HW 4: Stability, Controllability, Observability	Digital Control Systems, Spring 2021, NTU-EE
Name: 参考答案	Date: 4/24, 2021

Problem 4-1:

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

The characteristic equation of the closed-loop system is:

$$z(z-0.2)(z-0.4)+K=0 \quad (1.1)$$

where $K>0$

Using root locus, starting points are z=0, 0.2, 0.4.

To find the crossing point that roots cross the **unit circle**, let:

$$z=a+bi \qquad where\ \ a^2+b^2=1 \quad (1.2)$$

Apply Equation (1.2) to Equation (1.1), we can obtain:

$$(a+bi)(a+bi-0.2)(a+bi-0.4)=-K \quad (1.3)$$

Multiply with $(a-bi)$, and reorganize Equation (1.3):

$$a^2-b^2-0.6a+0.08+(2ab-0.6b)i=-K(a-bi) \quad (1.4)$$

Reorganize Equation (1.4), separate real and imaginary parts:

$$\begin{cases} a^2-b^2-0.6a+0.08=-Ka & (1.5) \\ b(2a-0.6)=Kb & (1.6) \end{cases}$$

For the case when $b\neq 0$, replace b^2 in Equation (1.5) by $b^2=1-a^2$, we get Equation (1.7)

$$a^2-(1-a^2)^2-0.6a+0.08=-a(2a-0.6) \quad (1.7)$$

Therefore,

$$a^2-0.3a-0.23=0 \quad (1.8)$$

So,

$$a\approx 0.652\ \ or-o.352 \quad (1.9)$$

From Equation (1.6), and apply Equation (1.9):

$$K=2a-0.6\approx 0.704\ \ or-1.304 \quad (1.10)$$

For another case that root locus may also cross the unit circle for $b = 0$ and $a = \pm 1$.

Follow Equation (1.1), when $b = 0, a = -1$

$-1(-1-0.2)(-1-0.4) + K = 0$ $K = 1.68$	(1.11)

Follow Equation (1.1), when $b = 0, a = 1$

$K = -0.48$	(1.12)

Therefore, consider Equation (1.10) and Equation (1.11) and $K > 0$:

$\begin{cases} K > 0 \\ K \le 0.704 \\ K \le 1.68 \end{cases}$	(1.13)

We can conclude the answer:

$\mathbf{0 < K \le 0.704}$	**Ans. of Pro. 4-1**

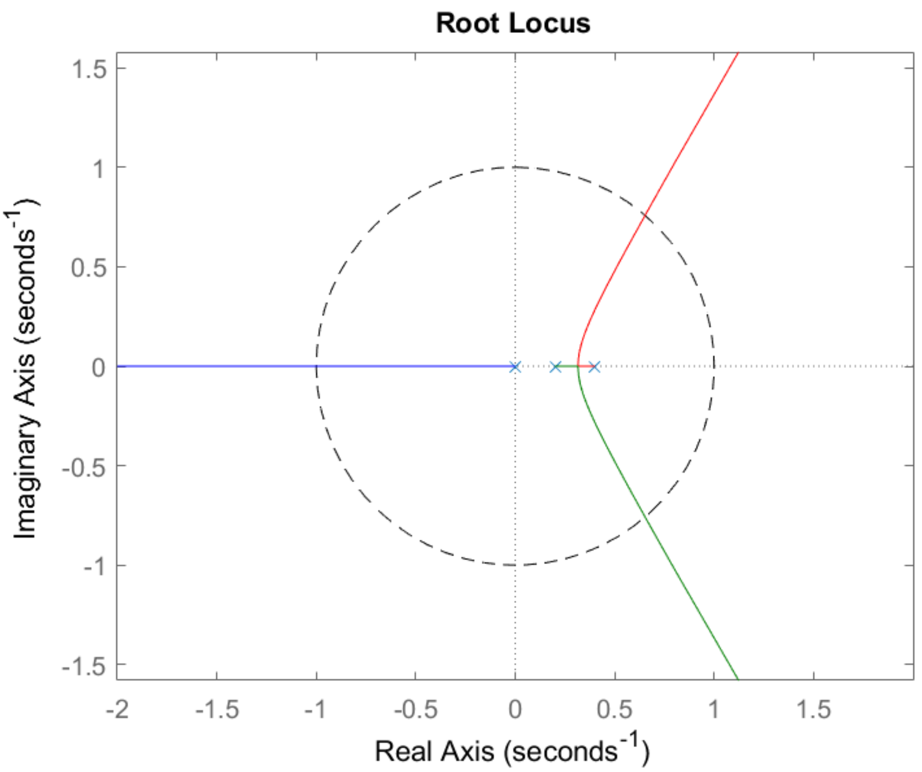


Figure 4-1. Root locus of problem 4-1 for K>0, and the unit disc in dashed line.

Problem 4-2:

[Ref: HW4_楊軒昱 楊軒昱_R07921006_DCS_HW4_Stability, Controllability, Observability]

As we know, the discrete time state-space model can be written like below:

$$x[k+1] = Fx[k] + Hu[k] \\ y[k] = Cx[k] + Du[k] \tag{2.1}$$

Compare Equation (2.1) to the question, we can get:

$$F = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix} \\ H = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \\ C = \begin{bmatrix} 2 & -4 \end{bmatrix} \tag{2.2}$$

(a)

To check whether the state-space model is stable, we have to find the eigenvalue of matrix F in Equation (2.2), which means we have to solve

$$\lambda I - F = 0 \tag{2.3}$$

And we can get

$$\lambda = \ 0.5, 0.25 \tag{2.4}$$

Because **the magnitude of both eigenvalues are smaller than 1**, which means **the eigenvalues are inside the unit circle**. **Hence, the system is stable according to Theorem 3.1 on page 11 of the lecture note: 107-2_dcs22_Stability.pdf.**

(b)

To check whether the state-space model is observable, we can check it by calculating W_o in Equation (2.5)

$$W_o = \begin{bmatrix} C \\ CF \end{bmatrix} = \begin{bmatrix} [2 \quad -4] \\ [2 \quad -4]\begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \qquad (2.5)$$

Because the **rank of W_o is 1**, which is **not full rank**, the **state-space model is not observable** **according to Theorem 3.8 on page 17 of the lecture note: 107-2_dcs23_ControllabilityObservability.pdf.**

(c)

To check whether the state-space model is controllable, we can check it by calculating W_c in Equation (2.6)

$$W_c = [H \quad FH] = \left[\begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix}\right] = \begin{bmatrix} 6 & 1 \\ 4 & 1 \end{bmatrix} \qquad (2.6)$$

Because the **rank of W_c is 2**, which is **full rank**, the **state-space model is controllable** **according to Theorem 3.7 on page 7 of the lecture note: 107-2_dcs23_ControllabilityObservability.pdf.**

Problem 4-3

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

(a)

The original state function is denoted in Equation (3.1)

$$x(k+1)=\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}x(k)+\begin{bmatrix}0\\1\\0\end{bmatrix}u(k)$$	(3.1)

Because we want to get a control sequence, first, we can take k = 0 and k = 1 into Equation (3.1), and we can get Equation (3.2) and Equation (3.3)

$$x(1)=\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}u(0)=\begin{bmatrix}3\\3+u(0)\\0\end{bmatrix}$$	(3.2)
$$x(2)=\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}\begin{bmatrix}3\\3+u(0)\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}u(1)=\begin{bmatrix}3+u(0)\\u(1)\\0\end{bmatrix}$$	(3.3)

Follow Equation (3.3), if we set $\begin{cases}\boldsymbol{u(0)=-3}\\\boldsymbol{u(1)=0}\end{cases}$, we can reach the origin from the initial state $x(0)$. That is, $x(2)=\begin{bmatrix}3+u(0)\\u(1)\\0\end{bmatrix}=\begin{bmatrix}3-3\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. So the answer for the control sequence is

$$\begin{cases}\boldsymbol{u(0)=-3}\\\boldsymbol{u(1)=0}\end{cases}$$	**Ans. of 4-3 (a)**

(b)

$$F^2=\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&3\\0&0&0\\0&0&0\end{bmatrix}$$	(3.4)
$$F^3=\begin{bmatrix}0&0&3\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$	(3.5)

In general, it would take 3 steps because it is a 3^{rd} order system.

However, we reach the origin by **2 steps** in this case.

(c) Let's check out the controllability,

$$W_c = [H \quad FH \quad F^2H] = \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

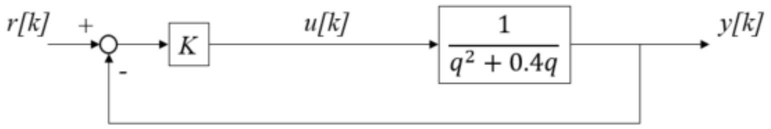
We can conclude from Equation (3.6) that W_c is **not full rank**, that is, this system is **uncontrollable, but may be controlled.**

$[1 \quad 1 \quad 1]^T$ is not in the column space of W_c and can therefore not be reached from the origin. It's easily seen from the state space description that $\boldsymbol{x_3}$ **can't be changed from its initial value**.

Problem 4-4

[Ref: HW4_林柏宇 林柏宇_DCS_HW04_20190426_stability, controllability, observability]

(a)



For closed-loop system, the transfer function is in Equation (4.1):

$$T = \frac{\frac{K}{q^2 + 0.4q}}{1 + \frac{K}{q^2 + 0.4q}} = \frac{K}{q^2 + 0.4q + K} \qquad (4.1)$$

The characteristic equation of the closed-loop system is in Equation (4.2):

$$q^2 + 0.4q + K = 0 \qquad (4.2)$$

Using **Jury's stability test** on Equation (4.2),

$$\begin{array}{ccc} 1 & 0.4 & K \\ K & 0.4 & 1 \\ \hline 1 - K^2 & & 0.4(1-K) \\ 0.4(1-K) & & 1 - K^2 \\ \hline 1 - K^2 - \frac{0.16(1-K)}{1+K} & & \end{array} \qquad (4.3)$$

The system is stable if $\begin{cases} 1 - K^2 > 0 \\ 1 - K^2 - \frac{0.16(1-K)}{1+k} > 0 \end{cases}$, that is,

$$\begin{cases} 1 - K^2 > 0 \\ (1-K)(1+K)(K+0.6)(K+1.4) > 0 \end{cases} \qquad (4.4)$$

Therefore, follow the Equation (4.3), we can obtain:

$$\begin{cases} -1 < K < 1 \\ -1.4 < K < -1 \;\; or - 0.6 < K < 1 \end{cases} \qquad (4.5)$$

Finally, we can conclude the answer in Equation (4.6):

$$\boldsymbol{-0.6 < K < 1} \qquad \textbf{Ans. of 4-4 a}$$

(b) Since $e(k) = r(k) - y(k)$, we can obtain:

$$E(z) = \big(1 - T(z)\big)R(z) \qquad (4.7)$$

When $r(k)$ is a step function, and $K = 0.5$, the system is stable, the final-value theorem can be used:

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}\frac{z-1}{z}E(z) = \lim_{z\to 1}\frac{z-1}{z}\left(1-\frac{K}{z^2+0.4z+K}\right)\left(\frac{z}{z-1}\right)$$ $$= \lim_{z\to 1}\frac{z^2+0.4z}{z^2+0.4z+K} = \frac{1.4}{1.4+K} = \mathbf{0.737}$$	**Ans. of 4-4 b**