113-1 Linear Algebra
Final exam solutions

1. (10%) Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by

$$T\left(p(x)\right) = 3p(0) + \left(-2p(1) + p'(0)\right)x + p(2)x^2,$$

where

$$p'(0) = \left.\frac{\mathrm{d}p(x)}{\mathrm{d}x}\right|_{x=0}.$$

Determine whether each of the following statements is true or false.

(**No explanation is needed.** Each correct answer gets 2% and each wrong answer gets 0%.)

(a) T is a linear transformation.

(b) T is an isomorphism.

(c) The eigenvalues of T are 0 and -3.

(d) A set of vectors consisting of the basis of each eigenspace for all eigenvalues of T constitutes a basis for $\mathcal{R}^3$.

(e) Let $\{1, x, x^2\}$ be a basis for $\mathcal{P}_2$. Then any vector $\mathbf{v}$ in vector space $\mathcal{P}_2$ can be uniquely represented as a linear combination of the vectors in $\{1, 1+2x, 1+2x+3x^2\}$.

Ans.

(a) **True**.

Let $p(x), q(x) \in \mathcal{P}_2$ and $a \in \mathcal{R}$, we have

$$\begin{aligned}
T(a \cdot p(x) + q(x)) &= 3 \cdot (a \cdot p(0) + q(0)) + (-2 \cdot (a \cdot p(1) + q(1)) + a \cdot p'(0) + q'(0))x + (a \cdot p(2) + q(2))x^2 \\
&= a \cdot \Big(3p(0) + (-2p(1) + p'(0))x + p(2)x^2\Big) + \Big(3q(0) + (-2q(1) + q'(0))x + q(2)x^2\Big) \\
&= a \cdot T(p(x)) + T(q(x)).
\end{aligned}$$

We then show that T is a linear transformation.

(b) **False**.

Since we can find an example where T is not one-to-one, so T is not an isomorphism. An example of T is not one-to-one is given as follows: Let $p \in \mathcal{P}_2$ and $p(x) = -2x + x^2$. We have

$$T(p(x)) = 3p(0) + (-2p(1) + p'(0))x + p(2)x^2 = 0.$$

Moreover, if $p(x) = 0$, we also have

$$T(p(x)) = 0.$$

From the above example, T is not one-to-one. Therefore, T is not an isomorphism.

(c) **False**.

To find the eigenvalues of T, we calculate its matrix representation with respect to the basis

For $\mathcal{B} = \{1, x, x^2\}$. $[T]_\mathcal{B} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ and

$$\begin{aligned} \mathbf{a}_1 &= [T(1)]_\mathcal{B} = [3 - 2x + x^2]_\mathcal{B} = [3\ \ -2\ 1]^T, \\ \mathbf{a}_2 = [T(x)]_\mathcal{B} &= [T(x)]_\mathcal{B} = [-x + 2x^2]_\mathcal{B} = [0\ \ -1\ 2]^T, \\ \mathbf{a}_3 &= [T(x^2)]_\mathcal{B} = [-2x + 4x^2]_\mathcal{B} = [0\ \ -2\ 4]^T. \end{aligned}$$

Thus,

$$[T]_\mathcal{B} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] = \begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & -2 \\ 1 & 2 & 4 \end{bmatrix}.$$

To find the eigenvalues, we solve $\det([T]_\mathcal{B} - tI) = 0$, where I is the identity matrix. We have

$$\det([T]_\mathcal{B} - tI) = \det \begin{bmatrix} 3-t & 0 & 0 \\ -2 & -1-t & -2 \\ 1 & 2 & 4-t \end{bmatrix} = (3-t)\det \begin{bmatrix} -1-t & -2 \\ 2 & 4-t \end{bmatrix} = -t(t-3)^2 = 0.$$

Thus, we find that the eigenvalues of T are $t = 0$ and $t = 3$, but rather 0 and -3. Therefore, the statement is false.

(d) **True**.

To verify if the set of vectors consisting of the basis of each eigenspace for all eigenvalues of T constitutes a basis for $\mathcal{R}^3$, we compute the dimensions of the eigenspaces corresponding to eigenvalues 0 and 3:

For the eigenvalue 0, we have

$$\text{rank}\ ([T]_\mathcal{B} - 0 \cdot I) = \text{rank}\ \left(\begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & -2 \\ 1 & 2 & 4 \end{bmatrix}. \right) = 2,$$

and we obtain

$$\text{nullity}\ ([T]_\mathcal{B} - 0 \cdot I) = 3 - \text{rank}\ ([T]_\mathcal{B} - 0 \cdot I) = 3 - 2 = 1.$$

Thus, the dimension of the eigenspace for eigenvalue 0 is 1.

For the eigenvalue 3, we have

$$\text{rank}\ ([T]_\mathcal{B} - 3 \cdot I) = \text{rank}\ \left(\begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & -2 \\ 1 & 2 & 1 \end{bmatrix} \right) = 1,$$

and we obtain

$$\text{nullity}\ ([T]_\mathcal{B} - 3 \cdot I) = 3 - \text{rank}\ ([T]_\mathcal{B} - 3 \cdot I) = 3 - 1 = 2.$$

Thus, the dimension of the eigenspace for eigenvalue 3 is 2.

Since the sum of the dimensions of the eigenspaces for the eigenvalues 0 and 3 is

$$1 + 2 = 3 = \dim(\mathcal{R}^3)$$

which equals to the dimension of $\mathcal{R}^3$. The set of vectors consisting of the basis of each eigenspace constitutes a basis for $\mathcal{R}^3$. Thus, the statement is true.

(e) **True**.

To determine whether any vector $\mathbf{v}$ in $\mathcal{P}_2$ can be uniquely represented as a linear combination of these vectors, we would like to verify whether $\{1, 1+2x, 1+2x+3x^2\}$ is also a basis for $\mathcal{P}_2$. For the set to be a basis, it must satisfy

(1) $\{1, 1+2x, 1+2x+3x^2\}$ is a linearly independent set.
(2) $\{1, 1+2x, 1+2x+3x^2\}$ is a generating set for $\mathcal{P}_2$.

In the following, we will address each of these two points.

(1) Let $a_1,\, a_2,\, a_3 \in \mathcal{R}$, it is obviously that $a_1 \cdot 1 + a_2(1+2x) + a_3(1+2x+3x^2) = 0$ only when $a_1 = a_2 = a_3 = 0$. Thus, $\{1, 1+2x, 1+2x+3x^2\}$ is a linearly independent set.

(2) To determine whether $\{1, 1+2x, 1+2x+3x^2\}$ is a generating set for $\mathcal{P}_2$, we have to show that whether $\text{Span}\{1, 1+2x, 1+2x+3x^2\} = \mathcal{P}_2$.

   (i) Let $a_1,\, a_2,\, a_3 \in \mathcal{R}$ and $q \in \text{Span}\{1, 1+2x, 1+2x+3x^2\}$, we have

   $$q(x) = a_1 \cdot 1 + a_2(1+2x) + a_3(1+2x+3x^2) = (a_1+a_2+a_3) + 2(a_2+a_3)x + 3a_3x^2.$$

   Since $q \in \mathcal{P}_2$, we show that $\text{Span}\{1, 1+2x, 1+2x+3x^2\} \subseteq \mathcal{P}_2$.

   (ii) Let $b_1,\, b_2,\, b_3 \in \mathcal{R}$ and $p \in \mathcal{P}_2$, we have

   $$p(x) = b_1 + b_2x + b_3x^2 = c_1 \cdot 1 + c_2(1+2x) + c_3(1+2x+3x^2),$$

   where $c_3 = \frac{a_3}{3}$, $c_2 = \frac{a_2}{2} - \frac{a_3}{3}$ and $c_1 = a_1 - \frac{a_2}{2}$. Since $p \in \{1, 1+2x, 1+2x+3x^2\}$, we show that $\text{Span}\{1, 1+2x, 1+2x+3x^2\} \supseteq \mathcal{P}_2$.

   We then show that $\text{Span}\{1, 1+2x, 1+2x+3x^2\} = \mathcal{P}_2$.

As a result, $\{1, 1+2x, 1+2x+3x^2\}$ is also a basis of $\mathcal{P}_2$ and any vector $\mathbf{v}$ in $\mathcal{P}_2$ can be uniquely represented as a linear combination of these vectors. Therefore, the statement is true.

Grading policy:

1. Correctly determining whether each statement is true or false earns 2 points.
2. No points are be awarded for incorrect answers, regardless of the explanations or calculations provided.

2. (10%) Let $\{\mathbf{v}, \mathbf{w}\}$ be an orthonormal basis for $\mathcal{R}^2$, and let $T : \mathcal{R}^2 \to \mathcal{R}^2$ be the function defined by

$$T(\mathbf{u}) = (\mathbf{u}\cdot\mathbf{v}\cos\theta + \mathbf{u}\cdot\mathbf{w}\sin\theta)\mathbf{v} + (-\mathbf{u}\cdot\mathbf{v}\sin\theta + \mathbf{u}\cdot\mathbf{w}\cos\theta)\mathbf{w}.$$

Show that T is an orthogonal operator.

Ans. We begin by claiming that for an orthonormal basis $\{\mathbf{b}_1, ..., \mathbf{b}_n\}$ of $\mathcal{R}^n$,

$$\sum_{i=1}^{n} \mathbf{b}_i\mathbf{b}_i^T = \mathbf{I}_n. \tag{1}$$

For any $\mathbf{v} \in \mathcal{R}^n$, let $\mathbf{v} = \sum_{j=1}^{n} c_j\mathbf{b}_j$. Then,

$$\sum_{i=1}^{n} \mathbf{b}_i\mathbf{b}_i^T(\mathbf{v}) = \sum_{i=1}^{n} \mathbf{b}_i\mathbf{b}_i^T \left(\sum_{j=1}^{n} c_j\mathbf{b}_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_j\mathbf{b}_i\mathbf{b}_i^T\mathbf{b}_j = \sum_{j=1}^{n} c_j\mathbf{b}_j = \mathbf{v}. \tag{2}$$

Returning to the problem, to prove that T is orthogonal, it suffices to show that $\langle T(\mathbf{a}), T(\mathbf{b})\rangle = \langle \mathbf{a}, \mathbf{b}\rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{R}^2$.

$$\begin{aligned}
\langle T(\mathbf{a}), T(\mathbf{b})\rangle &= \langle \mathbf{a}, \mathbf{v}\cos\theta + \mathbf{w}\sin\theta\rangle\langle \mathbf{b}, \mathbf{v}\cos\theta + \mathbf{w}\sin\theta\rangle + \langle \mathbf{a}, -\mathbf{v}\sin\theta + \mathbf{w}\cos\theta\rangle\langle \mathbf{b}, -\mathbf{v}\sin\theta + \mathbf{w}\cos\theta\rangle \\
&= \langle \mathbf{a}, \mathbf{v}\cos\theta + \mathbf{w}\sin\theta\rangle\langle \mathbf{v}\cos\theta + \mathbf{w}\sin\theta, \mathbf{b}\rangle + \langle \mathbf{a}, -\mathbf{v}\sin\theta + \mathbf{w}\cos\theta\rangle\langle -\mathbf{v}\sin\theta + \mathbf{w}\cos\theta, \mathbf{b}\rangle \\
&= \mathbf{a}^T(\mathbf{v}\cos\theta + \mathbf{w}\sin\theta)(\mathbf{v}\cos\theta + \mathbf{w}\sin\theta)^T\mathbf{b} + \mathbf{a}^T(-\mathbf{v}\sin\theta + \mathbf{w}\cos\theta)(-\mathbf{v}\sin\theta + \mathbf{w}\cos\theta)^T\mathbf{b} \\
&= \mathbf{a}^T\left(\mathbf{v}\mathbf{v}^T\cos^2\theta + \mathbf{w}\mathbf{w}^T\sin^2\theta + \mathbf{v}\mathbf{v}^T\sin^2\theta + \mathbf{w}\mathbf{w}^T\cos^2\theta\right)\mathbf{b} \\
&= \mathbf{a}^T(\mathbf{v}\mathbf{v}^T + \mathbf{w}\mathbf{w}^T)\mathbf{b} \\
&= \mathbf{a}^T\mathbf{I}_2\mathbf{b} \\
&= \mathbf{a}^T\mathbf{b} \\
&= \langle \mathbf{a}, \mathbf{b}\rangle.
\end{aligned} \tag{3}$$

The first equation follows from the axioms of inner product and the fact that $\{\mathbf{v}, \mathbf{w}\}$ is an orthonormal set. The second equation uses the axioms of the inner product. The sixth equation follows from our initial claim in (1).

Note that in the definition of T, we are using dot products, so the dot product is used as the inner product to define the orthogonal operator. In general, any other inner product can be used to define the orthogonal operator. The same result can be proven with a similar process, but care must be taken to ensure that the inner product used aligns with the one used in the definition of T.

Grading policy:

1. Write down the equation $\langle T(\mathbf{a}), T(\mathbf{b})\rangle = \langle \mathbf{a}, \mathbf{b}\rangle$ or provide any other equivalent definition of the orthogonal operator can earn 4 points.
2. Provide a detailed proof that T satisfies the definition of the orthogonal operator you stated gets 6 points.
3. Any errors in your proof will result in a deduction of at least 1 point.

3. (10%) Find a vector $\mathbf{z}$, such that $\|A\mathbf{z} - \mathbf{b}\|$ is a minimum, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$.

Ans. By computing the reduced row echelon form of A, we can have rank $A = 2$ and that the first two columns of A are linearly independent. Thus the first two columns of A form a basis for $W = \text{Col } A$. The vector $\mathbf{z}$ that minimize $||A\mathbf{z} - \mathbf{b}||$ are the solutions to

$$A\mathbf{z} = P_{\mathrm{W}}\mathbf{b}, \tag{4}$$

where $P_{\mathrm{W}} = A(A^TA)^{-1}A^T$. Then we have

$$\mathbf{z} = (A^TA)^{-1}A^T\mathbf{b} \tag{5a}$$

$$= \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \tag{5b}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{5c}$$

where

$$(A^TA)^{-1} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/6 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}. \tag{6}$$

Grading policy:

1. Correctly write (5a) earns 5 points.
2. Correctly answer (5c) earns 5 points.

4. (20%) Let $\mathcal{M}_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathcal{R} \right\}$ denote the set of all 2×2 matrices over the real numbers. Consider $\mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
   (a) (4%) Prove that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ forms a basis of $\mathcal{M}_{2\times 2}$.
   (b) (5%) Define a linear transformation $T : \mathcal{M}_{2\times 2} \to \mathcal{M}_{2\times 2}$ by
   $$T(X) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X.$$
   Find the matrix representation of T with respect to the basis $\mathcal{B}$ (i.e., $[T]_{\mathcal{B}}$).
   (c) (6%) Consider the Frobenius inner product of two matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ defined by $\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$. Apply the Gram-Schmidt process to $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ (in the given order) to find an orthonormal basis with respect to the Frobenius inner product.
   (d) (5%) Let W be the subspace spanned by $\{\mathbf{b}_1, \mathbf{b}_2\}$. Find the orthogonal projection of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ onto W with respect to the Frobenius inner product.

Ans.

(a) For $\mathcal{B}$ be a basis of $\mathcal{M}_{2\times 2}$, it must satisfy

  (1) $\mathcal{B}$ is a linearly independent set.
  (2) $\mathcal{B}$ is a generating set for $\mathcal{M}_{2\times 2}$.

  In the following, we will address each of these two points.

  (1) Let $x_1, x_2, x_3, x_4 \in \mathcal{R}$, it is obviously that $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4 = \begin{bmatrix} x_1 & x_2 + x_3 \\ x_2 & x_1 + x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ only when $x_1 = x_2 = x_3 = x_4 = 0$. Thus, $\mathcal{B}$ is a linearly independent set.
  (2) To determine whether $\mathcal{B}$ is a generating set for $\mathcal{M}_{2\times 2}$, we have to show that whether $\text{Span}\,\mathcal{B} = \mathcal{M}_{2\times 2}$.
    (i) Let $x_1, x_2, x_3, x_4 \in \mathcal{R}$ and $C \in \text{Span}\,\mathcal{B}$, we have
    $$C = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4 = \begin{bmatrix} x_1 & x_2 + x_3 \\ x_2 & x_1 + x_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
    where $a = x_1$, $b = x_2 + x_3$, $c = x_2$ and $d = x_1 + x_4$. Since $C \in \mathcal{M}_{2\times 2}$, we show that $\text{Span}\,\mathcal{B} \subseteq \mathcal{M}_{2\times 2}$.

(ii) Let y_1, y_2, y_3, $y_4 \in \mathcal{R}$ and $D \in \mathcal{M}_{2\times 2}$, we have

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} y_1 & y_2 + y_3 \\ y_2 & y_1 + y_4 \end{bmatrix} = y_1\mathbf{b}_1 + y_2\mathbf{b}_2 + y_3\mathbf{b}_3 + y_4\mathbf{b}_4,$$

where $y_1 = a$, $y_2 = c$, $y_3 = b - c$ and $y_4 = d - a$. Since $D \in$ Span $\mathcal{B}$, we show that Span $\mathcal{B} \supseteq \mathcal{M}_{2\times 2}$.

We then show that Span $\mathcal{B} = \mathcal{M}_{2\times 2}$.

Since $\mathcal{B}$ is a linearly independent and also a generating set for $\mathcal{M}_{2\times 2}$, $\mathcal{B}$ is a basis of $\mathcal{M}_{2\times 2}$.

Grading policy:

1. Show that $\mathcal{B}$ is a linearly independent set earns 1 point.
2. Show that Span $\mathcal{B} = \mathcal{M}_{2\times 2}$ and thus $\mathcal{B}$ is a generating set of $\mathcal{M}_{2\times 2}$
   (a) Show that Span $\mathcal{B} \subseteq \mathcal{M}_{2\times 2}$ earns 1 point.
   (b) Show that Span $\mathcal{B} \supseteq \mathcal{M}_{2\times 2}$ earns 1 point.
3. Earn 1 point if one state that $\mathcal{B}$ is a basis of $\mathcal{M}_{2\times 2}$ since $\mathcal{B}$ is a linearly independent and also a generating set for $\mathcal{M}_{2\times 2}$.

(b) Given $\mathcal{B}$, we have $[T]_{\mathcal{B}} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ and

$$\mathbf{a}_1 = [T(\mathbf{b}_1)]_{\mathcal{B}} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right]_{\mathcal{B}} = [\mathbf{b}_1 + 3\mathbf{b}_2 - \mathbf{b}_3 + 3\mathbf{b}_4]_{\mathcal{B}} = [1\ 3\ \ -1\ 3]^T,$$

$$\mathbf{a}_2 = [T(\mathbf{b}_2)]_{\mathcal{B}} = \left[\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \right]_{\mathcal{B}} = [2\mathbf{b}_1 + 4\mathbf{b}_2 - 3\mathbf{b}_3 + \mathbf{b}_4]_{\mathcal{B}} = [2\ 4\ \ -3\ 1]^T,$$

$$\mathbf{a}_3 = [T(\mathbf{b}_3)]_{\mathcal{B}} = \left[\begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \right]_{\mathcal{B}} = [0\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 + 3\mathbf{b}_4]_{\mathcal{B}} = [0\ 0\ 1\ 3]^T,$$

$$\mathbf{a}_4 = [T(\mathbf{b}_4)]_{\mathcal{B}} = \left[\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \right]_{\mathcal{B}} = [0\mathbf{b}_1 + 0\mathbf{b}_2 + 2\mathbf{b}_3 + 4\mathbf{b}_4]_{\mathcal{B}} = [0\ 0\ 2\ 4]^T.$$

The matrix representation of the linear transformation T with respect to the basis $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -3 & 1 & 2 \\ 3 & 1 & 3 & 4 \end{bmatrix}. \tag{7}$$

Grading policy:

1. Correctly answer (7) earns 5 points.
2. No points are awarded for incorrect answers, regardless of the explanations or calculations provided.

(c) Through the Gram-Schmidt process, we can obtain the **orthonormal** basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ with respect to the Frobenius inner product. By Gram-Schmidt process, we can have

$$\mathbf{x}_1 = \mathbf{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{8}$$

$$\mathbf{x}_2 = \mathbf{b}_2 - \frac{\langle\, \mathbf{b}_2, \mathbf{x}_1 \rangle}{||\mathbf{x}_1||^2}\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{9}$$

$$\mathbf{x}_3 = \mathbf{b}_3 - \frac{\langle\, \mathbf{b}_3, \mathbf{x}_2 \rangle}{||\mathbf{x}_2||^2}\mathbf{x}_2 - \frac{\langle\, \mathbf{b}_3, \mathbf{x}_1 \rangle}{||\mathbf{x}_1||^2}\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}, \tag{10}$$

$$\mathbf{x}_4 = \mathbf{b}_4 - \frac{\langle\, \mathbf{b}_4, \mathbf{x}_3 \rangle}{||\mathbf{x}_3||^2}\mathbf{x}_3 - \frac{\langle\, \mathbf{b}_4, \mathbf{x}_2 \rangle}{||\mathbf{x}_2||^2}\mathbf{x}_2 - \frac{\langle\, \mathbf{b}_4, \mathbf{x}_1 \rangle}{||\mathbf{x}_1||^2}\mathbf{x}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - 0 - 0 - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}. \tag{11}$$

We obtain $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ as an orthogonal basis of $\mathcal{B}$ with respect to the Frobenius inner product. Normalize $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and $\mathbf{x}_4$, we have

$$\mathbf{v}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{12}$$

$$\mathbf{v}_2 = \frac{\mathbf{x}_2}{||\mathbf{x}_2||} = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{13}$$

$$\mathbf{v}_3 = \frac{\mathbf{x}_3}{||\mathbf{x}_3||} = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{14}$$

$$\mathbf{v}_4 = \frac{\mathbf{x}_4}{||\mathbf{x}_4||} = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{15}$$

Lastly, we obtian $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ as an orthonormal basis of $\mathcal{B}$ with respect to the Frobenius inner product.

Grading policy:

1. Correctly answer (12), (13), (14) and (15) earns 6 points.
2. Deduct 2 points for each incorrect answer to (12), (13), (14), or (15), until the score reaches 0 points.
3. If you only provide the orthogonal basis without normalizing it
   (a) Correctly answer (8), (9), (10) and (11) earns 4 points.
   (b) Deduct 1 points for each incorrect answer to (8), (9), (10), or (11), until the score reaches 0 points.

(d) Find the projection of $\mathbf{h} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to $\mathbf{W} = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$. We have

$$\begin{aligned} \text{Proj}_{\text{W}}(\mathbf{h}) &= \frac{\langle\ \mathbf{h}, \mathbf{b}_1\rangle}{||\mathbf{b}_1||^2}\mathbf{b}_1 + \frac{\langle\ \mathbf{h}, \mathbf{b}_2\rangle}{||\mathbf{b}_2||^2}\mathbf{b}_2 && \text{(16a)} \\ &= \frac{5}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} && \text{(16b)} \\ &= \frac{5}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. && \text{(16c)} \end{aligned}$$

Grading policy:

1. Correctly answer (16c) gets 5 points.
2. If one correctly write (16a) but fail to obtain (16c) due to calculation errors, you will still earn 2 points.

5. (10%) Let A be an $m \times n$ matrix such that $m < n$ and rank $A = m$. It is known that $A\mathbf{x}$=$\mathbf{b}$, where $\mathbf{b} \in \mathcal{R}^m$, has infinitely many solutions $\mathbf{x} \in \mathcal{R}^n$.

   (a) (5%) Prove that Null AA^T = Null A^T.

   (b) (5%) Let $\mathbf{x}_0 = A^T(AA^T)^{-1}\mathbf{b}$, which is one of the solutions of $A\mathbf{x}$=$\mathbf{b}$. Prove that $||\mathbf{x}_0|| \leq ||\mathbf{x}||$ for every $\mathbf{x}$ satisfying $A\mathbf{x}$=$\mathbf{b}$.
   (Hint: First prove that $\mathbf{x}_0$ and $\mathbf{x}$-$\mathbf{x}_0$ are orthogonal. Then apply the Pythagorean theorem.)

Ans.

(a) (i) Prove that Null $A^T \subseteq$ Null AA^T:

$$\forall \mathbf{v} \in \text{Null } A^T,\ A^T\mathbf{v} = \mathbf{0} \ \Rightarrow\ AA^T\mathbf{v} = A(A^T\mathbf{v}) = \mathbf{0} \ \Rightarrow\ \mathbf{v} \in \text{Null } AA^T \ \Rightarrow\ \text{Null } A^T \subseteq \ \text{Null } AA^T$$

(ii) Prove that Null $AA^T \subseteq$ Null A^T:

$$\forall \mathbf{v} \in \text{Null } AA^T,\ AA^T\mathbf{v} = \mathbf{0} \ \Rightarrow\ \mathbf{v}^T AA^T\mathbf{v} = 0 \ \Rightarrow\ (A^T\mathbf{v})^T A^T\mathbf{v} = \|A^T\mathbf{v}\|^2 = 0$$

$$\Rightarrow\ \|A^T\mathbf{v}\| = 0 \ \Rightarrow\ A^T\mathbf{v} = \mathbf{0} \ \Rightarrow\ \mathbf{v} \in \text{Null } A^T \ \Rightarrow\ \text{Null } AA^T \subseteq \text{Null } A^T$$

Since Null $A^T \subseteq$ Null AA^T and Null $AA^T \subseteq$ Null A^T, Null AA^T = Null A^T.

Grading policy:

- 2 points will be deducted if only (i) or (ii) were correctly proved.
- If prove by showing that Null AA^T and Null A^T are both $\{\mathbf{0}\}$, the derivations must contain sufficient details / justifications. Otherwise some points will be deducted.

(b) (i) Prove that $\mathbf{x}_0$ and $\mathbf{x} - \mathbf{x}_0$ are orthogonal:

$$\mathbf{x}_0^T = (A^T(AA^T)^{-1}\mathbf{b})^T = \mathbf{b}^T((AA^T)^{-1})^T(A^T)^T = \mathbf{b}^T((AA^T)^T)^{-1}A = \mathbf{b}^T(AA^T)^{-1}A.$$

$$\mathbf{x}_0^T(\mathbf{x} - \mathbf{x}_0) = \mathbf{b}^T(AA^T)^{-1}A(\mathbf{x} - \mathbf{x}_0) = \mathbf{b}^T(AA^T)^{-1}(A\mathbf{x} - A\mathbf{x}_0) = \mathbf{b}^T(AA^T)^{-1}(\mathbf{b} - \mathbf{b}) = 0$$

Therefore $\mathbf{x}_0$ and $\mathbf{x} - \mathbf{x}_0$ are orthogonal.

(ii) By Pythagorean theorem:

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_0 + (\mathbf{x} - \mathbf{x}_0)\|^2 = \|\mathbf{x}_0\|^2 + \|\mathbf{x} - \mathbf{x}_0\|^2 \Rightarrow \|\mathbf{x}_0\|^2 \leq \|\mathbf{x}\|^2 \Rightarrow \|\mathbf{x}_0\| \leq \|\mathbf{x}\|$$

Grading policy:

- Correctly prove (i) earns 3 points.
- Correctly prove (ii) earns 2 points.
- Partially correct proof might still receive some points if it covers at least some of the key parts of the solution.

6. (40%) Throughout this problem, we consider the vector space $\mathcal{R}^n$ endowed with the usual dot product and its induced norm. Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ (counted with multiplicity). We denote the corresponding eigenvectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ and they form an orthonormal basis for $\mathcal{R}^n$. Answer the following questions. You may use the theorems or certain properties taught in class.

(a) (5%) Show that the matrix A can be written in the following form:

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^T. \tag{17}$$

(Hint: For any symmetric matrix A, there exist a diagonal matrix D and an orthogonal matrix P such that $A = PDP^{-1}$.)

Ans. Since the set of eigenvectors $\{\mathbf{v}_j\}_{j=1}^n$ forms a basis of $\mathcal{R}^n$, Theorem 5.2 of the textbook implies that $A = PDP^{-1}$, where D is a diagonal matrix with k-th diagonal entry being λ_k and

$$P := \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Furthermore, since $\{\mathbf{v}_j\}_{j=1}^n$ is orthonormal, the matrix P is an orthogonal matrix by definition. Theorem 6.9 of the textbook implies that $P^{-1} = P^T$. Then, by matrix multiplication formula, we have

$$\begin{aligned}
A &= PDP^T && \text{(18a)}\\
&= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}^T && \text{(18b)}\\
&= \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}^T && \text{(18c)}\\
&= \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^T && \text{(18d)}
\end{aligned}$$

as claimed.

Grading policy:

1. Correctly show $A = \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^T$ by (18) earns 5 points.
2. Any errors in your answers will result in a deduction of at least 1 point.

(b) (5%) Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be a unit vector in $\mathcal{R}^n$. Calculate

$$\sqrt{\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_j)^2}.$$

Express your answer in the simplest form.

Ans. For any $\mathbf{x} \in \mathcal{R}^n$, we have the following representation with respect to the orthonormal basis $\{\mathbf{v}_j\}_{j=1}^n$ (see for example p. 376 of the textbook):

$$\mathbf{x} = \sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_j)\mathbf{v}_j. \tag{19}$$

Since $\mathbf{x}$ is a unit vector, the orthonormal basis $\{\mathbf{v}_j\}_{j=1}^n$ guarantees that

$$1 = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_i)(\mathbf{x} \cdot \mathbf{v}_j)\mathbf{v}_i \cdot \mathbf{v}_j} = \sqrt{\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_j)^2} = 1. \tag{20}$$

Grading policy:

1. Correctly obtain $\sqrt{\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_j)^2} = 1$ by calculating (20) earns 5 points.
2. Failing to answer $\sqrt{\sum_{j=1}^{n} (\mathbf{x} \cdot \mathbf{v}_j)^2} = 1$ but correctly writing (19) earns 2 points.

(c) (5%) Consider a function

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \quad \mathbf{x} \in \mathcal{R}^n.$$

Show that there exists a unit vector $\mathbf{x}_1 \in \mathcal{R}^n$ such that $f(\mathbf{x}_1) \geq f(\mathbf{x})$ for all unit vectors $\mathbf{x} \in \mathcal{R}^n$. Find an instance of $\mathbf{x}_1$ and the corresponding $f(\mathbf{x}_1)$.

Ans. Using (17), we have

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{x}^T \sum_{j=1}^{n} \lambda_j \mathbf{v}_j \mathbf{v}_j^T \mathbf{x} && (21a) \\ &= \sum_{j=1}^{n} \lambda_j (\mathbf{x}^T \mathbf{v}_j)^2 && (21b) \\ &\leq \sum_{j=1}^{n} \lambda_1 (\mathbf{x}^T \mathbf{v}_j)^2 && (21c) \\ &= \lambda_1, && (21d) \end{aligned}$$

where the last line follows from (20) in Problem (b). This proves the first part.

Using (17) and the orthonormal basis $\{\mathbf{v}_j\}_{j=1}^n$, the eigenvector $\mathbf{v}_1$ saturates the above inequality, i.e. $\mathbf{x}_1 = \mathbf{v}_1$ and $f(\mathbf{x}_1) = f(\mathbf{v}_1) = \lambda_1$.

Grading policy:

1. Answering $f(\mathbf{x}_1) = \lambda_1$ and $\mathbf{x}_1 = \mathbf{v}_1$ along with providing correct reasons (e.g. using (21)) earns 5 points.
2. Provide correct reasons (e.g. using (21)) but fail to answer $f(\mathbf{x}_1) = \lambda_1$ and $\mathbf{x}_1 = \mathbf{v}_1$ earns 3 points.
3. Correctly answer $f(\mathbf{x}_1) = \lambda_1$ or $\mathbf{x}_1 = \mathbf{v}_1$ without providing reasons earns 1 points each.

(d) (5%) Let V be a subspace of $\mathcal{R}^n$. We say that V is an invariant subspace of an $n \times n$ matrix B if

$$B\mathbf{x} \in V, \quad \forall\, \mathbf{x} \in V.$$

Now suppose a subspace V of $\mathcal{R}^n$ is an invariant subspace of an $n \times n$ symmetric matrix A. Prove that its orthogonal complement, i.e., $V^\perp$, is also an invariant subspace of A.

Ans. We need to show that $A\mathbf{x} \in V^\perp$ for all $\mathbf{x} \in V^\perp$. Indeed, for all $\mathbf{y} \in V$,

$$(A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x}^T A\mathbf{y} = 0, \tag{22}$$

because V is an invariant subspace of A, i.e., $A\mathbf{y} \in V$, and recalling that $\mathbf{x} \in V^\perp$. This proves the claim.

Grading policy:

1. Complete the proof with (22) earn 5 points.
2. Any errors in your answers will result in a deduction of at least 1 point.

(e) (5%) For every integer $k \in \{1, 2, \dots, n\}$, find a subspace V of $\mathcal{R}^n$ with dimension $(n - k + 1)$ such that

$$\lambda_k = \max_{\mathbf{x} \in V : \|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x}. \tag{23}$$

Justify your answer.

Ans. The case of $k = 1$ has been shown in (c), where $V = \mathcal{R}^n$. For the case of $k = 2$, we may want to choose any subspace that is orthogonal to Span $\{\mathbf{v}_1\}$. On the other hand, Span $\{\mathbf{v}_1\}$ is an invariant subspace of A because $\mathbf{v}_1$ is an eigenvector of A. Plus, (d) implies that (Span $\{\mathbf{v}_1\})^\perp$ is also an invariant subspace of A. Hence, we choose the $(n-1)$-dimensional subspace $V = (\text{Span}\,\{\mathbf{v}_1\})^\perp = \text{Span}\,\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$. Applying the argument employed in (c) but now restricting the underlying vector space $\mathcal{R}^n$ to V, it holds that $\lambda_2 = \max_{\mathbf{x} \in V : \|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x}$.

For every integer k, we then choose an $(n-k+1)$-dimensional subspace $V = \text{Span}\,\{\mathbf{v}_k, \dots, \mathbf{v}_n\}$ so that $\lambda_k = \max_{\mathbf{x} \in V : \|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x}$ by following the similar reasoning.

Grading policy:

1. Answer a $(n - k + 1)$-dimensional subspace $V = \text{Span}\{\mathbf{v}_k, ..., \mathbf{v}_n\}$ and correctly justify (23) by using V (5 points).
2. Answer a $(n-k+1)$-dimensional subspace $V = \text{Span}\{\mathbf{v}_k, ..., \mathbf{v}_n\}$ but not correctly justify (23) by using V (3 points).

(f) (5%) Let k be an integer, K be an k-dimensional subspace of $\mathcal{R}^n$, and V be an $(n-k+1)$-dimensional subspace of $\mathcal{R}^n$. Prove that $K \cap V$ is a nonzero subspace of $\mathcal{R}^n$.

(Hint: Recall that $\mathcal{R}^n$ can only have at most n linearly independent vectors.)

Ans. We first show that $K \cap V$ is a subspace of $\mathcal{R}^n$. Firstly, $\mathbf{0} \in K \cap V$ since $\mathbf{0} \in K$ and $\mathbf{0} \in V$. For all $\mathbf{x}_1, \mathbf{x}_2 \in K \cap V$ and real scalars c_1 and c_2, we have

$$\begin{cases} c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \in K \\ c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \in V \end{cases},$$

because K and V are subspaces. Then, $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 \in K \cap V$, showing that $K \cap V$ is a subspace.

We pick k linearly independent vectors in K, say $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\} \subset K$, and pick $(n-k+1)$ linearly independent vectors in V, say $\{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{n-k+1}\} \subset V$. Since $\mathcal{R}^n$ can only have at most n linearly independent vectors, the union $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\} \cup \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{n-k+1}\}$ is linearly dependent. Then, there exists a set of non-all-zero scalars $\{a_1, a_2, ..., a_k\}$ and $\{b_1, b_2, ..., b_{n-k+1}\}$ such that $\sum_{i=1}^{k} a_i\mathbf{x}_i + \sum_{j=1}^{n-k+1} b_j\mathbf{y}_j = \mathbf{0}$. Then, $\sum_{i=1}^{k} a_i\mathbf{x}_i = -\sum_{j=1}^{n-k+1} b_j\mathbf{y}_j \neq \mathbf{0}$ because, by construction, linear independence of the set $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{n-k+1}\}$ implies $\sum_{j=1}^{n-k+1} b_j\mathbf{y}_j \neq \mathbf{0}$. Note that the left-hand side $\sum_{i=1}^{k} a_i\mathbf{x}_i$ belongs to K and the right hand side $\sum_{j=1}^{n-k+1} b_j\mathbf{y}_j$ belongs to V. This means that $K \cap V$ contains a nonzero vector $\sum_{i=1}^{k} a_i\mathbf{x}_i = -\sum_{j=1}^{n-k+1} b_j\mathbf{y}_j$, proving the claim.

Grading policy:
Part 1: $K \cap V$ is a subspace of $\mathcal{R}^n$ (2 points).

1. No criteria (implicitly) verified (0 points).
2. Other errors (e.g. incomplete statement, typo, conclusion missing...) (1 point).
3. Flawless (2 points).

Part 2: $K \cap V$ is nonzero (or nontrivial) (3 points). This part can be proved by contradiction, which is annotated in the following grading.

1. Invalid justification (e.g. wrong inference or unclear inference with no direct connections to the result) (0 points).
2. No substantial explanation of why the union of 2 linearly independent sets in K and L is still a linearly independent set in $\mathcal{R}^n$ if $K \cap V = \{\mathbf{0}\}$ (proof by contradiction) (1 point).
3. No substantial explanation of why a set of $n+1$ linearly independent vectors in $\mathcal{R}^n$ leads to a contradiction. (proof by contradiction) (1 point).
4. No substantial explanation of why the union of 2 linearly independent sets in K and L is a linearly dependent set in $\mathcal{R}^n$ (1 point).
5. No justification of why it is impossible that $K \cap V = \{\mathbf{0}\}$ given the union of 2 linearly independent sets in K and L is linearly dependent in $\mathcal{R}^n$ (1 point).
6. Other errors (e.g. incomplete statement, typo, conclusion missing...) (2 points)
7. Flawless (3 points).

Additional Note:

1. Only the trivial case $n = 1$ is proved with no scores in previous parts (1 point).

(g) (10%) Prove that for every integer $k \in \{1, 2, \ldots, n\}$,

$$\lambda_k = \min_{V \subseteq \mathcal{R}^n :\, \dim V = n-k+1} \ \max_{\mathbf{x} \in V :\, \|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x}. \tag{24}$$

Here, the minimization is over all $(n-k+1)$-dimensional subspaces of $\mathcal{R}^n$.

Ans. Problem (e) already showed the inequality '$\geq$' in (24). To prove the converse, we need to show that for any $(n-k+1)$-dimensional subspace V of $\mathcal{R}^n$, there exists a unit vector $\mathbf{x} \in V$ satisfying $\lambda_k \leq \mathbf{x}^T A \mathbf{x}$. To that end, we would like to find a unit vector $\mathbf{x} \in V$ of the form:

$$\mathbf{x} = \sum_{j=1}^{k} c_j \mathbf{v}_j \tag{25}$$

with $\sum_{j=1}^{k} c_j^2 = 1$ (by Problem (b)).If so, then

$$\mathbf{x}^T A \mathbf{x} = \sum_{j=1}^{k} \lambda_j c_j^2 \geq \sum_{j=1}^{k} \lambda_k c_j^2 = \lambda_k.$$

Since the $(n-k+1)$-dimensional subspace V of $\mathcal{R}^n$ is arbitrary, we have shown

$$\begin{aligned} \lambda_k &\leq \inf_{V \subseteq \mathcal{R}^n :\, \dim V = n-k+1} \ \sup_{\mathbf{x} \in V :\, \|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x} \\ &= \min_{V \subseteq \mathcal{R}^n :\, \dim V = n-k+1} \ \max_{\mathbf{x} \in V :\, \|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x}, \end{aligned} \tag{26}$$

where supremum in the last line can be attained because of the Extreme Value Theorem, the fact that $\mathbf{x} \mapsto \mathbf{x}^T A \mathbf{x}$ is a continuous function, and that the unit sphere $\mathbf{x} \in V :\, \|\mathbf{x}\| = 1$ is compact. The infimum can be attained because of the existence shown in Problem (e).(Note: One does not have to show that the inf-sup can be attained. Writing min-max in (26) is just fine.)

Finally, it remains to show that a vector $\mathbf{x} \in V$ of the form in (25) does exist. Indeed, the subspace $K := \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is of dimension k. By the result of Problem (f),we have $K \cap V \neq \{\mathbf{0}\}$. Hence, we can choose such a unit vector $\mathbf{x} \in K \cap V \subset V$, concluding the proof.

Grading policy :
Part 1: for any $(n-k+1)$-dimensional subspace V of $\mathcal{R}^n$, there exists $\mathbf{x} = \sum_{j=1}^{k} c_j \mathbf{v_j} \in V$ with $\sum_{j=1}^{k} c_j^2 = 1$. (6 points)

1. Invalid justification (e.g. wrong inference or unclear inference with no direct connections to the result) (0 points)
2. More than 1 implicit statement (3 points)
3. 1 implicit statement (4 points)
4. Typo (5 points)
5. Flawless (6 points)

Part 2: $\lambda_k \leq \mathbf{x}^T A \mathbf{x}$ (3 points).

1. No clear mention of Part 1 statement (0 points)
2. Wrong derivation or no derivation (0 points)
3. Implicit statement (1 point)
4. Typo (2 points)
5. Flawless (3 points)

Part 3: There exists a subspace V of $\mathcal{R}^n$ with dimension $(n-k+1)$ such that $\lambda_k = \max_{x \in V:||x||=1} \mathbf{x}^T A \mathbf{x}$, that is, Problem 6(e). (1 point)

1. No clear mention of statements for each part (0 points)
2. Otherwise (1 point)

Additional Note:

1. Only the trivial case $n = 1$ is proved with no scores in previous parts (1 point)