Chapter 1

Matrices, Vectors, and Systems of Linear Equations

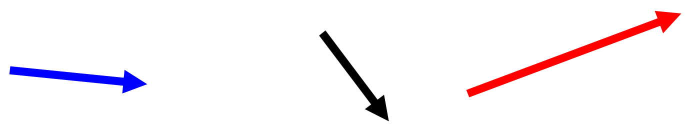
除了標註※之簡報外，其餘採用李宏毅教授之投影片教材

Vectors and Matrices

(Chapter 1.1)

What is a vector?

- Physics student: Vectors have lengths and directions.



- Math student: Vectors satisfy a set of rules:

  $\mathbf{u} + v$, $3u$ are vectors, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,

  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$, …

- EE/CS student: A vector **v** is a sequence of numbers

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

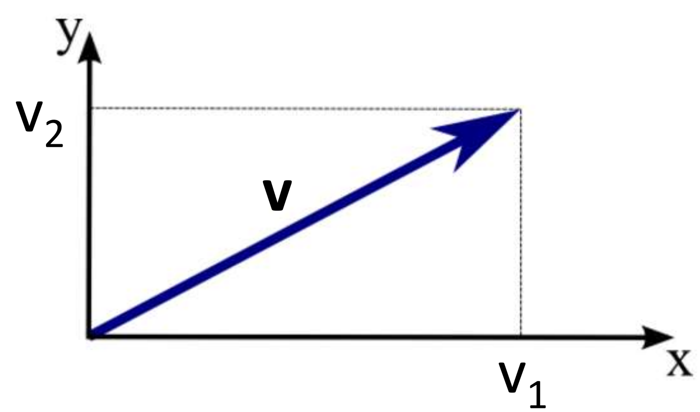
Vectors (from EE/CS viewpoint)

- A vector **v** is a sequence of numbers

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- ***Components***: the entries of a vector.
  - The i-th component of vector **v** refers to v_i
  - $v_1=1$, $v_2=2$, $v_3=3$
- If a vector only has less than four components, you can visualize it.

NOTE: Later we will see a more general definition of a "*vector*"

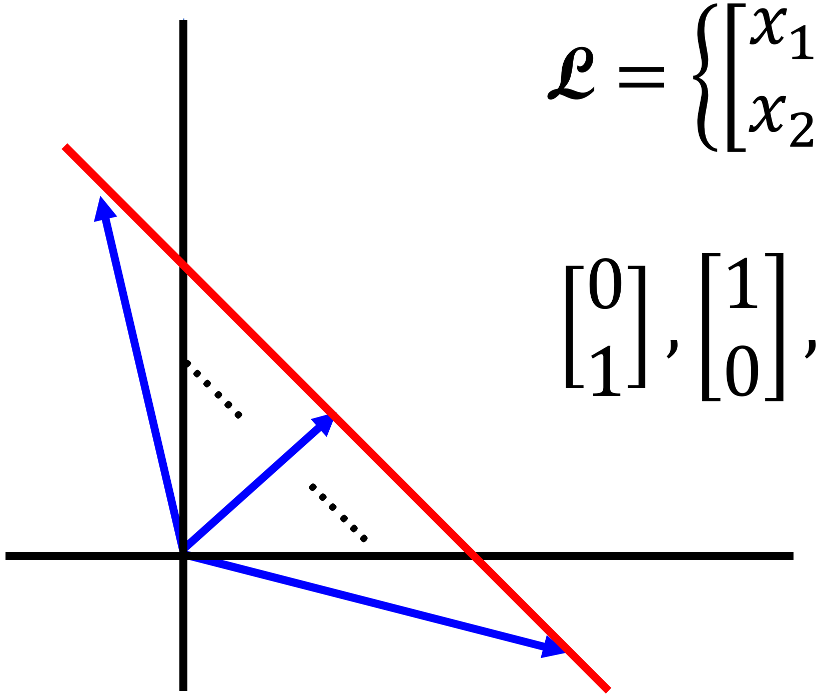


Vector Set

$$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}6\\8\\9\end{bmatrix},\begin{bmatrix}9\\0\\2\end{bmatrix}\right\}$$

A vector set with 4 elements

- A vector set can contain infinite elements

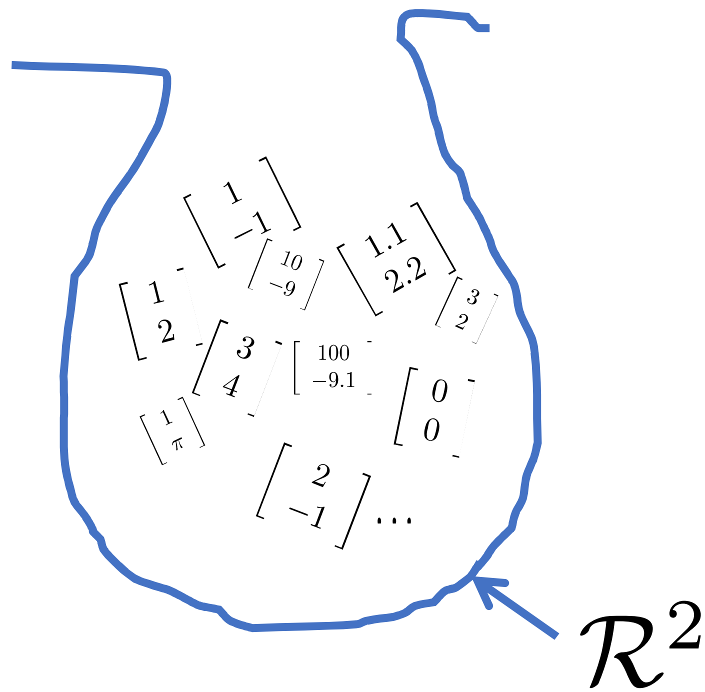


$$\mathcal{L}=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix}: x_1+x_2=1\right\}$$

$$\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0.3\\0.7\end{bmatrix},\begin{bmatrix}0.7\\0.3\end{bmatrix}\cdots\cdots$$

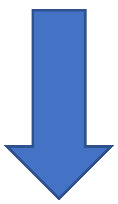
Vector Set

- $\mathcal{R}^n$:We denote the set of all **vectors** with *n* entries by $\mathcal{R}^n$

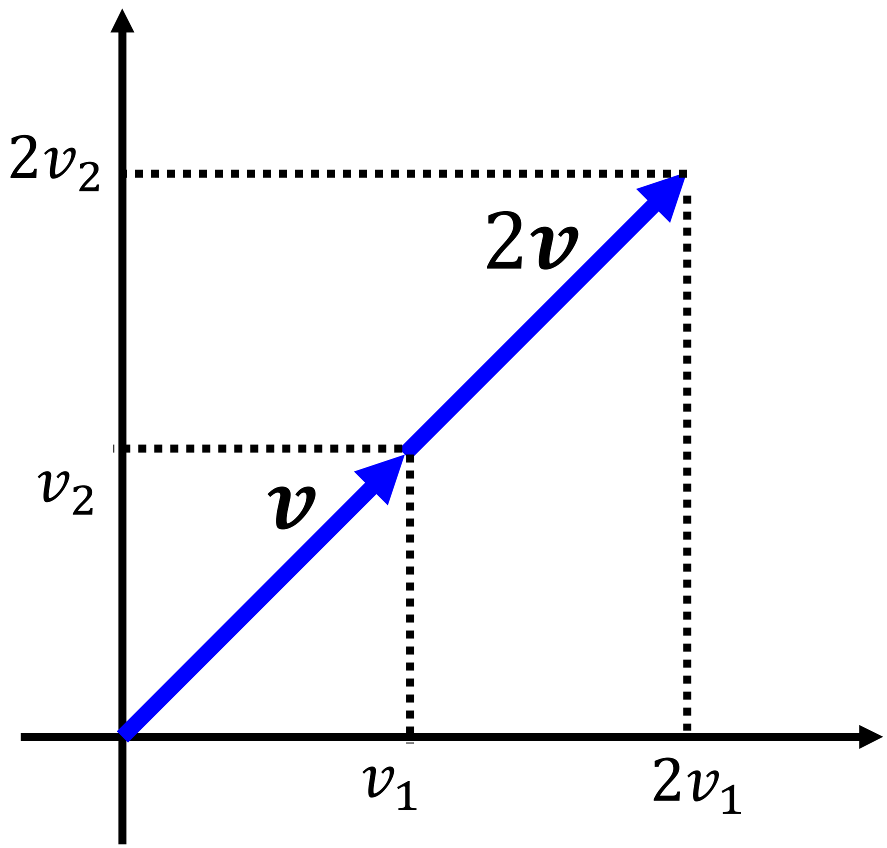


Scalar Multiplication

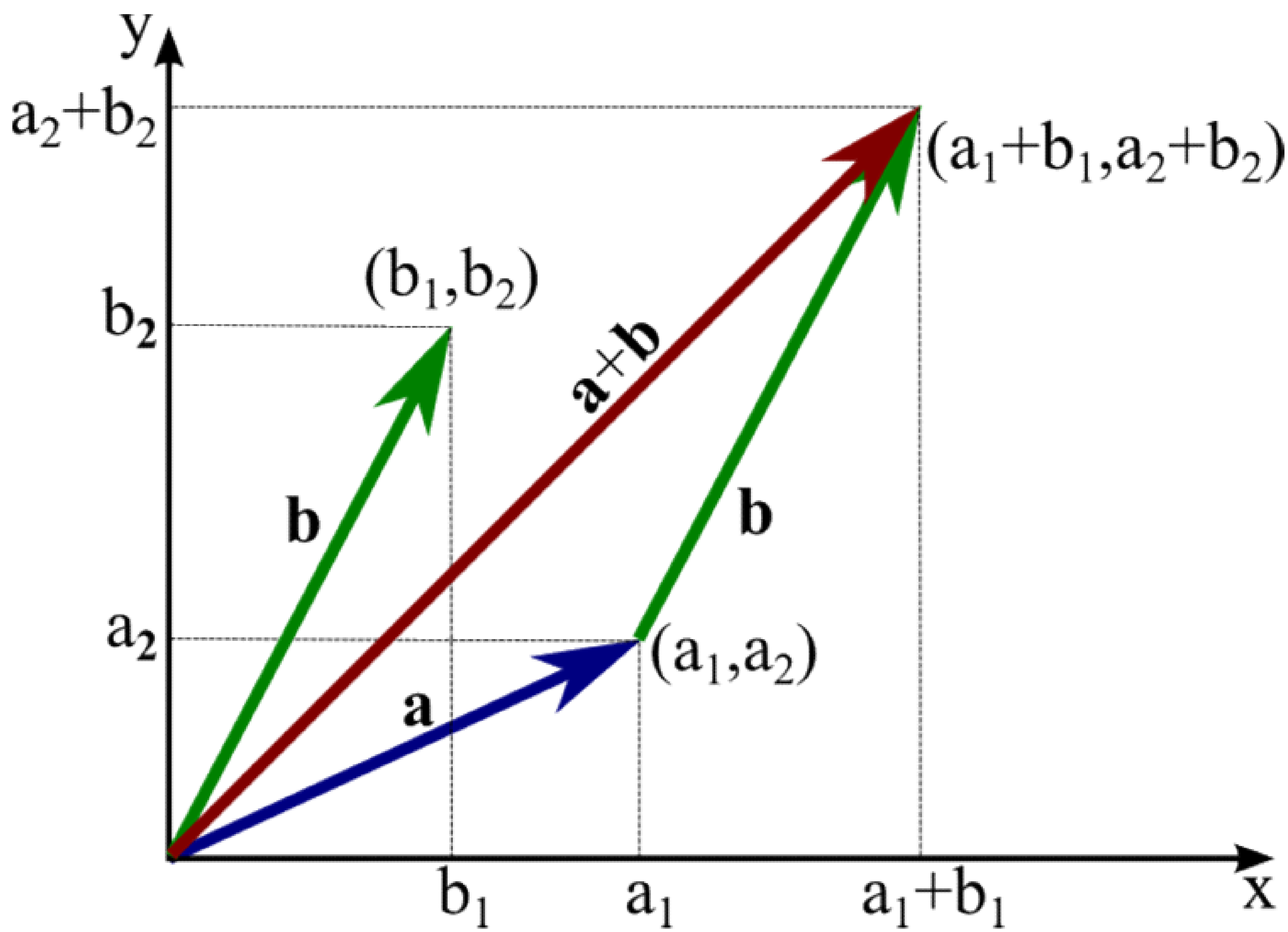
$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



$$c\boldsymbol{v}$$



Vector Addition



Properties of Vector

Objects having the following 8 properties are "vectors".

For any vectors **u**, **v** and **w** in $\mathcal{R}^n$, and any scalars a and b

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- There is an element $\mathbf{0}$ in $\mathcal{R}^n$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$
- There is an element $\mathbf{u}'$ in $\mathcal{R}^n$ such that $\boxed{\mathbf{u}'} + \mathbf{u} = \mathbf{0}$
  $= -\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$
- $(ab)\mathbf{u} = a(b\mathbf{u})$
- $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
- $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$ zero vector

In Chapter 7, the above will be generalized to "vector space"

Matrix

$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 3 X 1 $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 1 X 3

- If the matrix has m rows and n columns, we say the size of the matrix is m by n, written m x n
- We use $\mathcal{M}_{mxn}$ to denote the set that contains all matrices of size m x n

$$\begin{array}{cc} & \text{3 columns} \\ \text{2 rows} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array} \in \mathcal{M}_{2x3}$$

$\boxed{2 \text{ X } 3}$

$$\begin{array}{cc} & \text{2 columns} \\ \text{3 rows} & \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \end{array} \in \mathcal{M}_{3x2}$$

$\boxed{3 \text{ X } 2}$

先 Row 再 Column

Matrix

先 Row 再 Column

- ***Index of component***: the scalar in the i-th row and j-th column is called (i,j)-entry of the matrix

(1,2)-entry

$$A = \begin{bmatrix} 2 & \boxed{3} & 5 \\ 3 & 1 & -1 \\ \boxed{-2} & 1 & \boxed{1} \end{bmatrix}$$

(3,1)-entry

(3,3)-entry

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \dots \quad \boldsymbol{a_n}]$$

vectors

Matrix

- Two matrices with the same size can add or subtract.
- Matrix can be multiplied by a scalar

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 9 \\ 8 & 0 \\ 9 & 2 \end{bmatrix} \quad 9B$$

$$A + B \qquad A - B$$

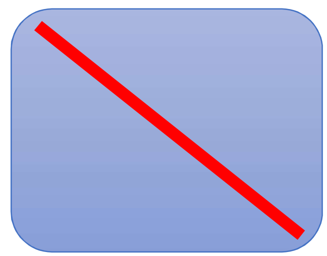
Properties

- A, B, C are mxn matrices, and s and t are scalars
  - $A + B = B + A$
  - $(A + B) + C = A + (B + C)$
  - $(st)A = s(tA)$
  - $s(A + B) = sA + sB$
  - $(s+t)A = sA + tA$

有名有姓的 Matrix

Square matrix

n

m  $m = n$

diagonal

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Triangular Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

Lower Triangular Matrix

有名有姓的 Matrix

Is I_3 a diagonal matrix?

YES

Is $O_{3\times3}$ a diagonal matrix?

YES

- Diagonal Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

All non-diagonal elements are "0".

- Identity Matrix

$$\mathrm{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Denoted by I (any size) or I_n

- Zero Matrix

$$O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Denoted by O (any size) or $O_{m\times n}$

Transpose

- If A is an mxn matrix, A^T (transpose of A) is an nxm matrix whose (i,j)-entry is the (j-i)-entry of A

(1,2)

$$A = \begin{bmatrix} 6 & \boxed{9} \\ 8 & 0 \\ 9 & \boxed{2} \end{bmatrix} \xrightarrow{\text{Transpose}} A^T = \begin{bmatrix} 6 & 8 & 9 \\ \boxed{9} & 0 & \boxed{2} \end{bmatrix}$$

(3,2) (2,1) (2,3)

Column 變成 Row ; Row 變成 Column

Why do we care about the transpose of a matrix?

Will explain later!

Transpose

- A and B are mxn matrices, and s is a scalar
- $(A^T)^T = A$
- $(sA)^T = sA^T$
- $(A + B)^T = A^T + B^T$

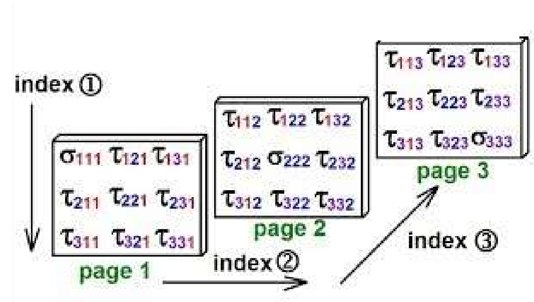
This is a linear system ☺

Symmetric Matrix $\quad A^T = A$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = A^T \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq B^T$$

Vectors and matrices are special cases of tensors?

$$S \qquad \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 4\\ 2 & 3 & -1\\ 4 & -1 & 5\end{bmatrix}$$



Scalar (Rank 0) Vector (Rank 1) Matrix (Rank 2) Rank 3 tensor

The above is an over-simplified definition of a tensor.

So what is a tensor?

- Informal Definition: An object that is invariant under a change of coordinates, and has components that change in a special, predictable way under a change of coordinates.

Matrix-Vector Product

(Chapter 1.2)

Matrix-Vector Product

$$A = \begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \boxed{a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}} \end{bmatrix}_{m \text{ X } n} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \text{ X } 1}$$

$$A\boldsymbol{x} = \begin{bmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} \\ \boxed{a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n} \\ \vdots \\ \boxed{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n} \end{bmatrix}_{m \text{ X } 1}$$

Dot Product with Row

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A\mathbf{x} = \begin{bmatrix} \quad \end{bmatrix}$$

Matrix-Vector Product

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

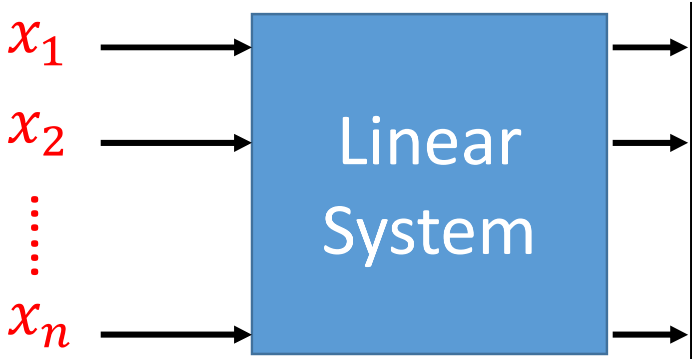
$$A\boldsymbol{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Dot Product with Row

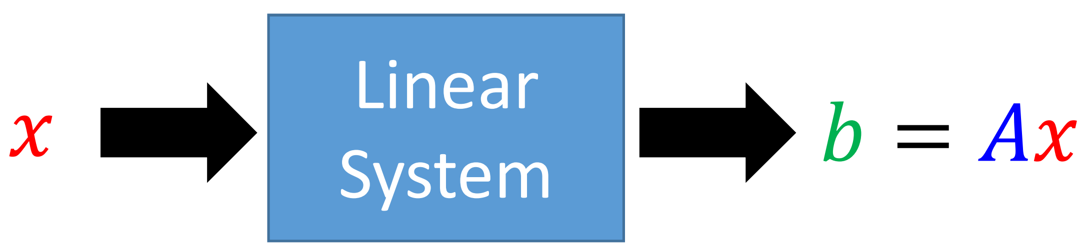
$$=$$

Weighted sum of Columns

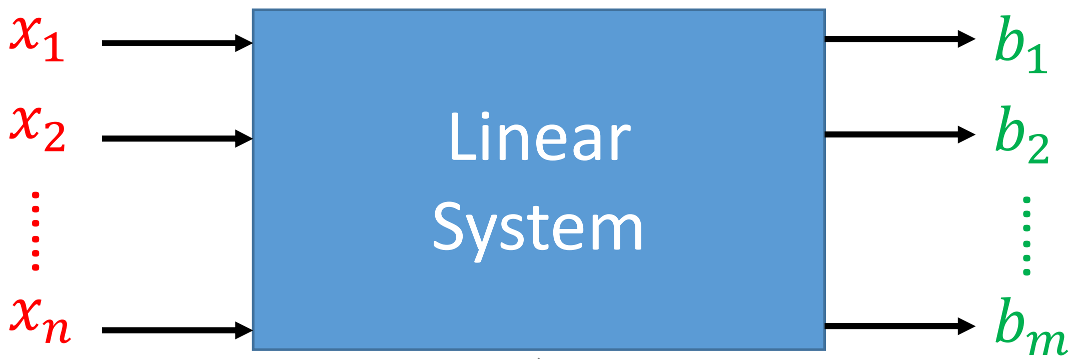
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$$\underbrace{\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{matrix}}_{Ax} \begin{matrix} = b_1 \\ = b_2 \\ \\ = b_m \end{matrix}$$



The matrix A represents the system.



$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

Properties of Matrix-Vector Product

(Chapter 1.2)

Matrix-vector Product

- The sizes of matrix and vector should match.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \mathrm{A}'' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$$

Properties of Matrix-vector Product

- A and B are mxn matrices, **u** and **v** are vectors in $\mathcal{R}^n$, and c is a scalar.
- $A(\boldsymbol{u}+\boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$
- $A(c\boldsymbol{u}) = c(A\boldsymbol{u}) = (cA)\boldsymbol{u}$
- $(A+B)\boldsymbol{u} = A\boldsymbol{u} + B\boldsymbol{u}$
- $A\mathbf{0}$ is the mx1 zero vector
- $\boldsymbol{O}\boldsymbol{v}$ is also the mx1 zero vector
- $I_n\boldsymbol{v} = \boldsymbol{v}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Properties of Matrix-vector Product

- A and B are mxn matrices. If $A\boldsymbol{w} = B\boldsymbol{w}$ for all $\boldsymbol{w}$ in $\mathcal{R}^n$. Is it true that $A = B$?

$A\boldsymbol{e_j} = \boldsymbol{a_j}$, where e_j is the j-th standard vector in $\mathcal{R}^n$

Column Aspect

$$\underset{j}{\boldsymbol{e_1}} = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \quad A\boldsymbol{e_1} = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_n}] \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = 1 \cdot \boldsymbol{a_1} + 0 \cdot \boldsymbol{a_2} + \cdots + 0 \cdot \boldsymbol{a_n}$$

$$= \underset{j}{\boldsymbol{a_1}}$$

$$\boxed{A\boldsymbol{e_1} = B\boldsymbol{e_1} \quad A\boldsymbol{e_2} = B\boldsymbol{e_2} \quad \cdots \quad A\boldsymbol{e_n} = B\boldsymbol{e_n}}$$

$$\Downarrow \qquad \Downarrow \qquad \cdots\cdots \qquad \Downarrow$$

$$\boldsymbol{a_1} = \boldsymbol{b_1} \quad \boldsymbol{a_2} = \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{a_n} = \boldsymbol{b_n} \quad \Rightarrow A = B$$

Linear Combination

(Chapter 1.2)

Linear Combination

- Given a vector set $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$
- The linear combination of the vectors in the set
  - $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$
  - $c_1, c_2, \cdots, c_k$ are scalars (coefficients of linear combination)

vector set: $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

coefficients: $\{-3,4,1\}$

$$-3\begin{bmatrix}1\\1\end{bmatrix} + 4\begin{bmatrix}1\\3\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}2\\8\end{bmatrix}$$

其實就是 weighted sum 啦 ☺

Column Aspect

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & & & \vdots & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \\
\boldsymbol{a_1} & & \boldsymbol{a_2} & & & & \boldsymbol{a_n} & &
\end{array}
$$

$$A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \cdots \quad \boldsymbol{a_n}]$$

Vector set

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$A\mathbf{x} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$$

Linear Combination

System of Linear Equations vs. Linear Combination

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

Non empty solution set?

Has solution or not?

Consistent?

The Same question

Column Aspect

$$A\mathbf{x} = b$$
$$= x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n}$$

linear combination of columns of A

Is $\boldsymbol{b}$ the linear combination of columns of A?

Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Is b the linear combination of columns of A?

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$$

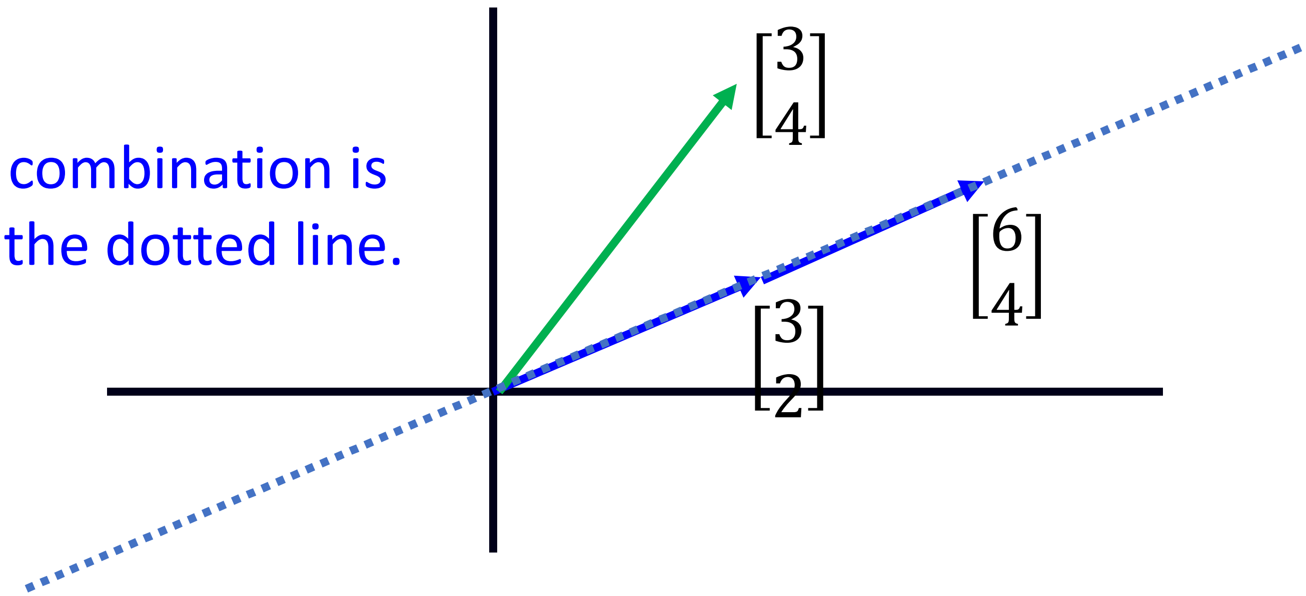
Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

- Vector set: $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$
- Is $\begin{bmatrix}3\\4\end{bmatrix}$ a linear combination of $\left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}\right\}$? **NO**

The linear combination is always on the dotted line.



Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Is b a linear combination of columns of A?

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

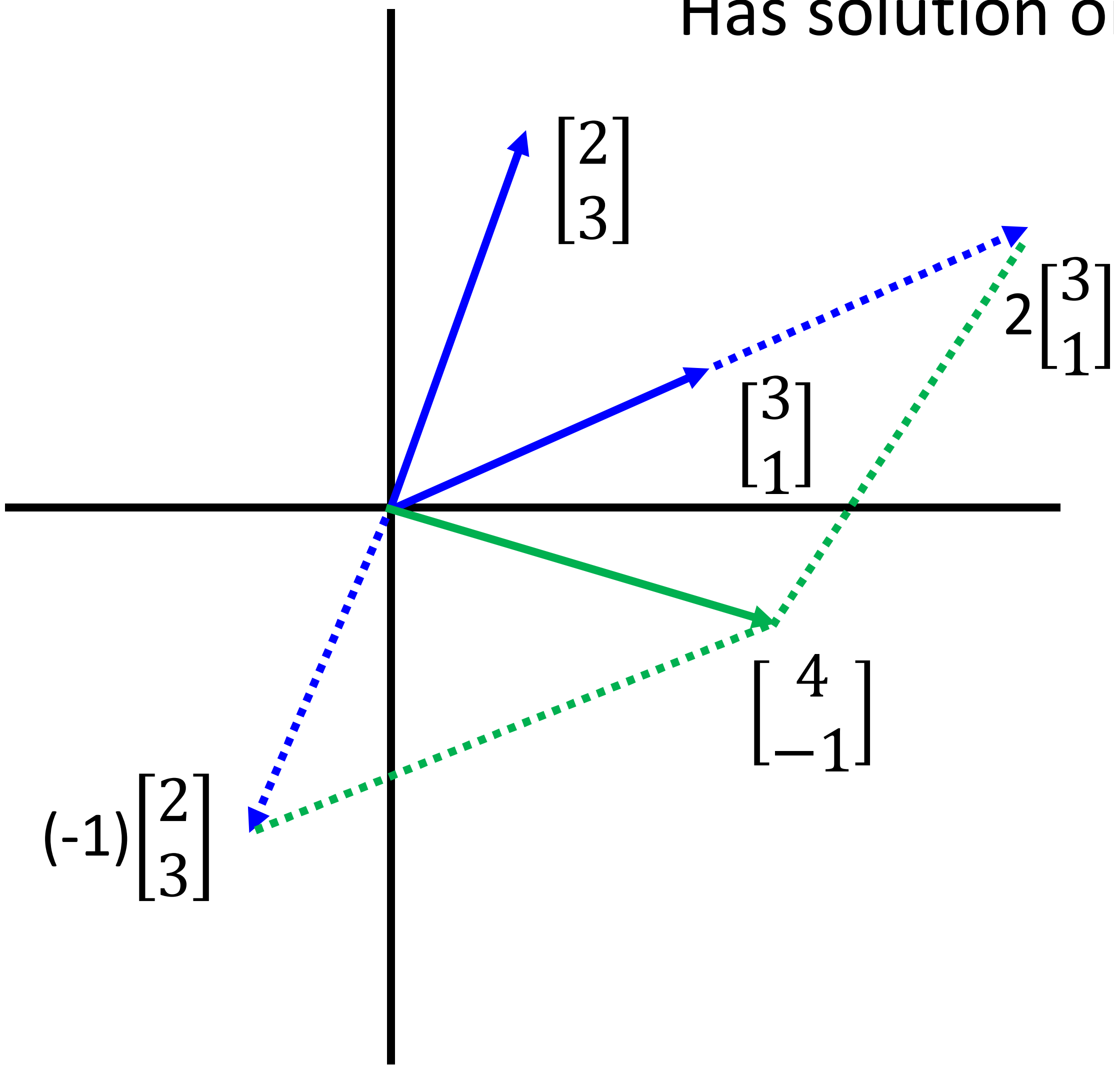
Example 2

$$\left\{\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}$$

$$\begin{bmatrix}4\\-1\end{bmatrix}$$

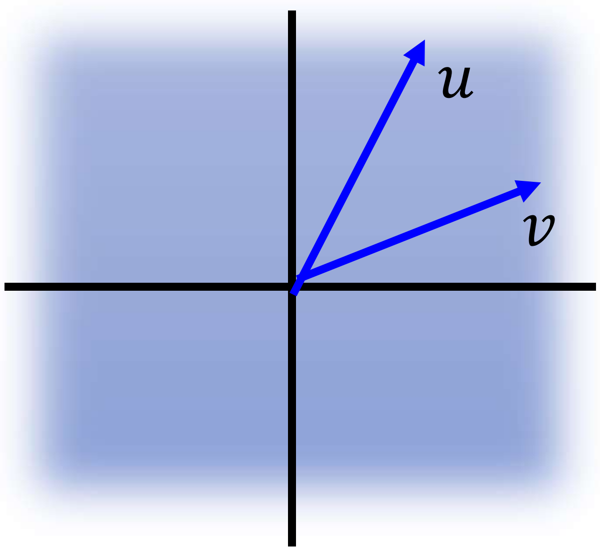
$$\boxed{\begin{aligned}2x_1 + 3x_2 &= 4\\ 3x_1 + 1x_2 &= -1\end{aligned}}$$

Has solution or not?



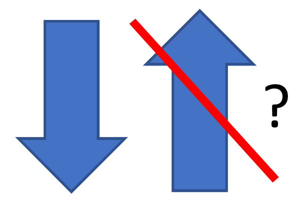
Example 2

- If **u** and **v** are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of **u** and **v**
  - Nonparallel: **u** and **v** are nonzero vectors, and **u** $\neq c$**v**.



$$\begin{aligned} u_1 x_1 + v_1 x_2 &= b_1 \\ u_2 x_1 + v_2 x_2 &= b_2 \end{aligned}$$

u and **v** are not parallel



Has solution

- If **u, v** and **w** are any nonparallel vectors in $\mathcal{R}^3$, then every vector in $\mathcal{R}^3$ is a linear combination of **u, v** and **w**? **NO**

Example 3

$$\begin{aligned} 2x_1 + 6x_2 &= -4 \\ 1x_1 + 3x_2 &= -2 \end{aligned}$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Is b the linear combination of columns of A?

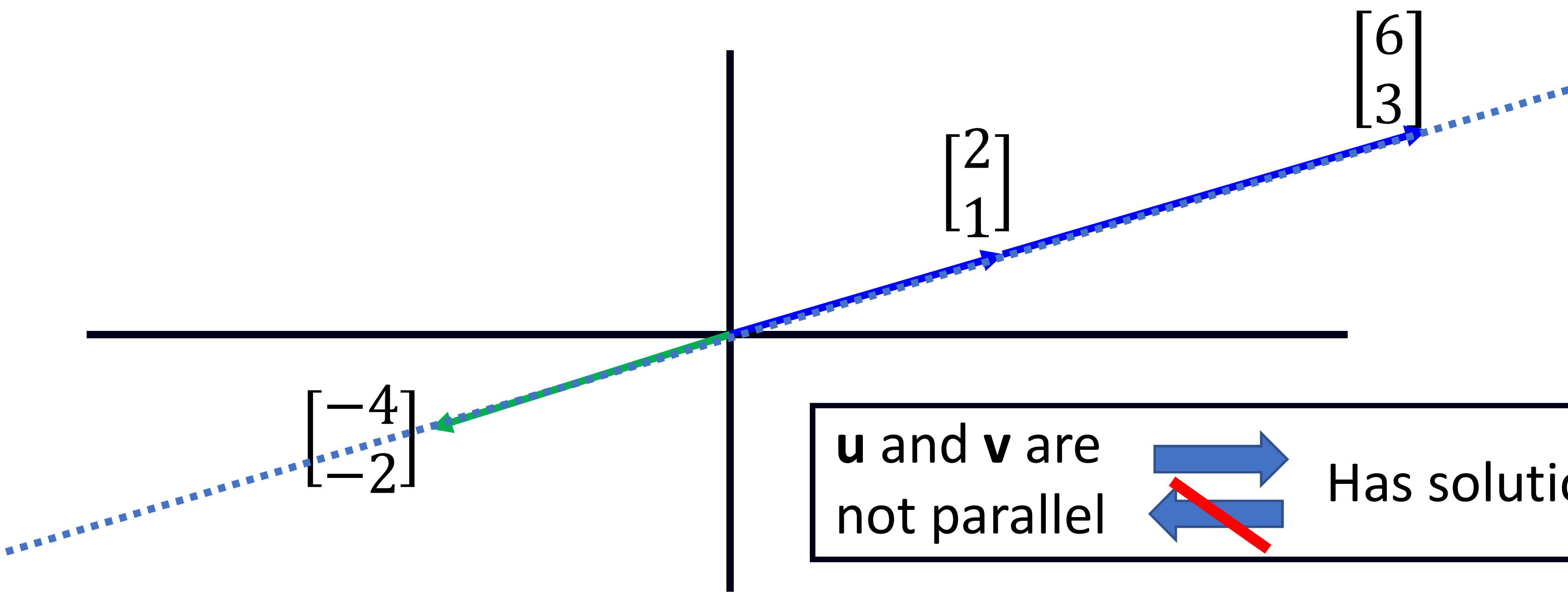
$$\begin{bmatrix} -4 \\ -2 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$$

Example 3

$$\begin{aligned} 2x_1 + 6x_2 &= -4 \\ 1x_1 + 3x_2 &= -2 \end{aligned}$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$
- Is $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$? Yes

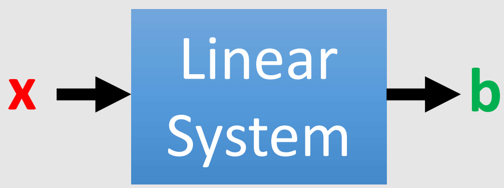


u and **v** are not parallel $\Rightarrow$ / $\cancel{\Leftarrow}$ Has solution

Having Solutions or Not

(Chapter 1.3)

Review



$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

System of Linear Equations

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

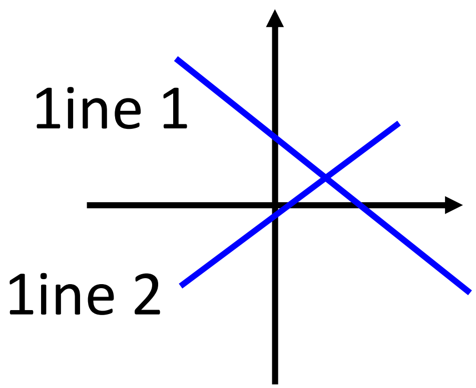
Given A and **b**, let's find **x**

$Zero, One, Infinity$ …

More Variables?

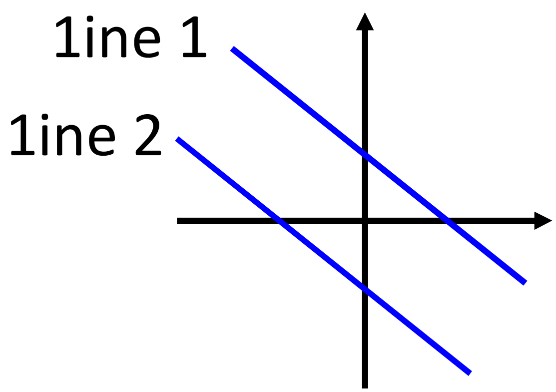
- Considering any system of linear equations with 2 variables and 2 equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{...... line 1}$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{...... line 2}$$



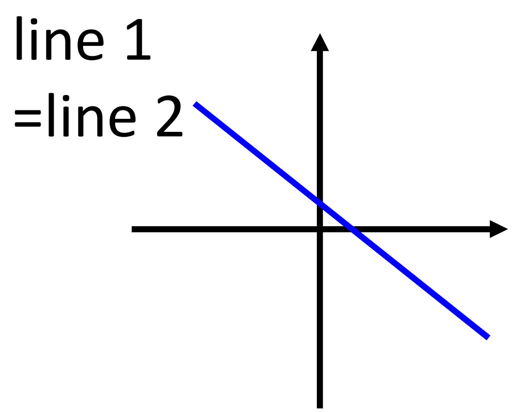
unique solution

consistent



no solution

inconsistent



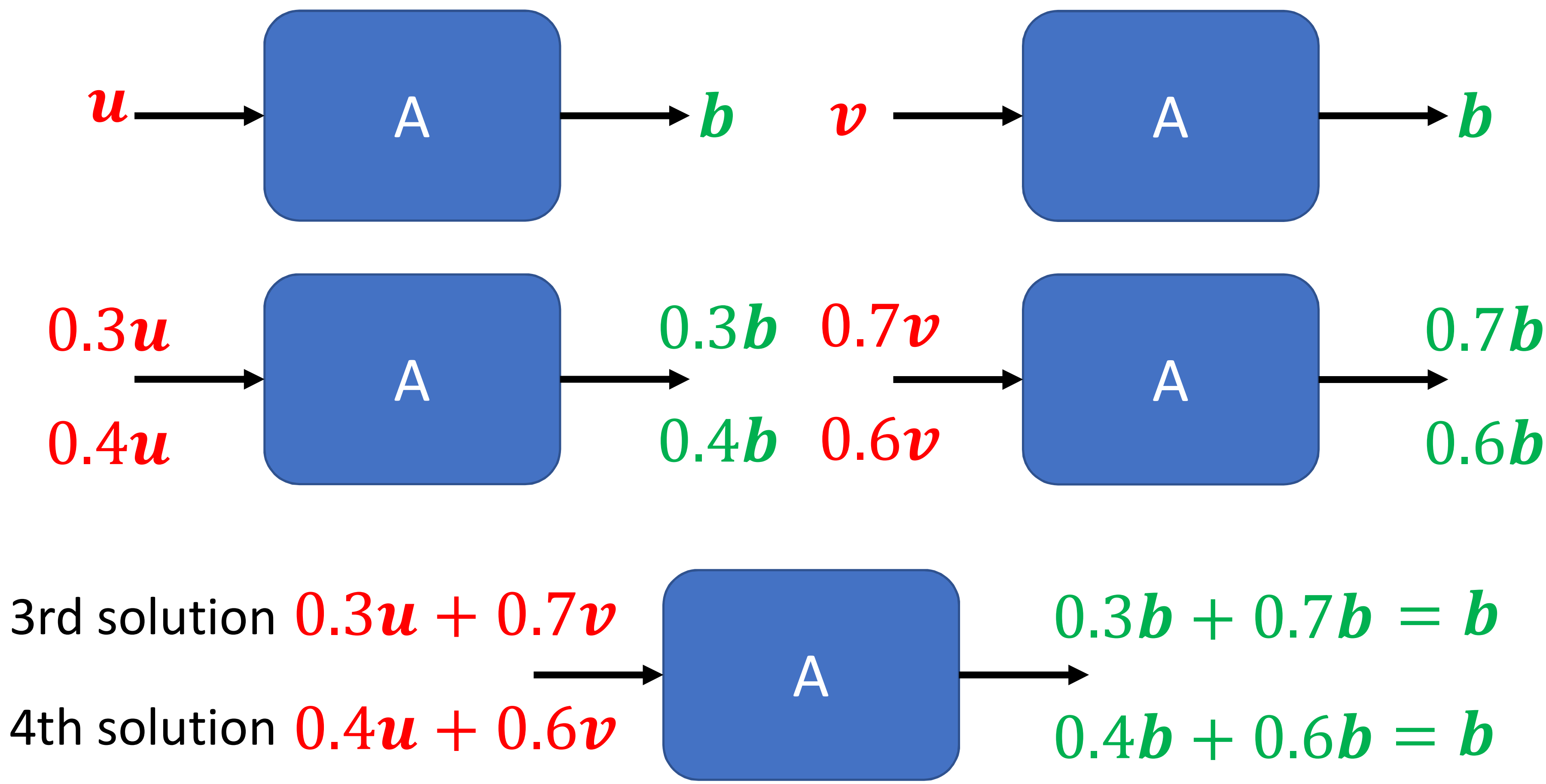
infinite solutions

consistent

Only Unique and Infinite?

為什麼不能只有兩個解？

一旦找到兩個解，
就可以找到無窮多解



Solving System of Linear Equations

(Chapter 1.4)

Equivalent

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

$\Longleftrightarrow$ equivalent

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

Equivalent

- Applying the following three operations on a system of linear equations will produce an equivalent one.
- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling (non zero)

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \text{X(-3)} \Rightarrow \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \text{X(-3)} \Rightarrow \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solving system of linear equation

- Strategy
  - We know how to transform a given system of linear equations into another equivalent one.
  - We do it again and again until the system of linear equation is very simple
  - Finally, we know the answer at a glance.

$$\begin{cases} x_1 \quad -3x_2 \quad = 0 & \text{X (-3)} \\ 3x_1 \quad +x_2 \quad = 10 \end{cases}$$

$$\begin{cases} x_1 \quad -3x_2 \quad = 0 \\ \qquad 10x_2 \quad = 10 & \text{X 1/10} \end{cases}$$

$$\begin{cases} x_1 \quad -3x_2 \quad = 0 \\ \qquad x_2 \quad = 1 & \text{X 3} \end{cases}$$

$$\begin{cases} x_1 \qquad = 3 \\ \qquad x_2 \quad = 1 \end{cases}$$

Augmented Matrix

- a system of linear equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$\Rightarrow \quad A\mathbf{x} = \mathbf{b}$$

$$
\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

coefficient matrix

Augmented Matrix

- a system of linear equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \Rightarrow A\mathbf{x} = \mathbf{b}$$

m x (n+1)

m x n   m x 1

$$\left[\ A \mid \mathbf{b}\ \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

augmented matrix

Back to Equivalent

- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling (non zero)

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \quad \Rightarrow \quad \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \quad \Rightarrow \quad \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Back to Equivalent

elementary row operations

- 1. Interchange Interchange any two rows of the matrix

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix}$$

- 2. Scaling (non zero) Multiply every entry of some row by the same nonzero scalar

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \text{X(-3)} \Rightarrow \begin{bmatrix} 3 & 1 & 10 \\ -3 & 9 & 0 \end{bmatrix}$$

- 3. Row Addition Add a multiple of one row of the matrix to another row

$$\begin{bmatrix} 3 & 1 & 10 \\ 1 & -3 & 0 \end{bmatrix} \text{X(-3)} \Rightarrow \begin{bmatrix} 0 & 10 & 10 \\ 1 & -3 & 0 \end{bmatrix}$$

Solving system of linear equation

A ***complex*** system of linear equations

A ***simple*** system of linear equations

$$A\mathbf{x} = \mathbf{b} \xleftrightarrow{\text{equivalent}} R'\mathbf{x} = \mathbf{b}'$$

$$A' = [\, A \;\; \mathbf{b} \,] \rightarrow A'' \rightarrow A''' \rightarrow \cdots\cdots \rightarrow R = [\, R' \;\; \mathbf{b}' \,]$$

elementary row operations

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

$$\begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases} \quad \text{X 3}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix} \quad \text{X 3}$$

$$\begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \end{cases} \quad \text{X 1/10}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 10 & 10 \end{bmatrix} \quad \text{X 1/10}$$

$$\begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \end{cases} \quad \text{X 3}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{X 3}$$

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Solving system of linear equation

A ***complex*** system of linear equations

????? A ***simple*** system of linear equations

$$A\mathbf{x} = \mathbf{b} \xleftrightarrow{\text{equivalent}} R'\mathbf{x} = \mathbf{b}'$$

$$A' = [\, A \ \ \mathbf{b} \,] \rightarrow A'' \rightarrow A''' \rightarrow \cdots\cdots \rightarrow R = [\, R' \ \mathbf{b}' \,]$$

Reduced Row Echelon Form (RREF)

elementary row operations:

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

Reduced Row Echelon Form (RREF)

(Chapter 1.4)

階層

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form (REF)***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form (REF)***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

NO

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

No zero rows

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form (RREF)***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form (RREF)***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading Entry

The pivot positions of A are (1,1), (2,3) and (3,4).

The pivot columns of A are 1st, 3rd and 4th columns.

Not going to prove

RREF is unique!

- A matrix can be transformed into multiple REFs by row operations, but only one RREF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

RREF is unique – Proof (by Induction)

R= $\begin{bmatrix} 1 & 2 & 0 & 3 & r_1 \\ 0 & 0 & 1 & 4 & r_2 \\ 0 & 0 & 0 & 0 & t_1 \end{bmatrix}$ **S**= $\begin{bmatrix} 1 & 2 & 0 & 3 & s_1 \\ 0 & 0 & 1 & 4 & s_2 \\ 0 & 0 & 0 & 0 & u_1 \end{bmatrix}$ **R-S**= $\begin{bmatrix} 0 & 0 & 0 & 0 & r_1 - s_1 \\ 0 & 0 & 0 & 0 & r_2 - s_2 \\ 0 & 0 & 0 & 0 & t_1 - u_1 \end{bmatrix}$

R, **S** are RREF of **A**. Consider x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

Ax=0 ⟷ **R**x=0 and **S**x=0;
hence (**R-S**)x=0 ⟹ x_5 =0

Claim: $t_1 \neq 0$.

If $t_1 = 0$, consider y= $\begin{bmatrix} r_1 \\ 0 \\ r_2 \\ 0 \\ -1 \end{bmatrix}$ ⟹ **R**y=0 -- a contradiction,
since $y_5 \neq 0$

If $t_1 \neq 0$ ⟹ $t_1 = 1$, $r_1 = r_2 = 0$

Likewise, we can show that $u_1 \neq 0$ ⟹ $u_1 = 1$, $s_1 = s_2 = 0$

※ **R = S**

How to find RREF

(Chapter 1.4)

Reduced Row Echelon Form (RREF)

- Gaussian elimination: an algorithm for finding the reduced row echelon form of a matrix.

Original augmented matrix

A row echelon form

The reduced row echelon form

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix} \underbrace{\to \cdots \to}_{\text{Elementary row operations}} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \underbrace{\to \cdots \to}_{\text{Elementary row operations}} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Because RREF of a matrix is unique, **the order of elementary row operations** in not important.

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ -2 \\ 3
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
-1 & -2 & 1 & 2 & 3 & 6 \\
2 & 4 & -3 & 2 & 0 & 3 \\
-3 & -6 & 2 & 0 & 3 & 9
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
$$

Example 1

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2\\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6\\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3\\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c}
\\ \\ -1 \\ \\
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2\\
0 & 0 & -1 & -2 & -2 & -1\\
0 & 0 & 0 & 4 & 4 & 8\\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2\\
0 & 0 & -1 & -2 & -2 & -1\\
0 & 0 & 0 & 4 & 4 & 8\\
0 & 0 & 0 & 8 & 8 & 16
\end{bmatrix}
$$

Example 1

$$
\begin{aligned}
x_1 \quad + 2x_2 - \ x_3 + 2x_4 + \ x_5 &= 2 \\
-x_1 - 2x_2 + \ x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 \qquad &= 3 \\
-3x_1 - 6x_2 + 2x_3 \qquad\quad + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c}
\\ \\ -2 \\ \\
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 8 & 8 & 16
\end{bmatrix}
$$

$$\Downarrow$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & \cancel{4}\ 1 & \cancel{4}\ 1 & \cancel{8}\ 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\begin{array}{l}
-2 \\
2 \\
\\
\\
\end{array}
$$

$$\Downarrow$$

$$
\begin{bmatrix}
1 & 2 & -1 & 0 & -1 & -2 \\
0 & 0 & -1 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$
\begin{aligned}
x_1 \quad + 2x_2 - \ x_3 + 2x_4 + \ x_5 &= 2 \\
-x_1 - 2x_2 + \ x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 \qquad &= 3 \\
-3x_1 - 6x_2 + 2x_3 \qquad + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 0 & -1 & -2 \\
0 & 0 & \cancel{-1}\ 1 & 0 & 0 & -3\ \cancel{3} \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad 1
$$

$$
\begin{bmatrix}
1 & 2 & 0 & 0 & -1 & -5 \\
0 & 0 & 1 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Move Free Variables to the right

$$\begin{aligned} x_1 + 2x_2 \quad + \quad - x_5 &= -5 \\ x_3 \quad &= -3 \\ x_4 + x_5 &= \ \ 2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

Solution in parametric form

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-5-2x_2+x_5\\ x_2 \quad 1\\ -3\\ 2-x_5 \quad 3\\ x_5 \quad -1\end{bmatrix} {-8} = \overset{1}{x_2}\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + \overset{-1}{x_5}\begin{bmatrix}1\\0\\0\\-1\\1\end{bmatrix} + \begin{bmatrix}-5\\0\\-3\\2\\0\end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 \quad + \quad -x_5 &= -5 \\ x_3 \quad &= -3 \\ x_4 + x_5 &= \;\; 2 \end{aligned}$$

$\Rightarrow$

$x_1 \; free$

$x_2 = -\frac{5}{2} - \frac{1}{2}x_1 + \frac{1}{2}x_5$

$\cancel{x_1 = -5 - 2x_2 + x_5}$

$\cancel{x_2 \quad \text{free}}$

$x_3 = -3$

$x_4 = 2 - x_5$

$x_5 \quad \text{free}$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}x_1 \quad -8\\ -5/2 - 1/2x_1 + 1/2\overset{1}{x_5}\\ -3\\ 2-x_5 \quad 3\\ x_5 \quad -1\end{bmatrix} = \overset{-8}{x_1}\begin{bmatrix}1\\-1/2\\0\\0\\0\end{bmatrix} + \overset{-1}{x_5}\begin{bmatrix}0\\1/2\\0\\-1\\1\end{bmatrix} + \begin{bmatrix}0\\-5/2\\-3\\2\\0\end{bmatrix}$$

Span

(Chapter 1.6)

Span

- A $\boxed{\text{vector set}}$ $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k\}$
- $Span\ S$ is the $\boxed{\text{vector set}}$ of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_k$

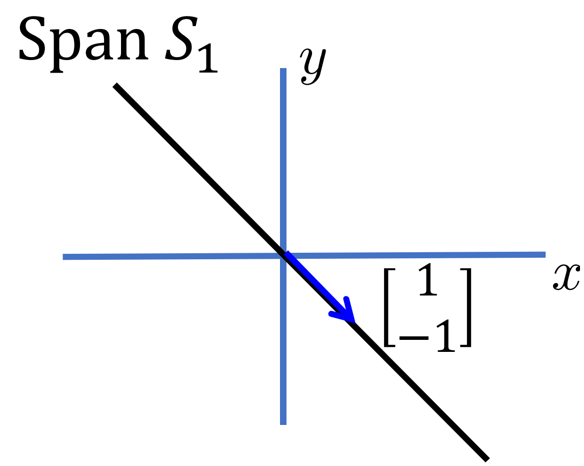
$$Span\ S = \{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \mid for\ all\ c_1, c_2, \cdots, c_k\}$$

- Vector set $V = Span\ S$
  - "S is a generating set for V" or "S generates V"
  - One way to describe a vector set with infinite elements

Span

$$c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Let $S_0 = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$, what is Span S_0?
  - Ans: $\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$ (only one member)



- Let $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$, what is Span S_1?
  - Span $S_1 = \{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \dots \dots \}$
  - If S contains a non zero vector, then Span S has infinitely many vectors

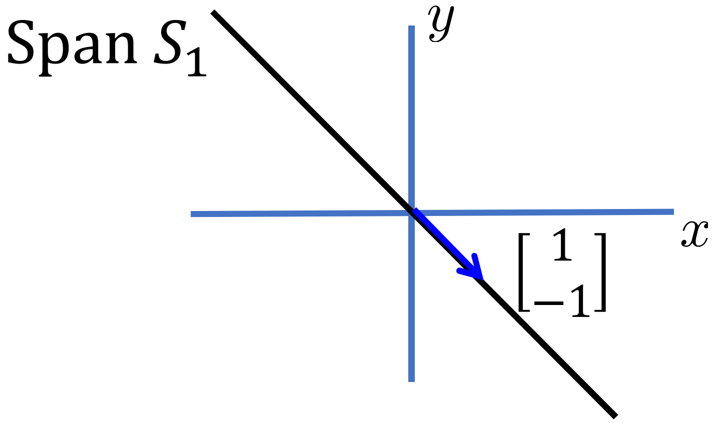
Span

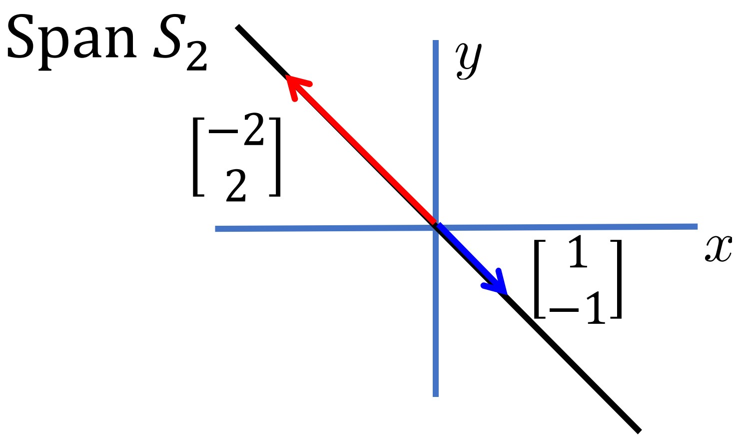
Let $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Span S_1 =?

Let $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

what is Span S_2?



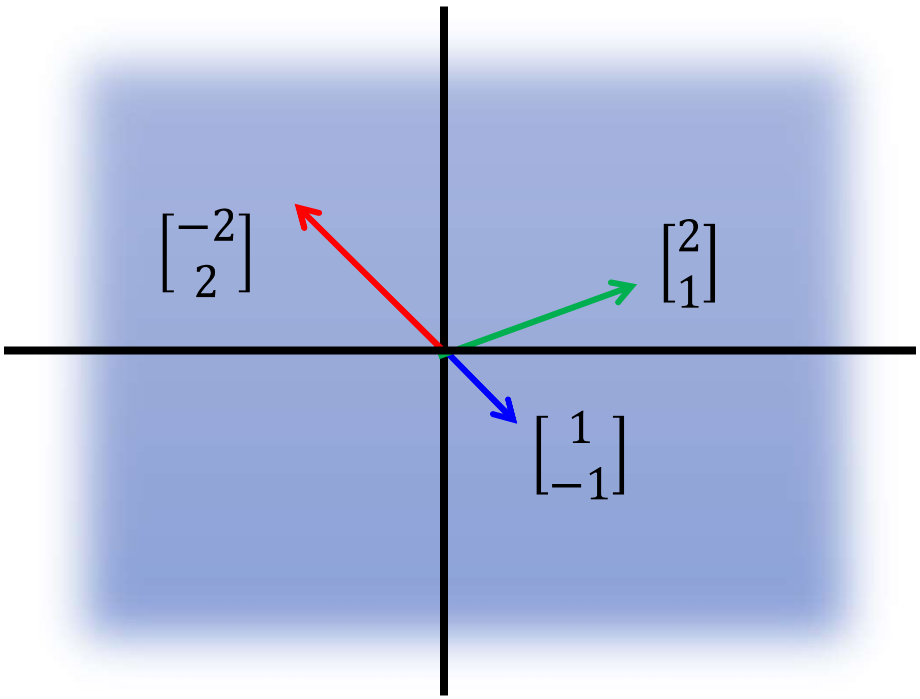


$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

Span

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is Span S_3?



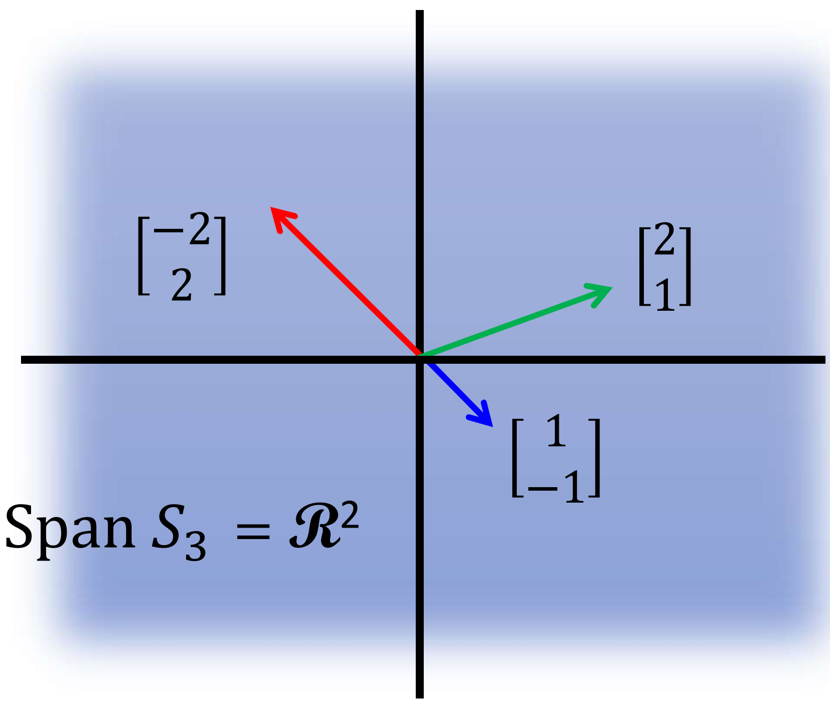
Every vector in $\mathcal{R}^2$ is their linear combination

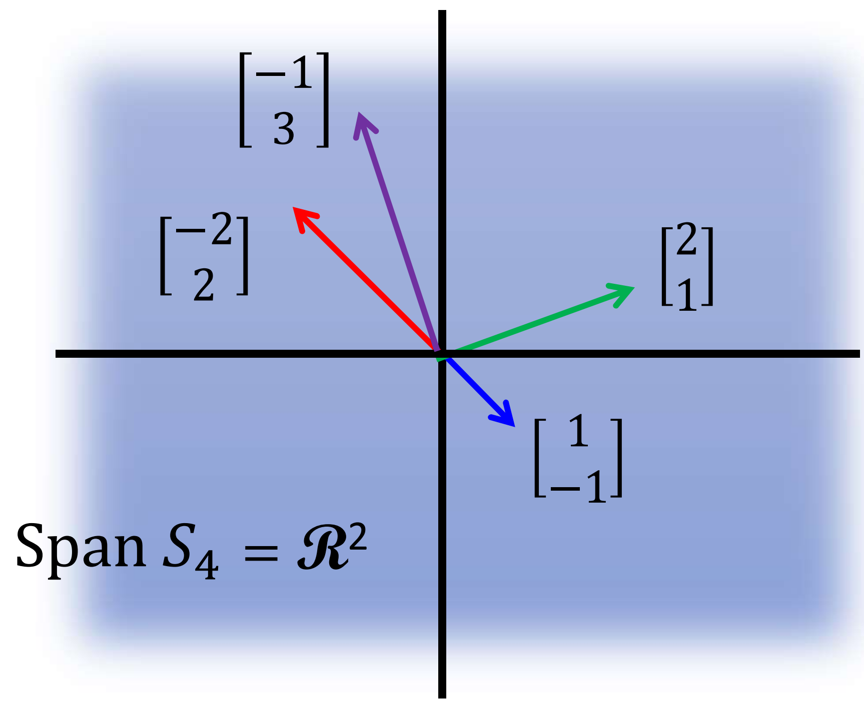
$$\text{Span } S_3 = \mathcal{R}^2$$

Span

Let $S_3 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$
what is Span S_3 =?

Let $S_4 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}$
what is Span S_4 =?





Useless Vector in Span

(Chapter 1.6)

Span

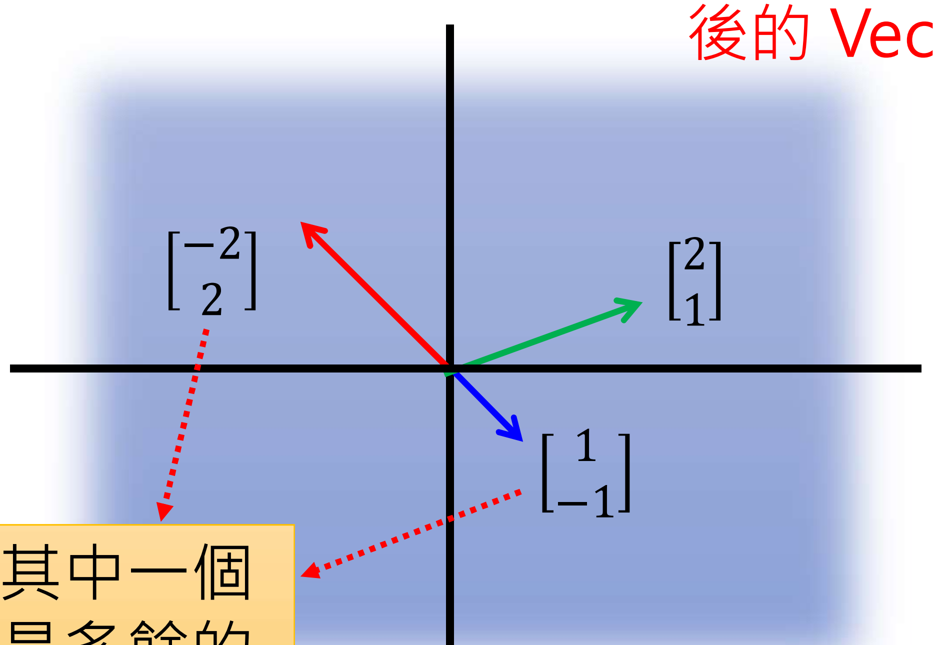
$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Every **b** has solution

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is Span S_3?

拿掉其中一個並不會影響 Span 後的 Vector Set



Every vector in $\mathcal{R}^2$ is their linear combination

$$\text{Span } S_3 = \mathcal{R}^2$$

其中一個是多餘的

只要其中一個在，另一個就是多餘的

多餘Vector 的特徵

Given vector set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_k, \boldsymbol{v}\}$

Given vector set $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_k\}$

$\boldsymbol{v}$ 是多餘:

$Span\ S = Span\ S'$ $\Leftrightarrow$ $\boldsymbol{v}$ 是 S 其餘成員的 linear combination ($\boldsymbol{v} \in Span\ S'$)

$$\boldsymbol{v} = \boxed{b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + \cdots + b_k\boldsymbol{u}_k}$$

Target

$\boldsymbol{w} \in Span\ S$

$\Updownarrow$

$\boldsymbol{w} \in Span\ S'$

$\boldsymbol{w} \in Span\ S$

$$\boldsymbol{w} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k + c\boxed{\boldsymbol{v}} \in Span\ S'$$

$$= (c_1 + cb_1)\boldsymbol{u}_1 + (c_2 + cb_2)\boldsymbol{u}_2 \ldots + (c_k + cb_k)\boldsymbol{u}_k$$

$\boldsymbol{w} \in Span\ S'$

$$\boldsymbol{w} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \qquad c = 0$$

$$= c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k + c\boldsymbol{v} \in Span\ S$$

多餘Vector 的特徵

Given vector set $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_k, \boldsymbol{v}\}$

Given vector set $S' = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_k\}$

$\boldsymbol{v}$ 是多餘:

$Span\ S = Span\ S'$

$\boldsymbol{v}$ 是 S 其餘成員的 linear combination ($\boldsymbol{v} \in Span\ S'$)

$$\boldsymbol{v} = 0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \cdots + 0\boldsymbol{u}_k + 1\boldsymbol{v}$$

$\boldsymbol{w} \in Span\ S$

$\boldsymbol{w} \in Span\ S'$

$\boldsymbol{v} \in Span\ S$

$\boldsymbol{v} \in Span\ S'$

$$\begin{array}{ccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?

The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is b the linear combination of columns of A?

The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in Span \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

Is b in the span of the columns of A?

Dependent and Independent

(Chapter 1.7)

Definition

Ax=**0**

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***dependent***

  Find one → Obtain many

  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

2 2 2

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$ unique

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***dependent***
  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$
- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix}-4\\12\\6\end{bmatrix}, \quad \begin{bmatrix}-10\\30\\15\end{bmatrix}\right\}$$

Dependent or Independent?

dependent

$$\underset{5}{x_1}\begin{bmatrix}-4\\12\\6\end{bmatrix} + \underset{-2}{x_2}\begin{bmatrix}-10\\30\\15\end{bmatrix} = \boldsymbol{0}$$

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***dependent***
  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$
- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix}6\\3\\3\end{bmatrix}, \begin{bmatrix}1\\8\\3\end{bmatrix}, \begin{bmatrix}7\\11\\6\end{bmatrix}\right\}$$

Dependent or Independent?

dependent

$$\underset{1}{x_1}\begin{bmatrix}6\\3\\3\end{bmatrix} + \underset{1}{x_2}\begin{bmatrix}1\\8\\3\end{bmatrix} + \underset{-1}{x_3}\begin{bmatrix}7\\11\\6\end{bmatrix} = \mathbf{0}$$

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***dependent***
  - If there exist scalars $x_1, x_2, \cdots, x_n$, **not all zero**, such that $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$
- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linearly ***independent***

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$

$$\left\{\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}\right\}$$ Dependent or Independent?

dependent

$$\underset{\mathbf{0}}{x_1}\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix} + \underset{\text{Any}}{x_2}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \underset{\mathbf{0}}{x_3}\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \mathbf{0}$$

Any set containing zero vector would be linearly dependent

Linearly Dependent

(for $n \geq 2$)

Given a vector set {$\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$}, there exists scalars x_1, x_2, …, x_n, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} \ldots + \boxed{x_i\boldsymbol{a_i}} + \cdots + x_n\boldsymbol{a_n} = \mathbf{0}$$

永遠可以找到某一項有非 0係數 $x_i \neq 0$

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} \ldots + x_n\boldsymbol{a_n} = -x_i\boldsymbol{a_i}$$

$$-\left(\frac{x_n}{x_1}\right)\boldsymbol{a_1} - \left(\frac{x_n}{x_2}\right)\boldsymbol{a_2} \ldots - \left(\frac{x_n}{x_i}\right)\boldsymbol{a_n} = \boldsymbol{a_i}$$

Given a vector set {$\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$}, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linearly Dependent

(for $n \geq 2$)

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

$$\boldsymbol{a_i} = c_1\boldsymbol{a_1} + c_2\boldsymbol{a_2} \ldots + c_n\boldsymbol{a_n}$$

$$-c_1\boldsymbol{a_1} - c_2\boldsymbol{a_2} \ldots + \boxed{\boldsymbol{a_i}} \ldots - c_n\boldsymbol{a_n} = \mathbf{0}$$

至少這項有非 0係數

$x_i \neq 0$

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linearly Dependent = Vector Set 中有多餘的

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

(for $n \geq 2$)

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Linearly Independent = Vector Set 中沒有多餘的

Ax=**0** has **infinite** solutions

- Columns of A are **dependent** → If Ax=b has a solution, it will have **Infinite** solutions

We can find non-zero solution **u** such that $A\boldsymbol{u} = \mathbf{0}$

There exists **v** such that $A\boldsymbol{v} = \mathbf{b}$

$$A(\boldsymbol{u} + \boldsymbol{v}) = \mathrm{b}$$

$\boldsymbol{u} + \boldsymbol{v}$ is another solution different to **v**

- If Ax=b has **Infinite** solutions → Columns of A are **dependent**

$$u \neq v$$

$$Au = \mathrm{b}$$

$$Av = \mathrm{b}$$

Ax=**0** has **infinite** solutions

$$A(\underbrace{u - v}_{\text{Non-zero}}) = \mathbf{0}$$

Column Correspondence Theorem

(Chapter 1.7)

Column Correspondence Theorem

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_n}] \xrightarrow{\text{RREF}} R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_n}]$$

If $\boldsymbol{a_j}$ is a linear combination of other columns of A ⇒ $\boldsymbol{r_j}$ is a linear combination of the ***corresponding*** columns of R with ***the same coefficients***

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\boldsymbol{a_j}$ is a linear combination of the ***corresponding*** columns of A with ***the same coefficients*** ⇐ If $\boldsymbol{r_j}$ is a linear combination of other columns of R

$\mathbf{a}_3 = 3\mathbf{a}_1 - 2\mathbf{a}_2$

$\mathbf{r}_3 = 3\mathbf{r}_1 - 2\mathbf{r}_2$

Column Correspondence Theorem - Example

$$
A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array}
\qquad
R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \Longleftrightarrow \quad \mathbf{r}_2 = 2\mathbf{r}_1$$

$$\mathbf{a}_5 = -\mathbf{a}_1+\mathbf{a}_4 \quad \Longleftrightarrow \quad \mathbf{r}_5 = -\mathbf{r}_1+\mathbf{r}_4$$

Augmented Matrix: $[A \quad b] \longrightarrow [R \quad b']$

- The RREF of matrix A is R

  $Ax = b$ and $Rx = b$ have the same solution set?

- The RREF of augmented matrix $[A \quad b]$ is $[R \quad b']$

  $Ax = b$ and $Rx = b'$ have the same solution set

- The RREF of matrix A is R

  $Ax = 0$ and $Rx = 0$ have the same solution set

If $b = 0$, then $b' = 0$.

How about Rows?

- Are there row correspondence theorem? **NO**

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \text{---} \; \boldsymbol{a}_1^T \; \text{---} \\ \text{---} \; \boldsymbol{a}_2^T \; \text{---} \\ \text{---} \; \boldsymbol{a}_3^T \; \text{---} \\ \text{---} \; \boldsymbol{a}_4^T \; \text{---} \end{bmatrix} \qquad R = \begin{bmatrix} \text{---} \; \boldsymbol{r}_1^T \; \text{---} \\ \text{---} \; \boldsymbol{r}_2^T \; \text{---} \\ \text{---} \; \boldsymbol{r}_3^T \; \text{---} \\ \text{---} \; \boldsymbol{r}_4^T \; \text{---} \end{bmatrix}$$

Check Independence

(Chapter 1.7)

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linearly independent or not?

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

matrix A

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

vector x

$Ax = \mathbf{0}$ have non-zero solution

Checking Independence

$$\mathcal{S} = \underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}_{\textbf{A}}$$

Linearly independent or not?

$Ax = \mathbf{0}$ have non-zero solution or not

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \hline \end{array}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \end{array} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Checking Independence

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

dependent

$$x_1 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_4 = 0$$

$$\Rightarrow$$

$$x_1 = -2x_3$$
$$x_2 = x_3$$
$$x_3 \; is \; free$$
$$x_4 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

setting $x_3 = 1$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linearly independent or not?

其實這題用看的就知道答案了!

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linearly dependent

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

matrix A

Given a vector set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

vector x

$Ax = \mathbf{0}$ have non-zero solution

Column Correspondence Theorem

pivot columns

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 2 \\ -3 \end{matrix}} & \begin{matrix} 2 \\ -2 \\ 4 \\ -6 \end{matrix} & \boxed{\begin{matrix} -1 \\ 1 \\ -3 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 2 \\ 2 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 3 \\ 0 \\ 3 \end{matrix} & \begin{matrix} 2 \\ 6 \\ 3 \\ 9 \end{matrix} \end{bmatrix}$$

linearly independent

Leading entries

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

linearly independent

The pivot columns are linearly independent.

Column Correspondence Theorem

pivot columns

Leading entries

$$A = \begin{bmatrix} \boxed{1} & 2 & \boxed{-1} & \boxed{2} & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{a}_6 = -5\mathbf{a}_1 - 3\mathbf{a}_3 + 2\mathbf{a}_4$

$\mathbf{r}_2 = 2\mathbf{r}_1$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\mathbf{r}_6 = -5\mathbf{r}_1 - 3\mathbf{r}_3 + 2\mathbf{r}_4$

The non-pivot columns are the linear combination of the previous pivot columns.

Independent

All columns are independent

Every column is a pivot column

Every column in RREF(A) is a standard vector.

Dependent

The column is the linear combination of left pivot column.

If a column is not pivot

Independent

All columns are independent

Every column is a pivot column

Every column in RREF(A) is standard vector.

3X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linearly independent

RREF

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Identity matrix

Independent

All columns are independent

⬇

Every column is a pivot column

⬇

Every column in RREF(A) is standard vector.

4X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linearly independent

RREF ⬇

$$\begin{bmatrix} \\ \\ \end{bmatrix} \quad \begin{bmatrix} I \\ O \end{bmatrix}$$

Independent

All columns are independent

↓

Every column is a pivot column

↓

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linearly independent

RREF

Cannot be a pivot column

Independent

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}$$

dependent

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \quad \Rightarrow \quad \not\Leftarrow$$

(矮胖型)

The columns are dependent

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Dependent or Independent?

dependent

More than 3 vectors in R^3 must be dependent.

More than m vectors in R^m must be dependent.

Rank of a Matrix

(Chapter 1.7)

Rank

Maximum number of Independent Columns

||

Number of Pivot Column

||

Number of Non-zero rows

Rank R = Rank A

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Rank = ? **3**

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = ? **3**

Rank

Maximum number of Independent Columns

||

Number of Pivot Column

||

Number of Non-zero rows

Rank A $\leq$ Number of columns

Rank A $\leq$ Min(Number of columns, Number of rows)

Rank A $\leq$ Number of rows

Rank

Matrix A is ***full rank***
if Rank A = min(m,n)

Matrix A is ***rank deficient***
if Rank A < min(m,n)

- Given an mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because "the columns of A are independent" is equivalent to "rank A = n"
    - If m < n, the columns of A are dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors belonging to R^3 is dependent

In R^m, you cannot find more than m vectors that are independent.

Basic, Free Variables vs. Rank

$Ax = b$

$$\begin{aligned} x_1 \;+2x_2 - x_3 + 2x_4 + 5x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 \qquad &= 3 \\ -3x_1 - 6x_2 + 2x_3 \qquad + 3x_5 &= 9 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 5 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right] \qquad \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A $\quad b \qquad$ RREF(A) $\quad b'$

$$\begin{aligned} x_1 + 2x_2 \qquad - x_5 &= -5 \\ x_3 \qquad &= -3 \\ x_4 + x_5 &= 2 \\ \cancel{0} &= \cancel{0} \end{aligned}$$

3 useful equations

$x_1 = -5 - 2x_2 + x_5$

$x_3 = -3$

$x_4 = 2 - x_5$

rank = non-zero row = 3 basic variables

nullity = No. column – non-zero row = 2 free variables

Rank

Rank

Maximum number of Independent Columns

Number of Pivot Columns

Number of Non-zero rows of RREF

Number of Basic Variables

Nullity = no. column - rank

~~Number of zero rows of RREF~~

Number of Free Variables

RREF vs. Span

(Chapter 1.7)

Consistent or not

- Given Ax=b, if the reduced row echelon form of [*A* b] is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent

b is in the span of the columns of A

- Given Ax=b, if the reduced row echelon form of [*A* b] is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

inconsistent

b is NOT in the span of the columns of A

Consistent or not

Ax =b is inconsistent (no solution)

The RREF of [A b] is

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad d \neq 0$$

Only the last column is non-zero

3

4

Rank A $\neq$ Rank [A b]

Need to know b

Ax =b is consistent for ***every*** b $\qquad A: m \times n$

$\parallel$

Every b is in the span of the columns of A= $\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

$\parallel$

Every b belongs to $Span\{a_1, \cdots, a_n\}$

$\parallel$

$Span\{a_1, \cdots, a_n\} = R^m$

$\parallel$

RREF of [A b] cannot have a row whose only non-zero entry is at the last column

$\parallel$

RREF of A cannot have zero row 沒有任何破綻

$\parallel$

Rank A = no. of rows

Consistent or not

$A: m \times n$

$Span\{a_1, \quad \cdots \quad, a_n\} = R^m$ = Rank A = no. of rows

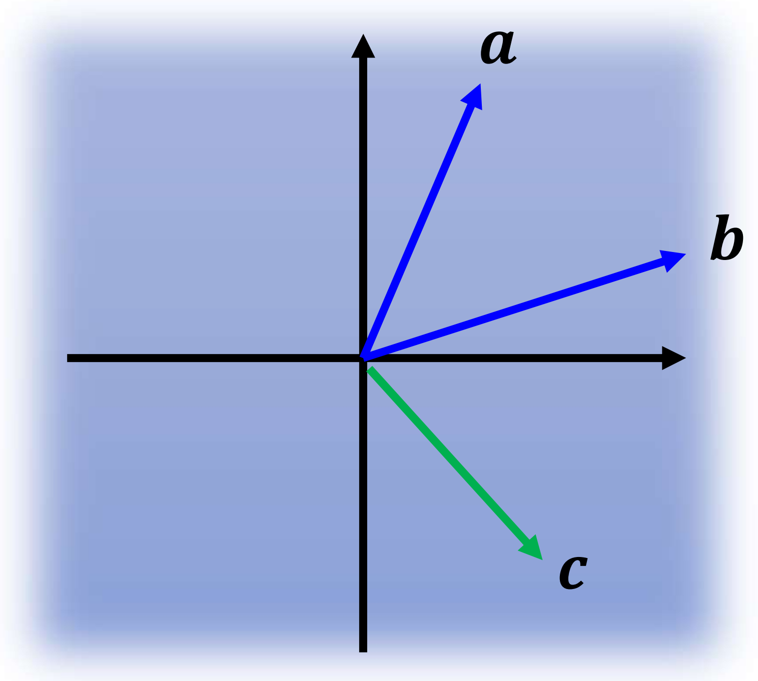
m independent vectors can span R^m

More than m vectors in R^m must be dependent.

Example

m independent vectors can span R^m

Consider R^2



$$\text{Does } \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ generate } \mathcal{R}^3?$$

yes

independent

The Big Picture of $\color{blue}{A}\color{red}{x} = \color{green}{b}$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

Square Matrix:

方程式數 = 變數個數

唯一解? 無窮多解? 無解?

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

Fat Matrix:

方程式數 <<變數個數

很可能有無窮多解

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}$$

Tall Matrix:

方程式數 > 變數個數

很可能無解

※

Number of Solutions of $Ax = b$

One Solution

No Solution

Infinite Solutions

可否找到好的近似解?
(Linear regression)

可否找到最小的解?

Pseudo-inverse Matrix
(based on SVD)

※

Solutions of $Ax = b$

$Zero, One, Infinity$ ...

No Solution

- b is NOT a linear combination of column vectors of A

One Solution

- b is a linear combination of column vectors of A
- Ax=0 only has zero solution

Infinite Solutions

- b is a linear combination of column vectors of A
- Ax=0 has a non-zero solution

Is $\boldsymbol{b}$ in the ***span*** of the columns of A?

Summary

$A: m \times n \quad \boldsymbol{x} \in R^n \quad \boldsymbol{b} \in R^m$

Is b a ***linear combination*** of columns of A?

Is $\boldsymbol{b}$ in the ***span*** of the columns of A?

NO

No solution

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

Summary

$A: m \times n$

$\boldsymbol{x} \in R^n \quad \boldsymbol{b} \in R^m$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}$$

NO

Is b a ***linear combination*** of columns of A?

Is b in the ***span*** of the columns of A?

NO

No solution

YES

Infinite solution

YES

Is b a ***linear combination*** of columns of A?

Is b in the ***span*** of the columns of A?

NO

No solution

YES

Unique solution