{-1}$$

Inverse for matrix transpose

- If A is invertible, is A^T invertible?

$$(A^T)^{-1} =? \quad (A^{-1})^T$$

$$\boxed{(AB)^T = B^T A^T}$$

$$A^{-1}A = I \Rightarrow (A^{-1}A)^T = I \Rightarrow A^T(A^{-1})^T = I$$

$$AA^{-1} = I \Rightarrow (AA^{-1})^T = I \Rightarrow (A^{-1})^T A^T = I$$

How to prove $(AB)^T = B^T A^T$?

- Method 1:

Express (*i*, *j*)-entry of $(AB)^T$ and $B^T A^T$ directly.

- Method 2:

First prove: $A\,x \cdot y = x \cdot A^T y$, which is not difficult
(A is n x n, x and y are n x *1*)

1. $(AB)\,x \cdot y = A(Bx) \cdot y = Bx \cdot A^T y = x \cdot B^T(A^T y)$
   $= x \cdot (B^T A^T) y$

2. $(AB)\,x \cdot y = x \cdot (AB)^T\, y$

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Application of Matrix Inverse

(Chapter 2.3-2.4)

Solving Linear Equations

- The inverse can be used to solve system of linear equations.

$$A\mathbf{x} = \mathbf{b}$$

If A is invertible.

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$\underbrace{\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 5x_2 &= 7 \end{aligned}}_{Ax = b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

However, this method is computationally inefficient.

Invertible

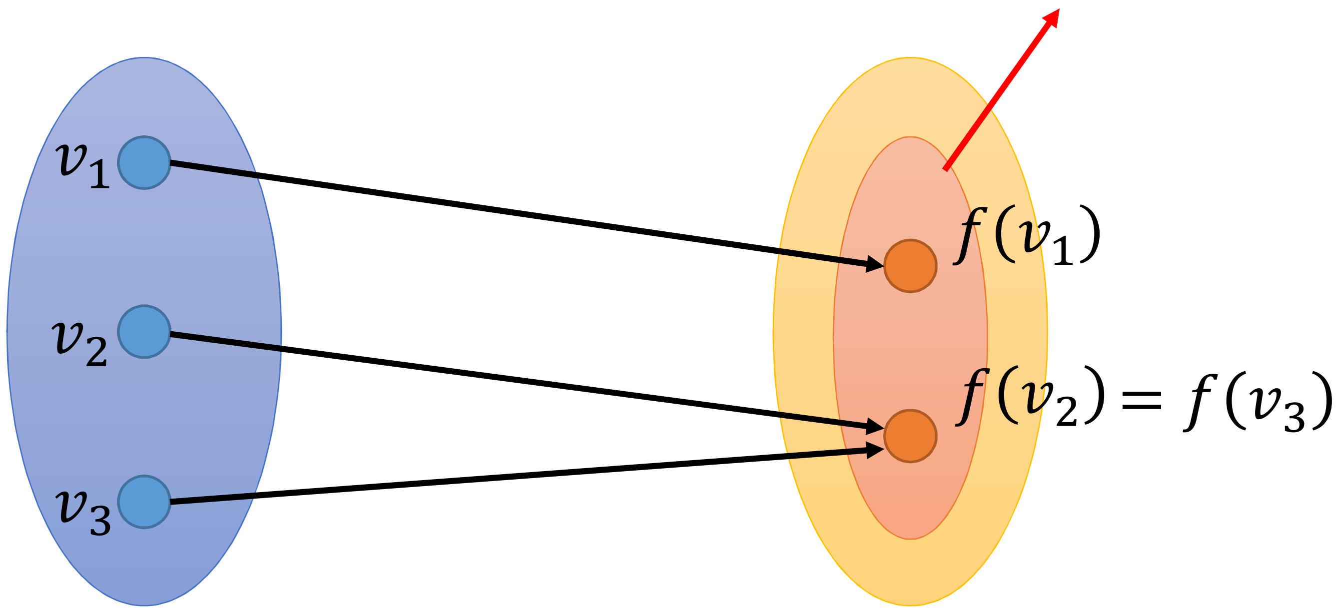
(Chapter 2.3-2.4)

Review - Terminology

What ***actually*** come out of function f

- Given a function f

Range (值域)



Domain (定義域)

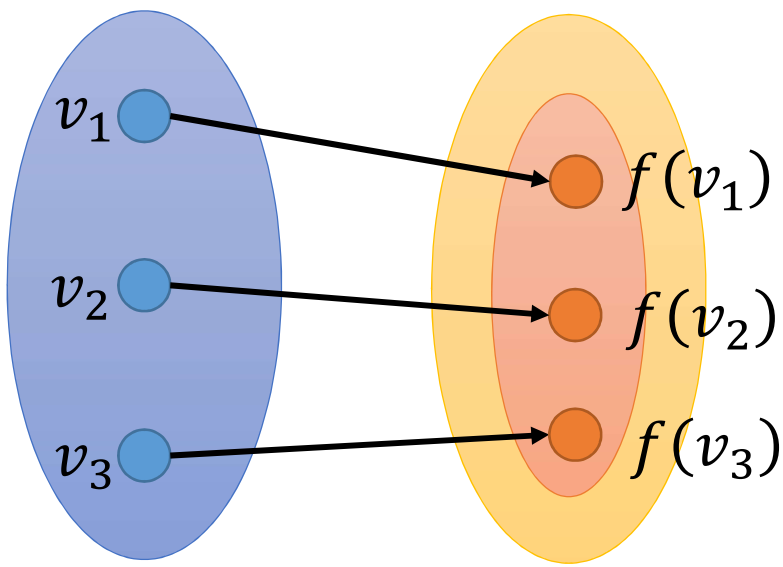
What can go into function f

Co-domain (對應域)

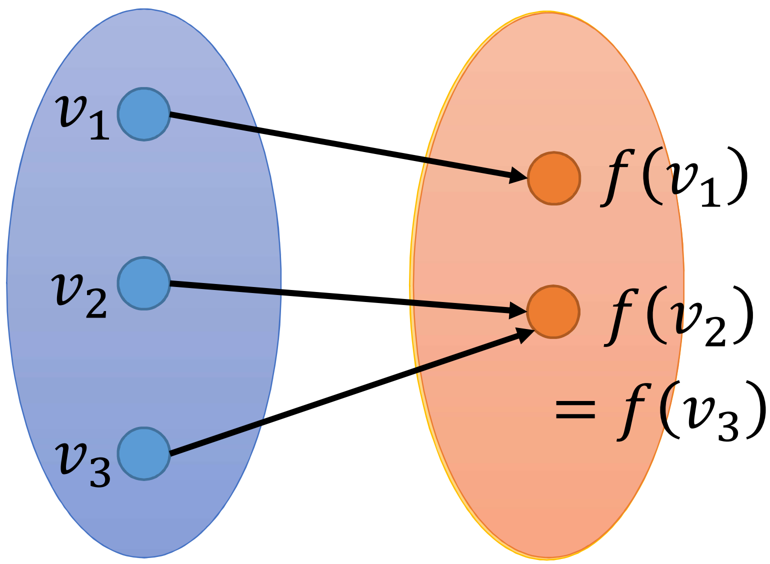
What ***may possibly*** come out of function f

Review - Terminology

- one-to-one (一對一)



- Onto (映成)

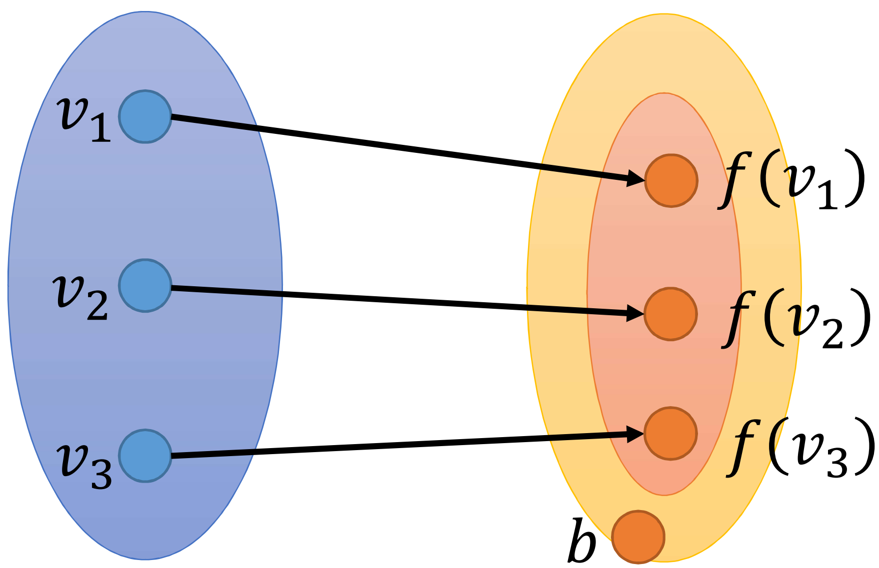


Co-domain = range

One-to-one

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- A function f is one-to-one



~~$f(x) = b$ has one solution~~

$f(x) = b$ has at most one solution

If co-domain is “smaller” than the domain, linear function f cannot be one-to-one.

If a matrix A is 矮胖, it cannot be one-to-one.

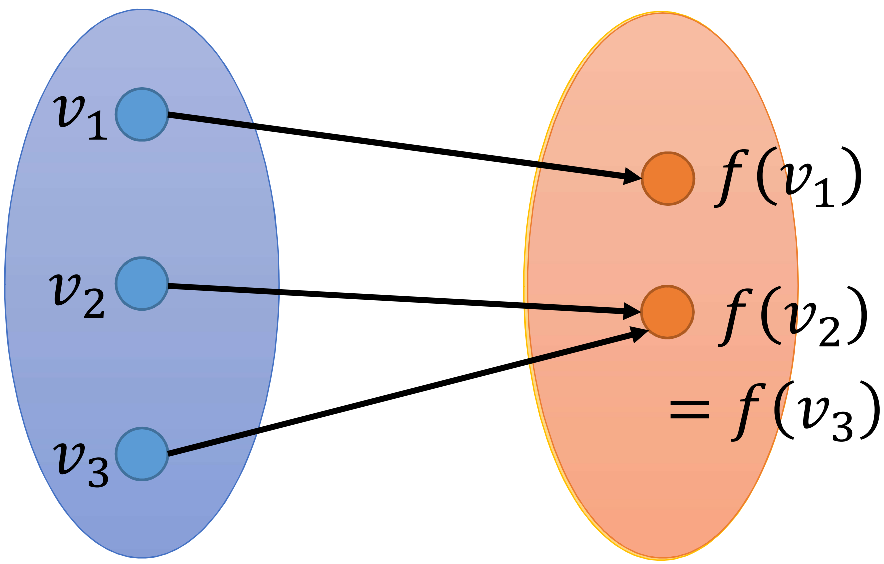
The reverse is not true.

If a matrix A is one-to-one, its columns are independent.

Onto

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

- A function f is onto



Co-domain = range

$f(x) = b$ always have solution

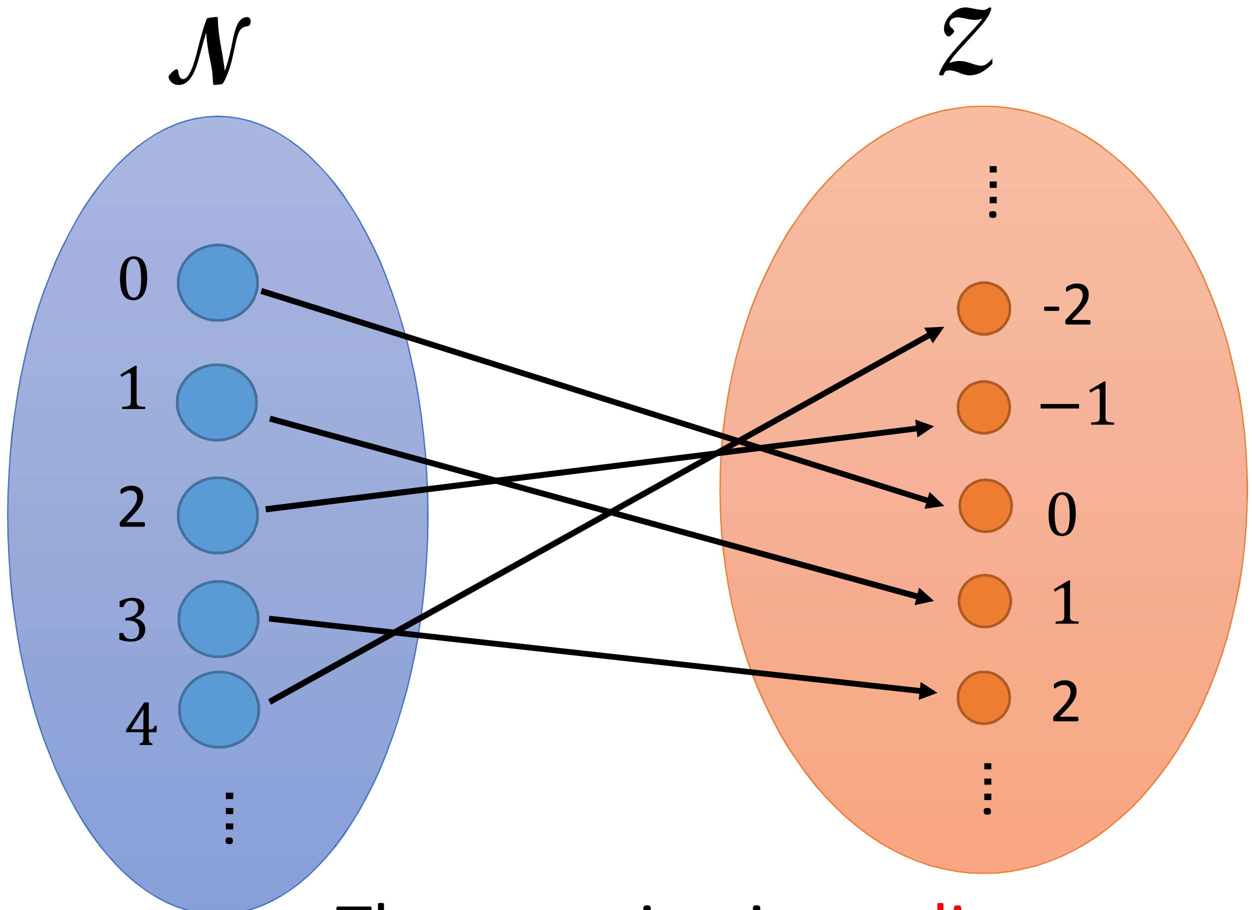
If co-domain is "larger" than the domain, linear function f cannot be onto.

If a matrix A is 高瘦, it cannot be onto.

The reverse is not true.

If a matrix A is onto, rank A = no. of rows, i.e., no zero row in RREF

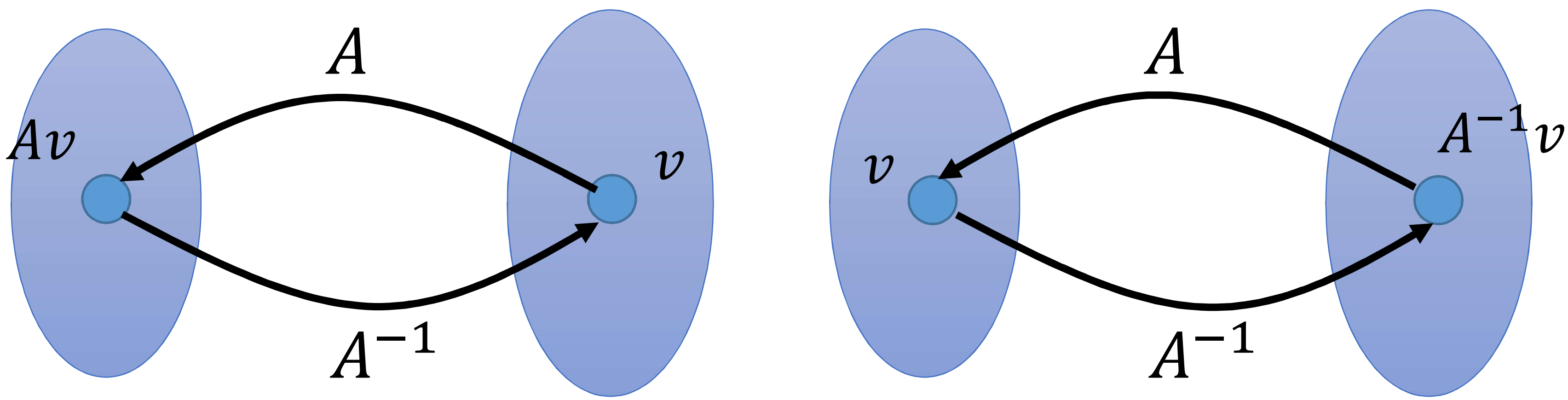
1-1 and Onto Function from $\mathcal{N}$ (Natural Numbers) to $\mathcal{Z}$ (Integers)



The mapping is not linear

Invertible

- A is called invertible if there is a matrix B such that $AB = I$ and $BA = I$ ($B = A^{-1}$)



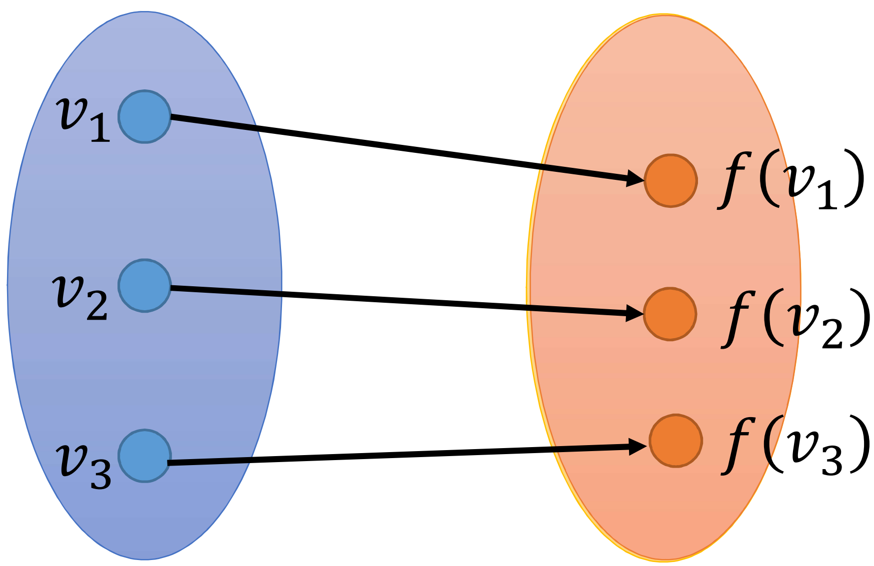
A must be one-to-one

A must be onto
(不然 A^{-1} 的 input 就會有限制)

One-to-one and onto

An invertible matrix A is always square.

- A linear function f is one-to-one and onto



The domain and co-domain must have “the same size”.

The corresponding matrix A is square.

One-to-one ⟷ Onto

在 Square 的前提下，要就都成立，要就都不成立

Equivalent Conditions of Invertibility

- Let A be an n x n matrix. A is invertible if and only if
  - The columns of A span R^n
  - For every b in R^n, the system Ax=b is consistent
  - The rank of A is n
  - The columns of A are linearly independent
  - The only solution to Ax=0 is the zero vector
  - The nullity of A is zero
  - The reduced row echelon form of A is I_n
  - A is a product of elementary matrices
  - There exists an n x n matrix B such that BA = I_n
  - There exists an n x n matrix C such that AC = I_n

Invertible

- Let A be an n x n matrix.
  - Onto → One-to-one → invertible
    - The columns of A span R^n
    - For every b in R^n, the system Ax=b is consistent
    - The rank of A is the number of rows
  - One-to-one → Onto → invertible
    - The columns of A are linear independent
    - The rank of A is the number of columns
    - The nullity of A is zero
    - The only solution to Ax=0 is the zero vector
    - The reduced row echelon form of A is I_n

Rank A = n

Is A Invertible?

- Let A be an n x n matrix. A is invertible if and only if
  - The reduced row echelon form of A is I_n

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} I_n$$

Invertible

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Not Invertible

Summary

- Let A be an n x n matrix. A is invertible if and only if

onto

- The columns of A span R^n
- For every b in R^n, the system Ax=b is consistent
- The rank of A is n

One-to-one

- The columns of A are linear independent
- The only solution to Ax=0 is the zero vector
- The nullity of A is zero
- The reduced row echelon form of A is I_n
- A is a product of elementary matrices
- There exists an n x n matrix B such that BA = I_n
- There exists an n x n matrix C such that AC = I_n

=

square matrix

Invertible

A is n x n

(def) $\exists\, B,\ AB = I$ and $BA = I$

A is invertible.

There exists an n x n matrix B such that BA = I_n

?

The only solution to Ax=0 is the zero vector

If $Av = 0$, then ....

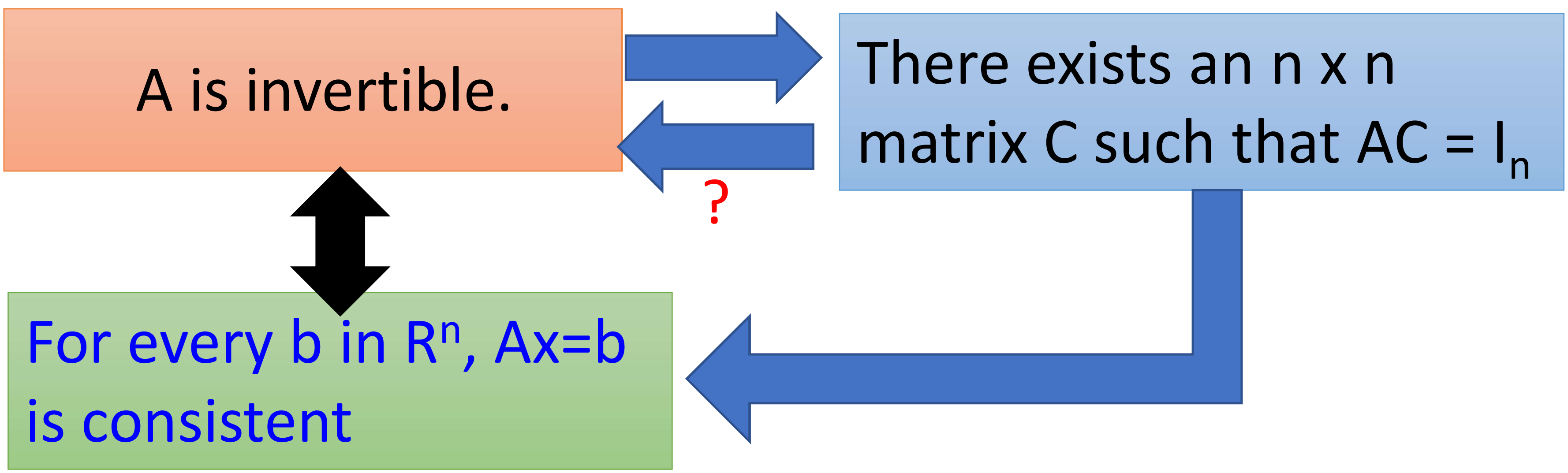
$$BA = I_n \Rightarrow v = 0$$

$$BAv = 0 \qquad I_n v = v$$

Invertible

A is n x n

(def) $\exists\, C,\ AC = I$ and $CA = I$



For any vector b,

$$AC = I_n$$

$$ACb \qquad I_n b = \mathrm{b}$$

Cb is always a solution for b

Summary

- Let A be an n x n matrix. A is invertible if and only if
  - The columns of A span R^n
  - For every b in R^n, the system Ax=b is consistent
  - The rank of A is n
  - The columns of A are linear independent
  - The only solution to Ax=0 is the zero vector
  - The nullity of A is zero
  - The reduced row echelon form of A is I_n
  - A is a product of elementary matrices
  - There exists an n x n matrix B such that BA = I_n
  - There exists an n x n matrix C such that AC = I_n

onto

One-to-one

=

square matrix

$AC = I_n \Rightarrow CA = I_n$?

Theorem: Let A be an n x n matrix. If there exists an n x n matrix C such that $AC = I_n$, then $CA = I_n$.

(Proof) We first prove that the columns of C ($\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$) are linear independent. Suppose $d_1\boldsymbol{c}_1 + d_2\boldsymbol{c}_2 + \ldots + d_n\boldsymbol{c}_n = 0$, then

$$d_1A\boldsymbol{c}_1 + d_2A\boldsymbol{c}_2 + \ldots + d_nA\boldsymbol{c}_n = A0 = 0.$$

$AC=I_n$ implies $d_1\boldsymbol{e}_1 + d_2\boldsymbol{e}_2 + \ldots + d_n\boldsymbol{e}_n = 0$, which is only true if $d_1, d_2, \ldots, d_n$ are all zero, since $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are standard bases.

Let $x= k_1\boldsymbol{c}_1 + k_2\boldsymbol{c}_2 + \ldots + k_n\boldsymbol{c}_n = Cy$, for $y= (k_1, k_2, \ldots, k_n)^T$.

Thus $Ax=ACy=y$ (sinca $AC=I_n$). $CAx=Cy=x$, for arbitrary x. Hence $CA=I_n$

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$Exercise$

AB invertible $\Rightarrow$ A and B are invertible

(Proof)

AB invertible $\Rightarrow \exists C, (AB)C = I$

$\Rightarrow A(BC) = I$

$\Rightarrow A$ is invertible

AB invertible $\Rightarrow \exists C, C(AB) = I$

$\Rightarrow (CA)B = I$

$\Rightarrow B$ is invertible

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$Exercise$

$I - BA$ invertible $\Rightarrow I - AB$ invertible

(Proof)

Suppose $I - AB$ is not invertible

$\exists\ u \neq 0, (I - AB)\ u = 0 \Rightarrow u = ABu \Rightarrow Bu \neq 0$

Consider $(I - BA)Bu$

$= B(I - AB)u = 0$

Let $B\ u = v\ \ (\neq 0)$.

$(I - BA)v = 0$

$\Rightarrow\ \ (I - BA)$ not invertible -- contradiction

※

Inverse of Elementary Matrices

(Chapter 2.3-2.4)

Elementary Row Operation

- Every elementary row operation can be performed by matrix multiplication.
- 1. Interchange

elementary matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- 2. Scaling

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

- 3. Adding *k* times row i to row j:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

Exchange the 2^{nd} and 3^{rd} rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Multiply the 2^{nd} row by -4

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Elementary Matrix

- How to find elementary matrix?
- Apply the desired elementary row operation on Identity matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$E_1 A =$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2 A =$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 A =$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Inverse of Elementary Matrix

Reverse elementary row operation

Exchange the 2nd and 3rd rows

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longleftrightarrow$$

Exchange the 2nd and 3rd rows

$$E_1^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Multiply the 2nd row by -4

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow$$

Multiply the 2nd row by -1/4

$$E_2^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Adding 2 times row 1 to row 3

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Longleftrightarrow$$

Adding -2 times row 1 to row 3

$$E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

RREF vs. Elementary Matrix

- Let A be an mxn matrix with reduced row echelon form R.

$$E_k \cdots E_2 E_1 A = R$$

- There exists an invertible m x m matrix P such that PA=R

$$P = E_k \cdots E_2 E_1$$

$$P^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Invertible

An n x n matrix A is invertible.

⇕

The reduced row echelon form of A is I_n

⇕

A is a product of elementary matrices

R=RREF(A)=I_n

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$= E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Find Inverse of a Matrix

(Chapter 2.3-2.4)

2 X 2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Find e, f, g, h

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, A is not invertible.

Algorithm for Matrix Inversion

- Let A be an n x n matrix. A is invertible if and only if the reduced row echelon form of A is I_n

$$\underbrace{E_k \cdots E_2 E_1}_{A^{-1}} A = R = I_n$$

$$A^{-1} = E_k \cdots E_2 E_1$$

Algorithm for Matrix Inversion

- Let A be an n x n matrix. Transform [A I_n] into its RREF [R B]
    - R is the RREF of A
    - B is a nxn matrix (not RREF)
- If R = I_n, then A is invertible
    - B = A^{-1}

$$E_k \cdots E_2 E_1 [A \quad I_n]$$

$$= [\underbrace{R}_{I_n} \quad \underbrace{E_k \cdots E_2 E_1}_{A^{-1}}]$$

Algorithm for Matrix Inversion

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \xrightarrow[\text{RREF}]{} I_n$$

Invertible

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\boxed{\begin{matrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{matrix}}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

Algorithm for Matrix Inversion

- Let A be an n x n matrix. Transform [A I_n] into its RREF [R B]
  - R is the RREF of A
  - B is a nxn matrix (not RREF)
- If R = I_n, then A is invertible
  - B = A^{-1}
- To find $A^{-1}C$, transform [A C] into its RREF [R C']
  - C' = $A^{-1}C$

$$E_k \cdots E_2 E_1 [A \quad C] = [\underbrace{R}_{I_n} \quad \overbrace{\underbrace{E_k \cdots E_2 E_1}_{A^{-1}} C}^{A^{-1}C}]$$

Linear Transformation

(Chapter 2.6)

Linear Transformation

- A mapping (function) T is called linear if for all "vectors" u, v and scalars c:
- Preserving vector addition:

$$T(u+v) = T(u) + T(v)$$

- Preserving vector multiplication: $T(cu) = cT(u)$

Is matrix transpose linear?

Input: m x n matrices, output: n x m matrices

Linear Transformation

- Derivative: linear?

function f e.g. x^2 → Derivative → function f' e.g. 2x

- Integral from a to b linear?

function f e.g. x^2 → Integral (from a to b) → scalar $\int_a^b f(t)dt$

e.g. $\frac{1}{3}(b^3 - a^3)$

Linear Transformation and Matrix

Let $T : R^n \rightarrow R^m$ be a linear transformation such that

$$\boldsymbol{e_1} = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\cdots$$

$$\boldsymbol{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix whose n columns correspond to $T(e_i)$

$$A = \left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

※

Linear Transformation and Matrix

Proof:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$$\begin{aligned} T \text{ is a L.T.} \Rightarrow T(\boldsymbol{v}) &= T(v_1 e_1 + v_2 e_2 + \cdots + v_n e_n) \\ &= T(v_1 e_1) + T(v_2 e_2) + \cdots + T(v_n e_n) \\ &= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n) \end{aligned}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

※

Linear Transformation and Matrix

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

Linear Transformation and Matrix

$$T(x) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix}$$

$$\mathrm{Ax} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Linear Transformation and Matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m.

- Note:

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \overset{R^n \text{ vector}}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} = \overset{R^m \text{ vector}}{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

※ $$T : R^n \longrightarrow R^m$$

Rotation Matrix

- (Rotation in the plane)

Show that the L.T. $T: R^2 \rightarrow R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

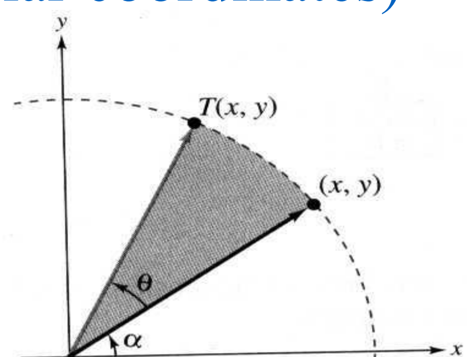
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$v = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

r : the length of v

α : the angle from the positive x-axis counterclockwise to the vector v



Rotation in the Plane

※

Rotation Matrix

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r : the length of $T(\mathbf{v})$

$\theta+\alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ.

※

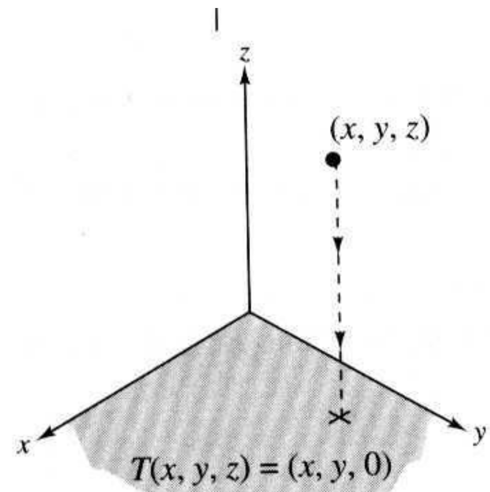
Projection Matrix

- A projection in R^3

The linear transformation $T: R^3 \rightarrow R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3.

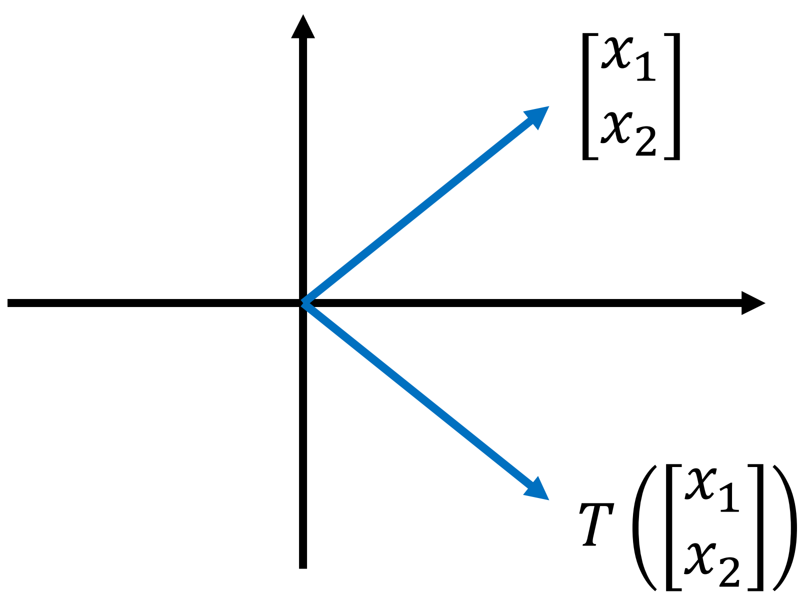


Projection onto xy-plane

Linear transformation and matrix

- Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

special case: $\mathcal{L}$ is the *horizontal axis*



$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =? \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$T(e_1)$
$= e_1$

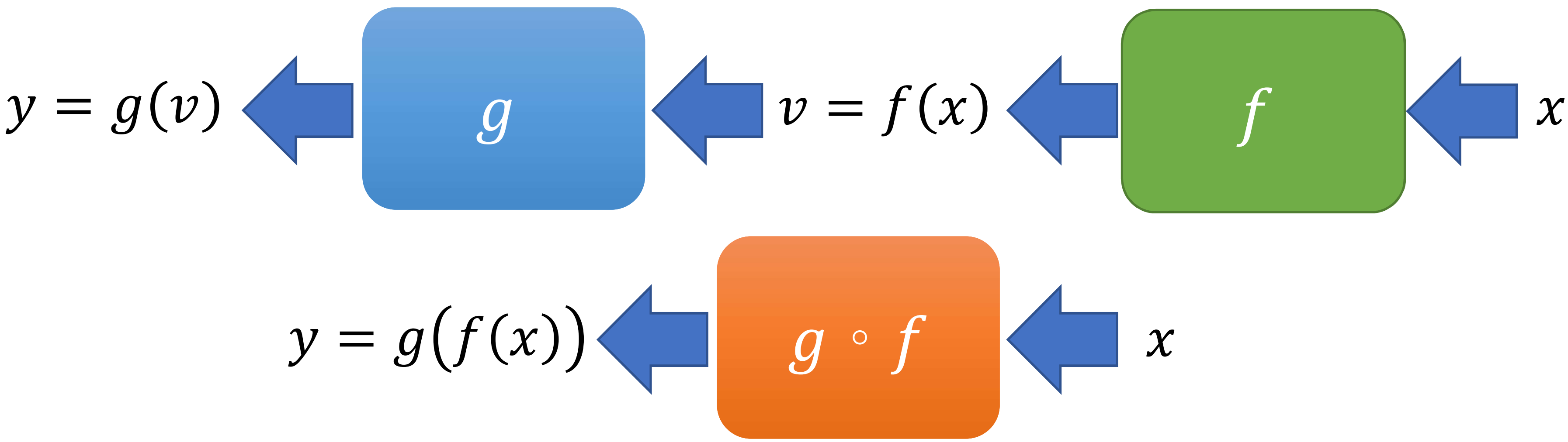
$T(e_2)$
$= -e_2$

Composition of Linear Transformations

(Chapter 2.7)

Matrix Multiplication - Meaning

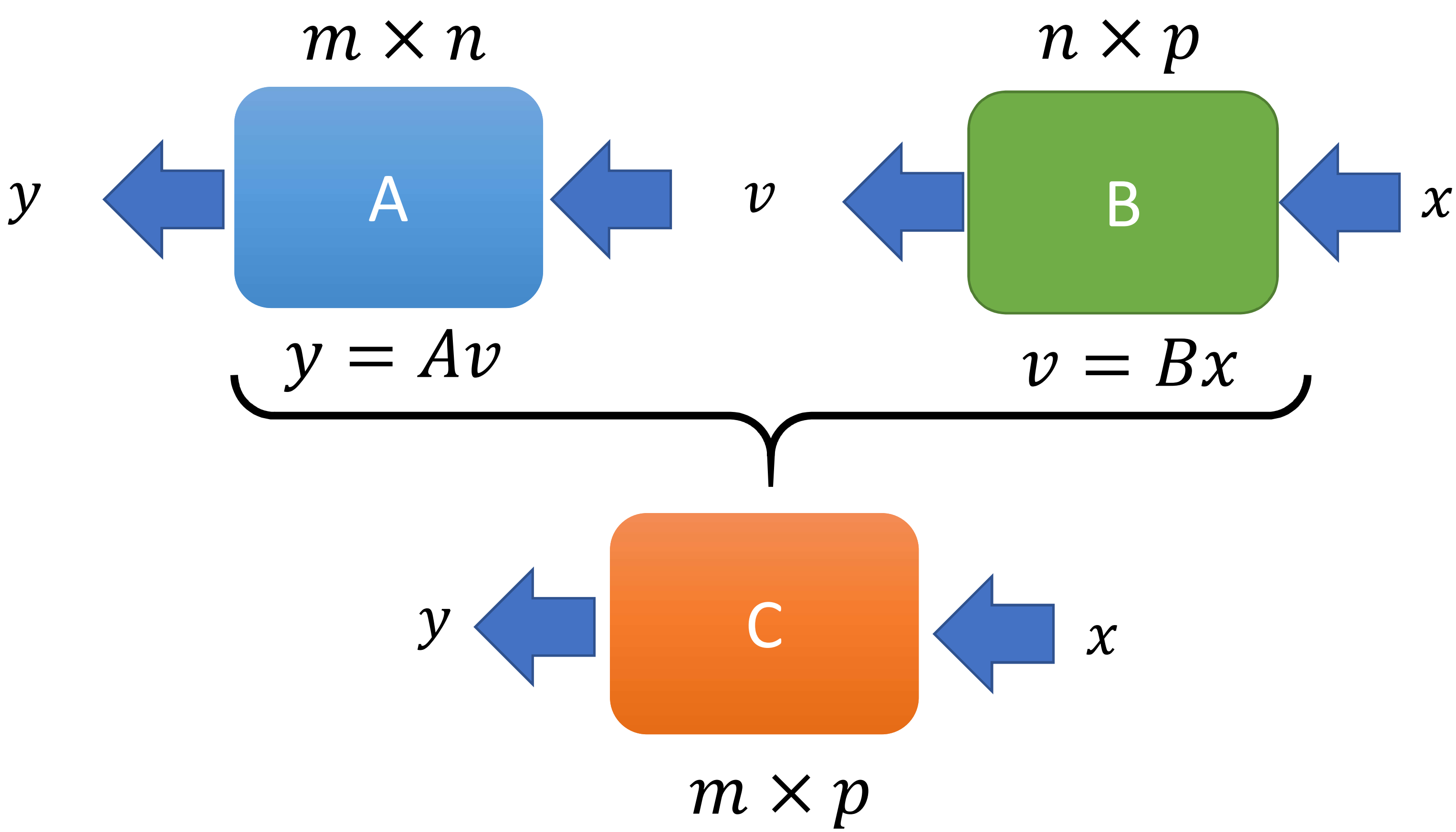
- ***Composition***
  - Given two transformations f and g, the transformation $g(f(.))$ is the composition $g \circ f$.



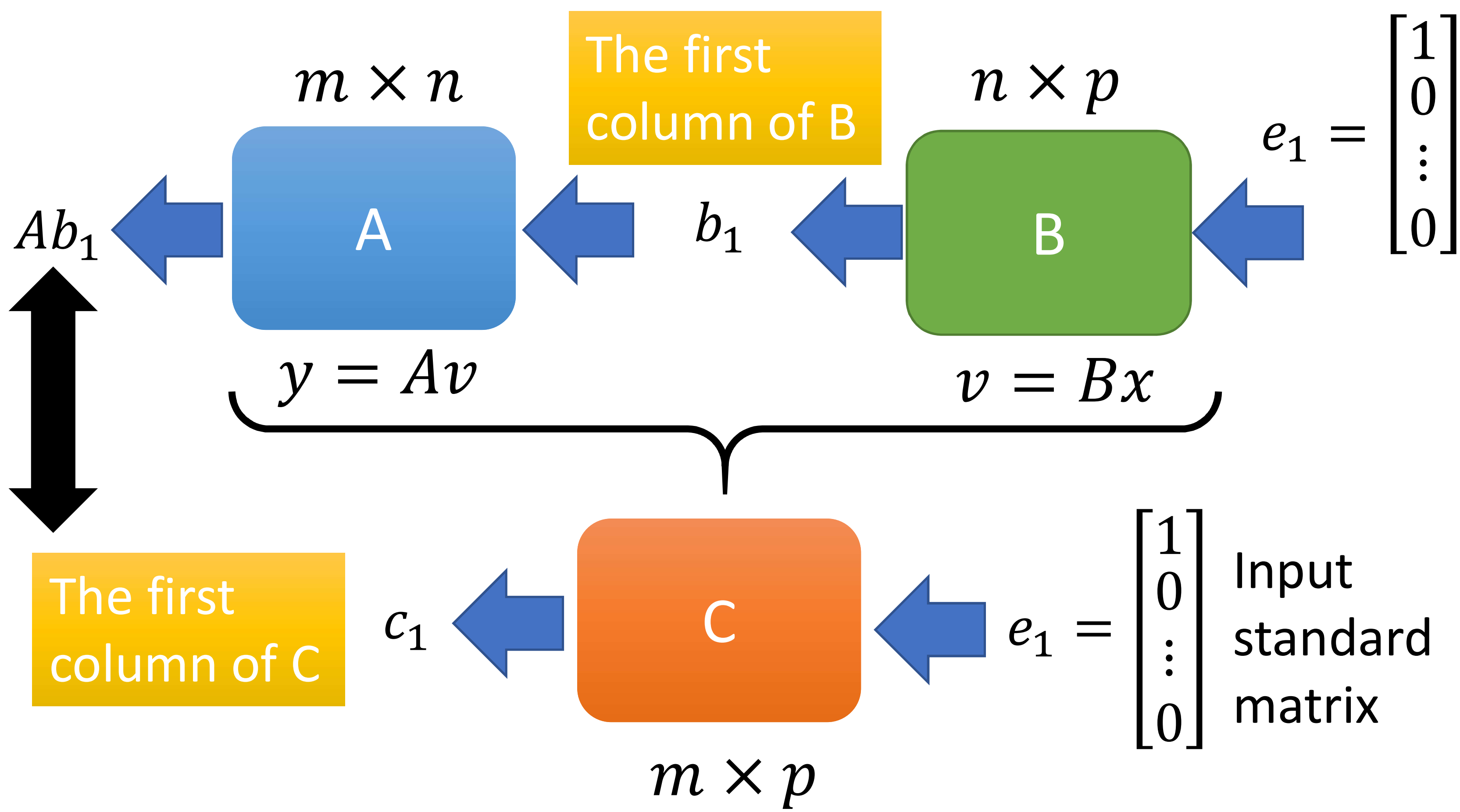
Matrix multiplication is the composition of two linear transformations.

Matrix Multiplication - Composition

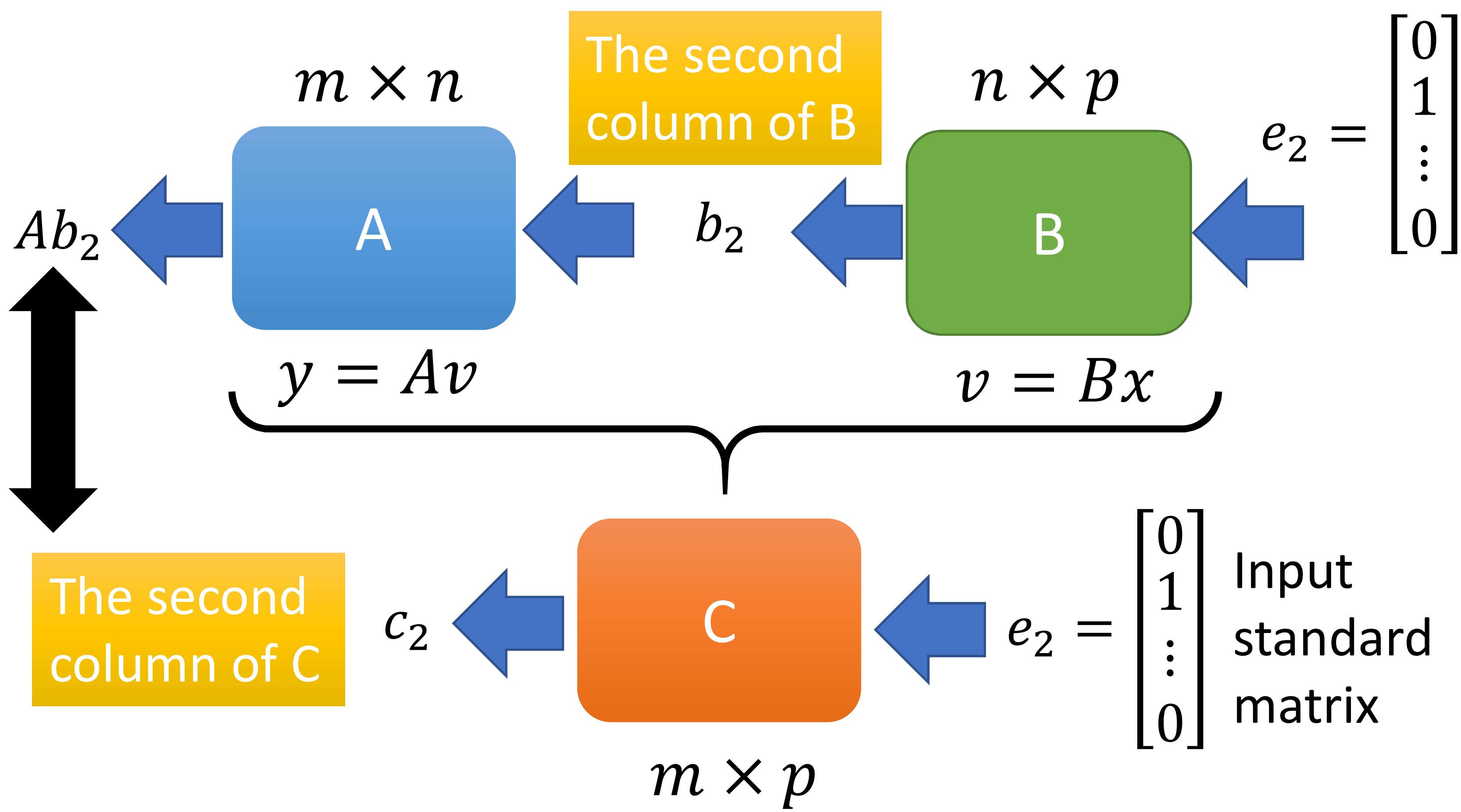
- ***Composition***

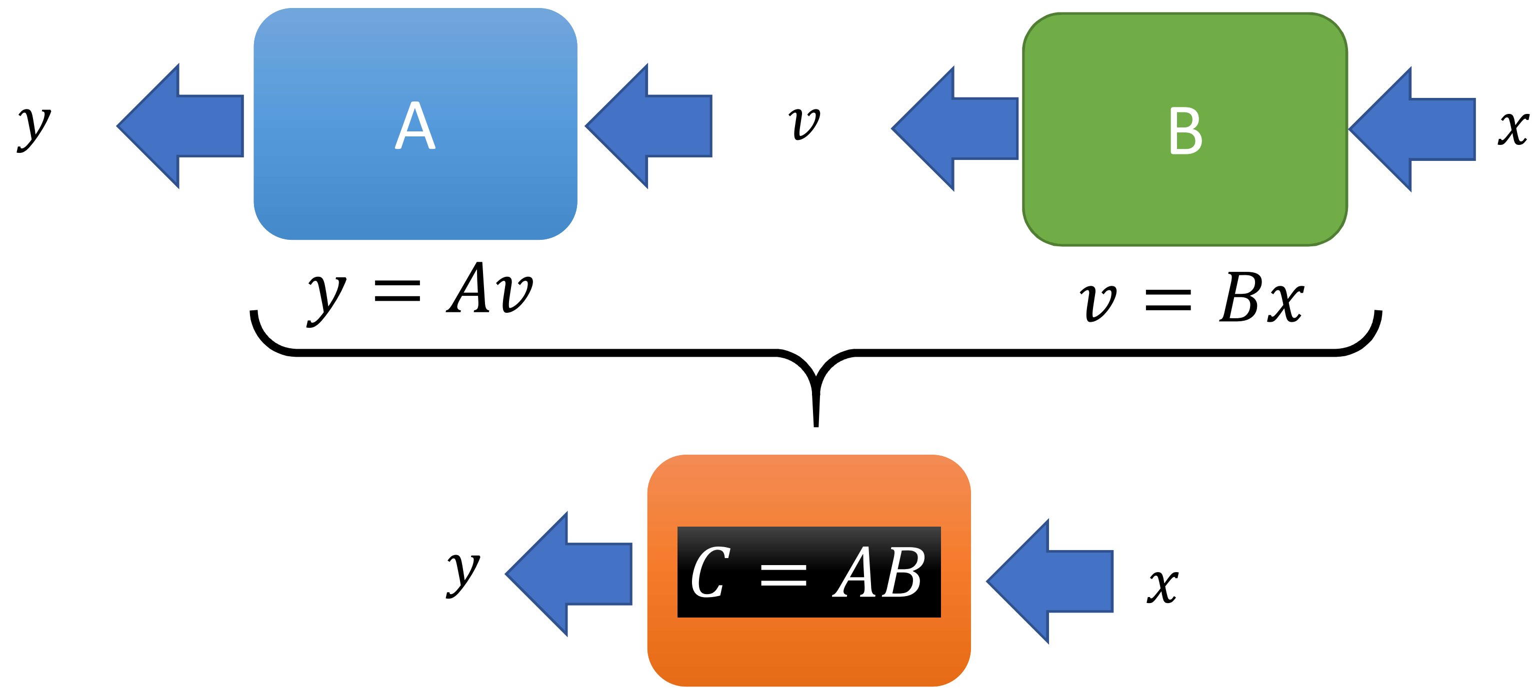


Matrix Multiplication - Meaning



Matrix Multiplication - Meaning



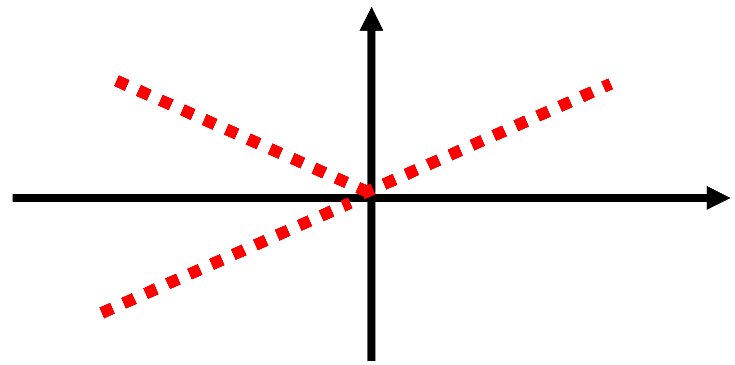


The composition of A and B is

$$C = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p\,]$$

Matrix Multiplication

Example



reflection about the x-axis

rotation by 180°

R^2 $\quad$ R^2 $\quad$ R^2

$$y \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leftarrow v \leftarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \leftarrow x$$

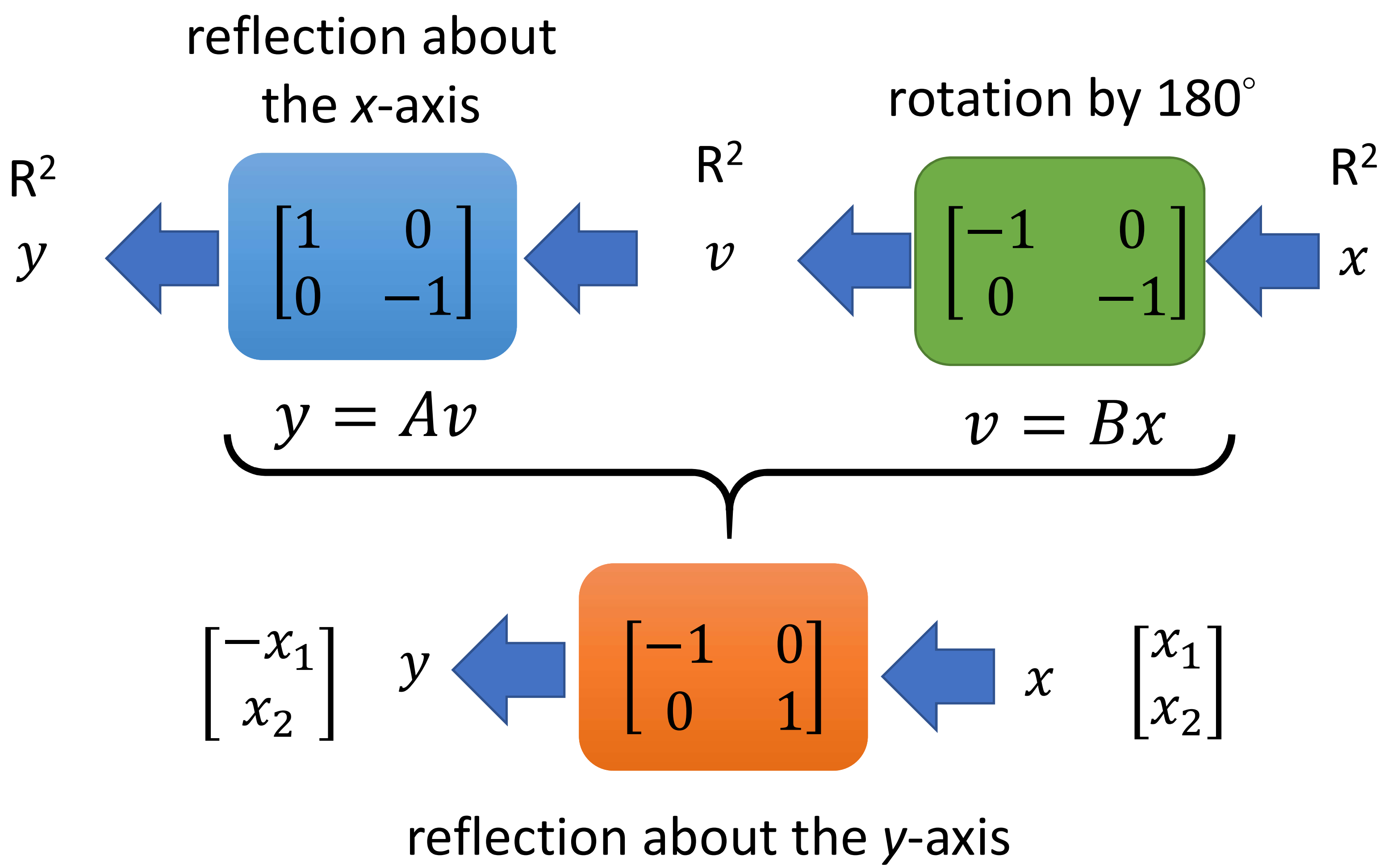
$y = Av$ $\quad$ $v = Bx$

$$y \leftarrow \boxed{} \leftarrow x$$

reflection about the y-axis

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$



LU Decomposition

(Chapter 2.6*)

LU Decomposition

Let *A* be an $m \times m$ nonsingular square matrix. There exist two *L* and *U* such that A=LU, where *L* is a lower triangular matrix and *U* is an upper triangular matrix (assuming no row exchange in doing RREF on A).

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{mm} \end{bmatrix}$$

※

How to do LU decomposition?

$$A = \begin{bmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 2 \end{bmatrix}$$

執行 Elementary Row Operations

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & -12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 2 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & -5 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}}_{E_1} \underbrace{\begin{bmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 2 \end{bmatrix}}_{A}$$

$A \approx \ldots \approx U$ (upper triangular)

$\Rightarrow U = E_k \cdots E_1 A \quad \Rightarrow A = (E_1)^{-1} \cdots (E_k)^{-1} U$

※

How to do LU decomposition?

Compute

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 3 & 1 \end{bmatrix}$$

We have $A = \begin{bmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & -5 \end{bmatrix} = \boldsymbol{LU}$

Use LU decomposition to solve system of linear equations

Based on the above, we have $A = LU$,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

To solve $A\boldsymbol{X} = \boldsymbol{b}$, we first solve $L\boldsymbol{Y} = \boldsymbol{b}$

($AX = LUX = b$; Let $UX=Y$)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1/2 & 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ -17 \end{bmatrix}$$

Then $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -15 \end{bmatrix}$

Now, we solve $U\boldsymbol{X} = \boldsymbol{Y}$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -15 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

LU Decomposition vs. Gaussian Elimination

What is the challenge of solving $A\mathbf{x} = \mathbf{b}$

!!! Huge matrix A !!!

Time complexity for solving systems of linear equations (on n x n matrices)

- Gaussian Elimination: $O(n^3)$ time
- LU Decomposition: $O(n^3)$ time
  - Given L and U, solving $(LU)x = b$: $O(n^2)$ time

$$\begin{array}{cc} & \overbrace{\hphantom{-1\quad -2\quad -1\quad 2\quad 1\quad 2}}^{n} \\ \begin{array}{c} \\ 1 \\ -2 \\ 3 \end{array} & \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array}$$

n^2 time

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

Suppose we need to solve $Ax=b_1, b_2, \ldots b_m$

- Naïve Gaussian Elimination: $O(mn^3)$ time
- LU Decomposition: $O(n^3) + mO(n^2)$ time

- Other matrix decomposition: Cholesky, QR, spectral, singular value

Cholesky Decomposition

A simplified version of LU decomposition for symmetric matrices.

$$A = LL^T,$$

where L is a lower triangular matrix

E.g.

$$\begin{bmatrix} 2 & 4 & -3 \\ 4 & 14 & -9 \\ -3 & -9 & 12 \end{bmatrix} =$$

$$\begin{bmatrix} 2^{1/2} & 0 & 0 \\ 8^{1/2} & 6^{1/2} & 0 \\ -(\frac{9}{2})^{1/2} & -(\frac{3}{2})^{1/2} & 6^{1/2} \end{bmatrix} \begin{bmatrix} 2^{1/2} & 8^{1/2} & -(\frac{9}{2})^{1/2} \\ 0 & 6^{1/2} & -(\frac{3}{2})^{1/2} \\ 0 & 0 & 6^{1/2} \end{bmatrix}$$

Matrix Decomposition A=XYZ

- Decomposing a matrix into the product of a sequence of "nice" matrices (normally 2 or 3) is very useful in Linear Algebra.
- **Analogy**: Given the product of two prime numbers m=p*q, decomposing m into p and q is considered computationally difficult, on which the famous RSA crypto system is based.
- There are several important matrix decomposition approaches in Linear Algebra, including "Singular Value Decomposition" which is behind the success of Google search.

Some Important Matrix Decompositions

Method	Form	Property	Restriction
LU	A=LU	L: lower-triangular U: upper-triangular	A: square matrix; no interchange in RREF
Cholesky	$A=LL^T$	L: lower-triangular	A: symmetric square matrix
Eigenvalue	$A=PDP^{-1}$	P: columns are eigenvectors D: diagonal (eigenvalues)	A is square with complete eigenvectors
Schur	$A=UTU^{-1}$	U: orthonormal T: upper triangular (eigenvalues along diagonal)	A is square, U and T might be complex matrices
QR	A=QR	R: upper triangular Q: orthonormal columns	A has linearly independent column vectors
SVD	$A=USV^T$	U, V: orthogonal S: diagonal	None

More on Matrix Rank

Recall

Column Correspondence Theorem

$$
\begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array}
\qquad\qquad
\begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array}
$$

$$
A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}
\qquad
R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4 \iff \mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$$

※ # of ind. columns of A = # of ind. columns of R

Recall

Row operations preserve "span"

$$-1 \quad \begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{s}_4 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{bmatrix} \begin{array}{l} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{t}_4 \end{array}$$

$$\mathbf{t}_4 = -\mathbf{s}_1 + \mathbf{s}_4$$

$$3\mathbf{s}_1 + 2\mathbf{s}_2 - 5\mathbf{s}_3 + 1\mathbf{s}_4 \qquad = \qquad 3\mathbf{s}_1 + 2\mathbf{s}_2 - 5\mathbf{s}_3 + 1(\mathbf{t}_4 + \mathbf{s}_1)$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# of ind. rows of A = # of ind. rows of R

Review: Rank A

def

- Maximum Number of Independent Columns
- Number of Pivot Columns
- Number of Non-zero rows
- Number of Basic Variables
- Maximum Number of Independent Rows

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

※ # of ind. Rows (Columns) in A = # of ind. Rows (Columns) in R

Properties of Rank

- A is a m x n matrix.

$$Rank\ \mathrm{A}\ \leq \min(\, m\, ,\, n\,)$$

  - A is said to have **full rank** if Rank A = m or Rank A = n.
  - A is said to be **rank deficient** if it does not have full rank.
- Rank A = Rank A^T

Note: Rows of A = Columns of A^T

\# ind. Rows of A = # ind. Columns of A^T

Rank A ... Rank A^T

Properties of Rank

- Let E be an elementary matrix

$$Rank(EA) = \text{Rank}(\ A)$$

(proof) Elementary row operations preserve row independency.

- If A is a m x n matrix, and Q is a m x m invertible matrix. $Rank(QA) = \text{Rank}(A)$

(Invertible matrix is a product of elementary matrices.)

※

Properties of Rank

- (1) If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$$

- (2) If B is a matrix of rank n, then

$$Rank(AB) = Rank(A)$$

- (3) If A is a matrix of rank n, then

$$Rank(AB) = Rank(B)$$

Properties of Rank

(1a)

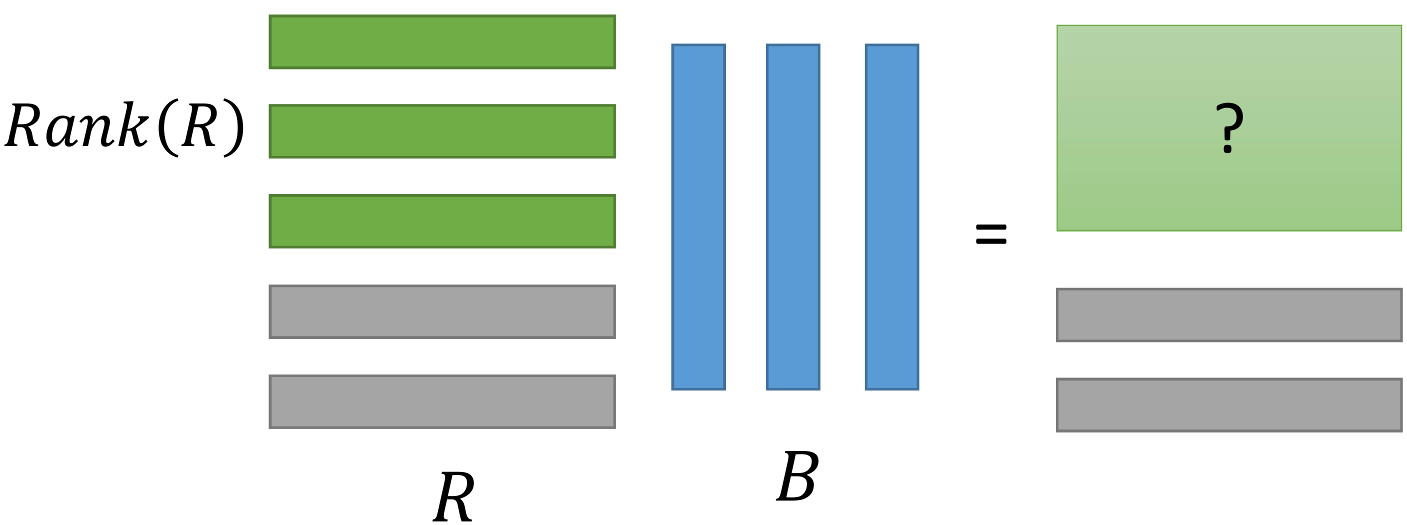
$$Rank(AB) \leq Rank(A)$$

$PA = R$ in RREF

P is an invertible matrix

$$Rank(AB) = Rank(PAB) = Rank(RB)$$

$$Rank(A) = Rank(PA) = Rank(R)$$



It can also have zero rows

Properties of Rank

(1b)

$$Rank(AB) \leq Rank(A) \Rightarrow Rank(AB) \leq Rank(B)$$

(proof)

$$\text{rank}(AB) = \underbrace{\text{rank}(B^T A^T) \leq \text{rank}(B^T)}_{From\ (1a)} = \text{rank}(B)$$

Properties of Rank

Suppose A is a m x n matrix, and B is a n x k matrix.

We know $Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$

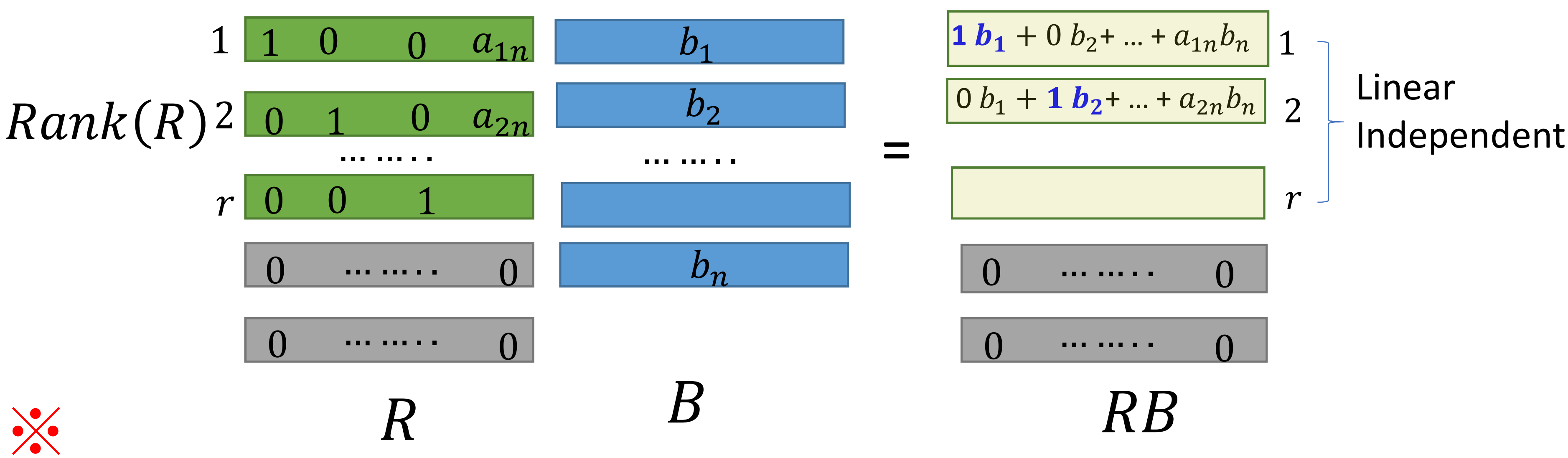
(2) If B is a matrix of rank n, then $Rank(AB) = Rank(A)$

$PA = R$ in RREF

P is an invertible matrix

$Rank(AB) = Rank(PAB) = Rank(RB)$

$Rank(A) = Rank(PA) = Rank(R)$



Properties of Rank

Suppose A is a m x n matrix, and B is a n x k matrix.

We know $Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$

(3) If A is a matrix of rank n, then $Rank(AB) = Rank(B)$

(proof)

$$rank(AB) = \underbrace{rank(B^T \overset{rank\ n}{\downarrow}{A^T}) = rank(B^T)}_{From\ (2)} = rank(B)$$

※

Properties of Rank

Suppose A is a m x n matrix.

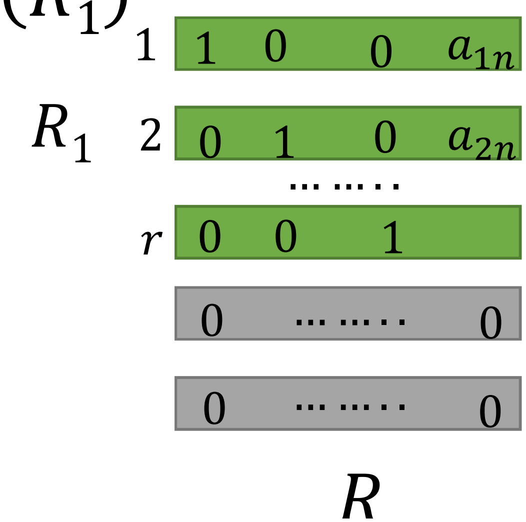
$$Rank(A^{\top}A) = Rank(A)$$

$(Proof)\ PA^{\top} = R,\ \ P$ is an invertible matrix, $R_{n \times n}$ is in RREF. Hence, $Rank(R) = Rank(A)$

$Rank(A^{\top}A) = Rank(P^{-1}RR^{\top}(P^{-1})^{\top}) =$

$Rank(RR^{\top}) = Rank(R_1 R_1{}^{\top}) = Rank(R_1)$

$= Rank(R) = Rank(A)$

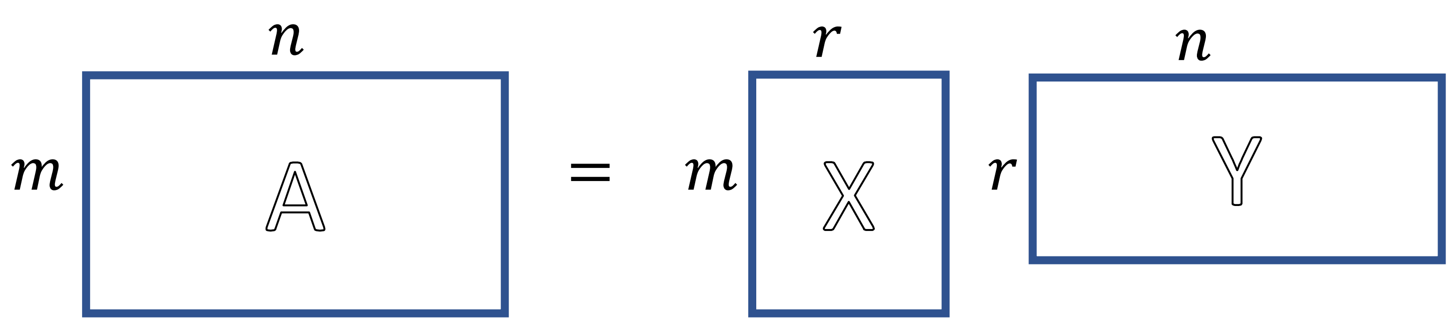


$R^{\top} = [(R_1{}^{\top}); 0]$; R_1 is full rank.

Properties of Rank

Rank A is the minimum r such that

$$A = XY$$



size $= mn$

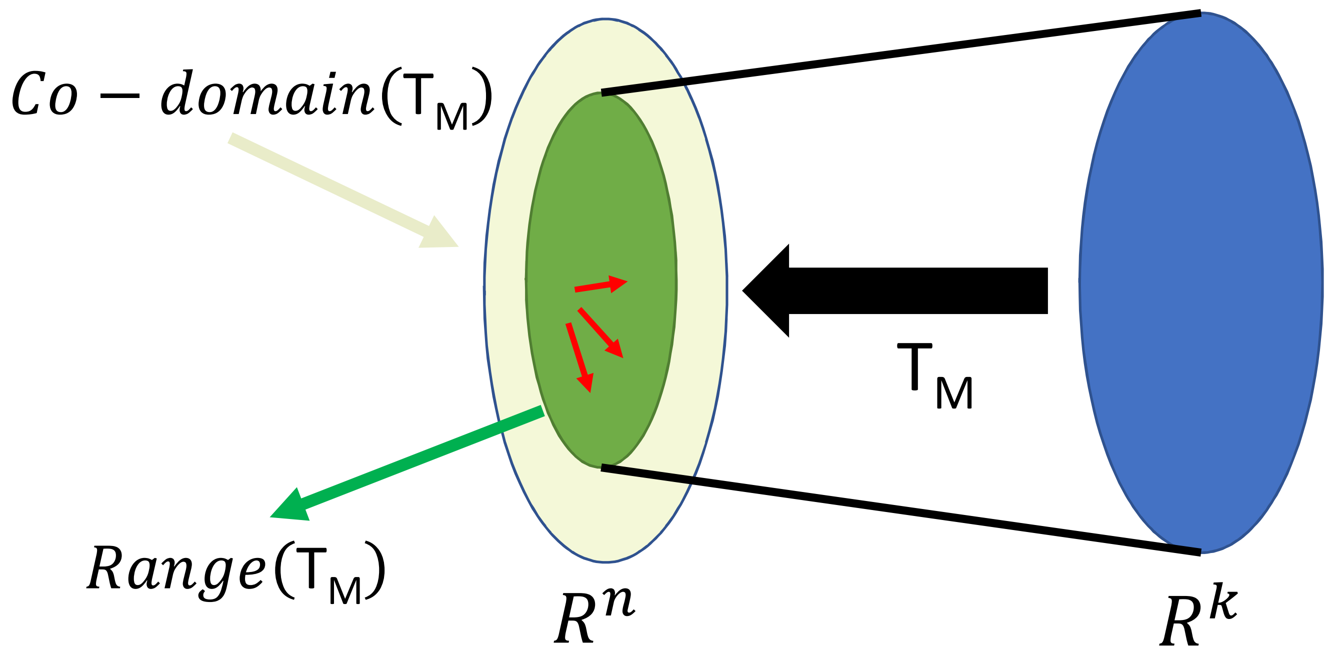
size $= r(m+n)$

For small r, $mn > r(m+n)$

Properties of Rank

Given a n x k matrix M, let T_M be its linear transformation.

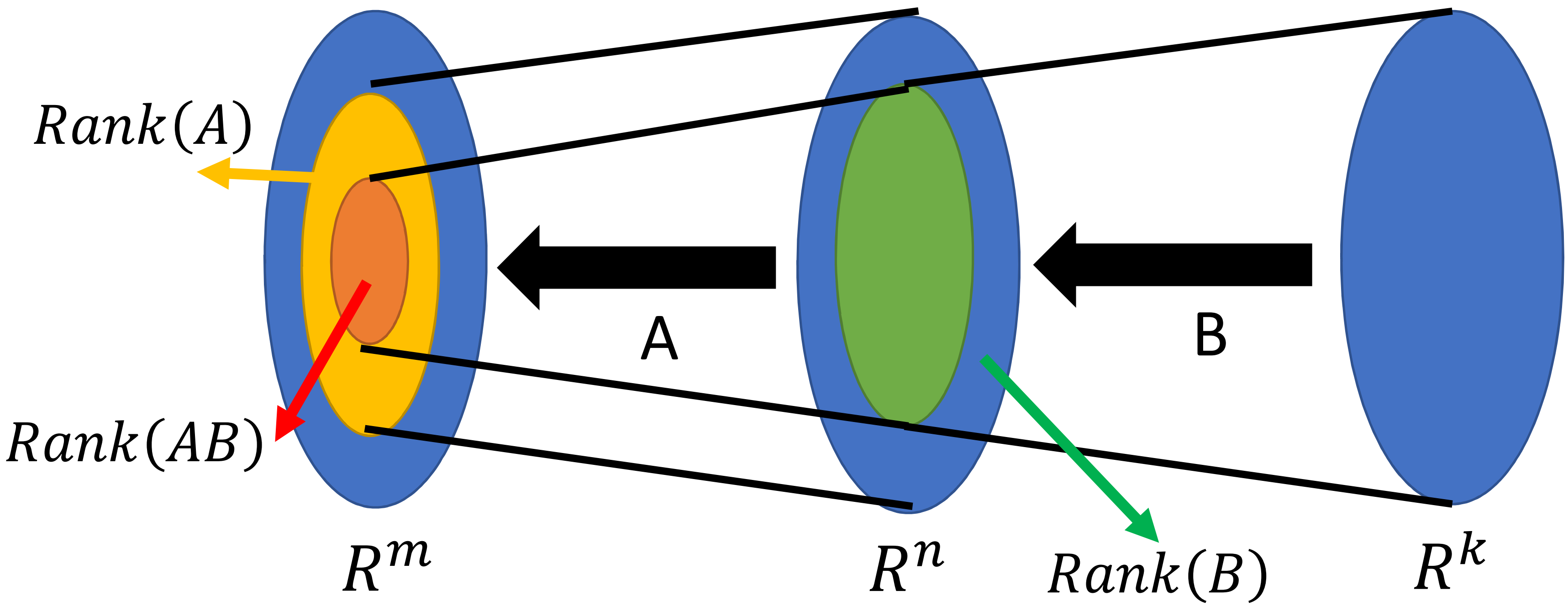
$Rank(\mathrm{M})$ is the maximum number of linear independent vectors in range(T_M).



Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq Rank(A)$$



HW: Proof $Rank(A + B) \leq Rank(A) + Rank(B)$