Control System: Homework 10 for Units 6I-6K: Bode Plot

Assigned: Dec 9, 2022

Due: Dec 15, 2022 (11:59pm)

1. (Lead compensation)

46. For the system shown in Fig. 6.100, suppose that

$$G(s) = \frac{5}{s(s+1)(s/5+1)}.$$

Design a lead compensation $D(s)$ with unity DC gain so that PM $\geq 40°$ using Bode plot sketches. What is the approximate bandwidth of the system?

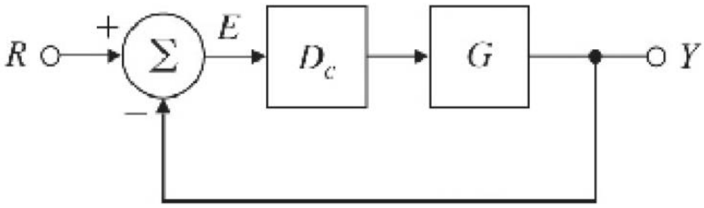


Figure 6.89: Fig. 6.100 Control system for Problem 6.46

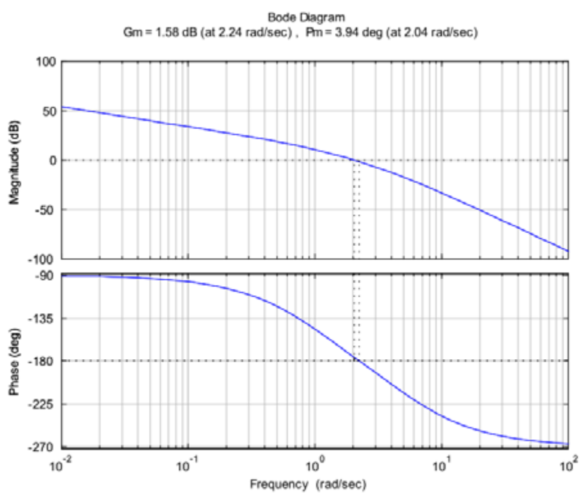
Solution :

Start with a lead compensator design with :

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$$

which has unity DC gain with $\alpha < 1$.

The Bode plot of the given system is :



Since $PM = 3.9°$, let's add phase lead $\geq 60°$. From Fig. 6.53,

$$\frac{1}{\alpha} \simeq 20 \Longrightarrow \text{choose } \alpha = 0.05$$

To apply maximum phase lead at $\omega = 10$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha}T} = 10 \Longrightarrow \frac{1}{T} = 2.2,\ \frac{1}{\alpha T} = 45$$

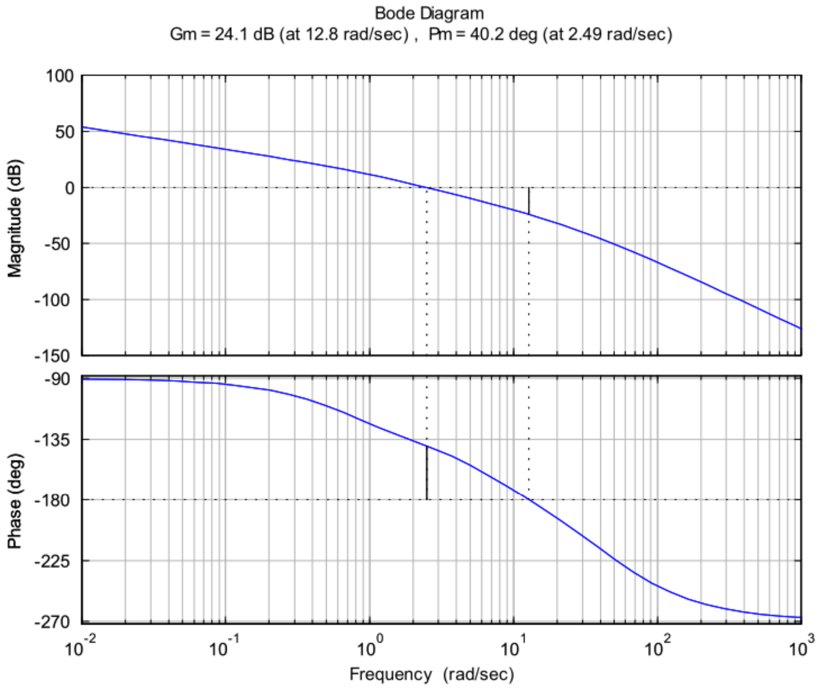
Therefore by applying the lead compensator :

$$D(s) = \frac{\frac{s}{2.2} + 1}{\frac{s}{45} + 1}$$

we get the compensated system with :

$$PM = 40°,\ \omega_c = 2.5$$

The Bode plot with designed compensator is :



From Fig. 6.50, we see that $\omega_{BW} \simeq 2 \times \omega_c \simeq 5$ rad/sec.

2. (Lag compensation)

56. For a system with open-loop transfer function $_c$

$$G(s) = \frac{10}{s[(s/1.4)+1][(s/3)+1]},$$

design a lag compensator with unity DC gain so that PM $\geq$35°. What is the approximate bandwidth of this system?

Solution :

Lag compensation design :

Use

$$D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$$

K=1 so that DC gain of $D_c(s) = 1$.

(a) Find the stability margins of the plant without compensation by plotting the Bode, find that:

$$\begin{aligned} PM &= -20^\circ \ (\omega_c = 3.0 \text{ rad/sec}) \\ GM &= 0.44 \ (\omega = 2.05 \text{ rad/sec}) \end{aligned}$$

(b) The lag compensation needs to lower the crossover frequency so that a $PM \simeq 35^\circ$ will result, so we see from the uncompensated Bode that we need the crossover at about

$$\Longrightarrow \omega_{c,new} = 1.$$

where

$$|G(j\omega_c)| \simeq 8.5$$

so the lag needs to lower the gain at $\omega_{c,new}$ from 7.5 to 1.

(c) Pick the zero breakpoint of the lag to avoid influencing the phase at $\omega = \omega_{c,new}$ by picking it a factor of 20 below the crossover, so

$$\frac{1}{T_D} = \frac{\omega_{c,new}}{20}$$

$$\Longrightarrow T_D = 20$$

(d) Choose α :

Since $D_c(j\omega) \cong \frac{1}{\alpha}$ for $\omega >> \frac{1}{T}$, let

$$\frac{1}{\alpha} = \frac{1}{|G(j\omega_{c,new})|}$$

$$\alpha = |G(j\omega_{c,new})| = 8.5$$

(e) Compensation :

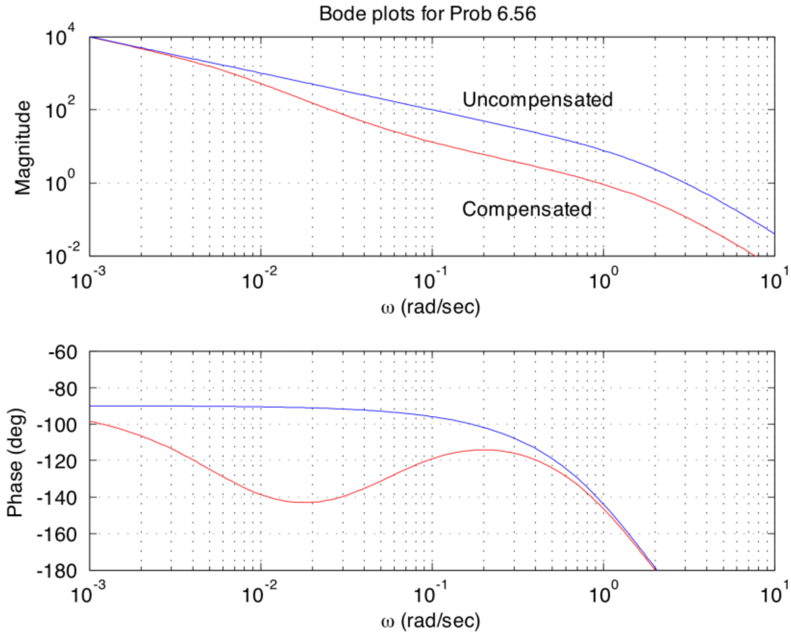
$$D_c(s) = \frac{\frac{s}{0.05} + 1}{\frac{s}{0.0059} + 1}$$

(f) Stability margins of the compensated system :

$$\begin{aligned} PM &= 36^\circ \ (\omega_c = 0.8 \text{ rad/sec}) \\ GM &= 3.6 \ (\omega = 2.0 \text{ rad/sec}) \end{aligned}$$

Approximate bandwidth ω_{BW}:

$$PM \cong 42^\circ \implies \omega_{BW} \cong 2\omega_c = 2 \text{ (rad/sec)}$$



3. (Lead-Lag compensation)

62. Consider the system in Fig. 6.100 with the plant transfer function

$$G(s) = \frac{10}{s(s/10+1)}.$$

We wish to design a compensator $D(s)$ that satisfies the following design specifications:

(a) i. $K_v = 100$,

ii. $\text{PM} \geq 45°$,

iii. sinusoidal inputs of up to 1 rad/sec to be reproduced with $\leq 2\%$ error,

iv. sinusoidal inputs with a frequency of greater than 100 rad/sec to be attenuated at the output to $\leq 5\%$ of their input value.

(b) Create the Bode plot of $G(s)$, choosing the open-loop gain so that $K_v = 100$.

(c) Show that a *sufficient* condition for meeting the specification on sinusoidal inputs is that the magnitude plot lies outside the shaded regions in Fig. 6.102. Recall that

$$\frac{Y}{R} = \frac{KG}{1+KG} \quad \text{and} \quad \frac{E}{R} = \frac{1}{1+KG}.$$

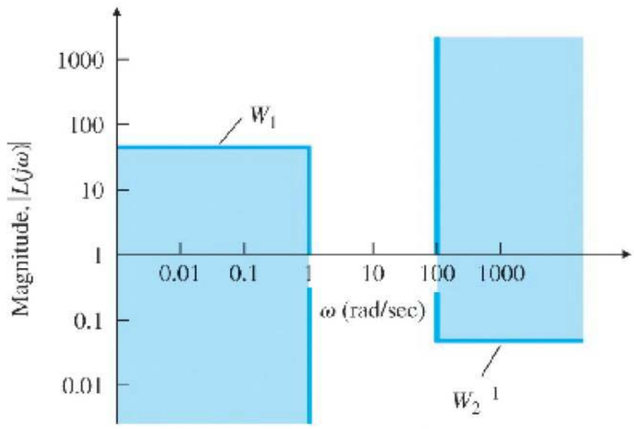


Fig. 6.102 Control system constraints for Problem 62

(d) Explain why introducing a lead network alone cannot meet the design specifications.

(e) Explain why a lag network alone cannot meet the design specifications.

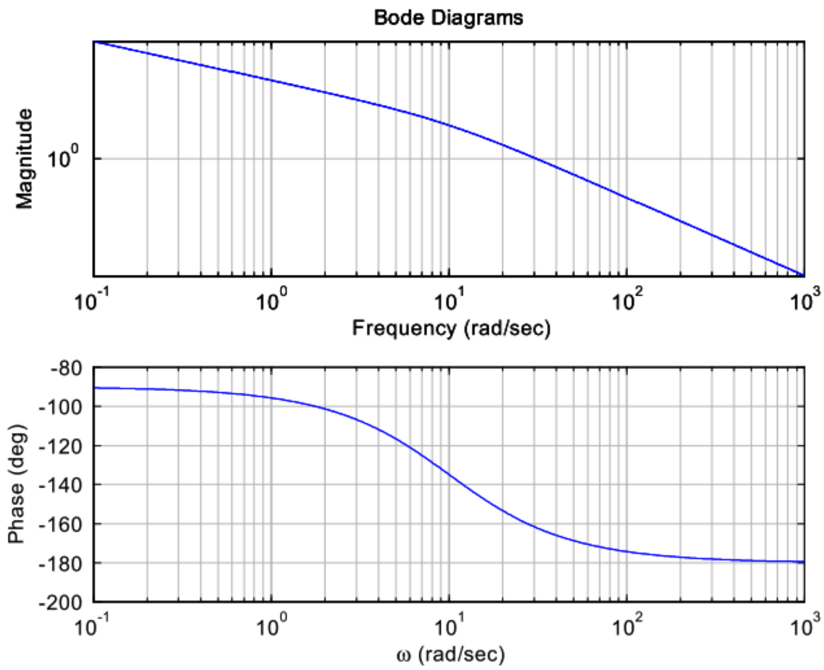
(f) Develop a full design using a lead-lag compensator that meets all the design specifications, without altering the previously chosen low frequency open-loop gain.

Solution :

(a) To satisfy the given velocity constant K_v,

$$K_v = \lim_{s\to 0} sKG(s) = 10K = 100$$
$$\implies K = 10$$

(b) The Bode plot of $G(s)$ with the open-loop gain $K = 10$ is :



(c) From the 3rd specification,

$$\left|\frac{E}{R}\right| = \left|\frac{1}{1+KG}\right| < 0.02\ (2\%)$$
$$\implies |KG| > 49 \text{ (at } \omega < 1 \text{ rad/sec)}$$

From the 4th specification,

$$\left|\frac{Y}{R}\right| = \left|\frac{KG}{1+KG}\right| < 0.05\ (5\%)$$
$$\implies |KG| < 0.0526 \text{ (at } \omega > 100 \text{ rad/sec)}$$

which agree with the figure.

(d) A lead compensator may provide a sufficient PM, but it increases the gain at high frequency so that it violates the specification above.

(e) A lag compensator could satisfy the PM specification by lowering the crossover frequency, but it would violate the low frequency specification, W_1.

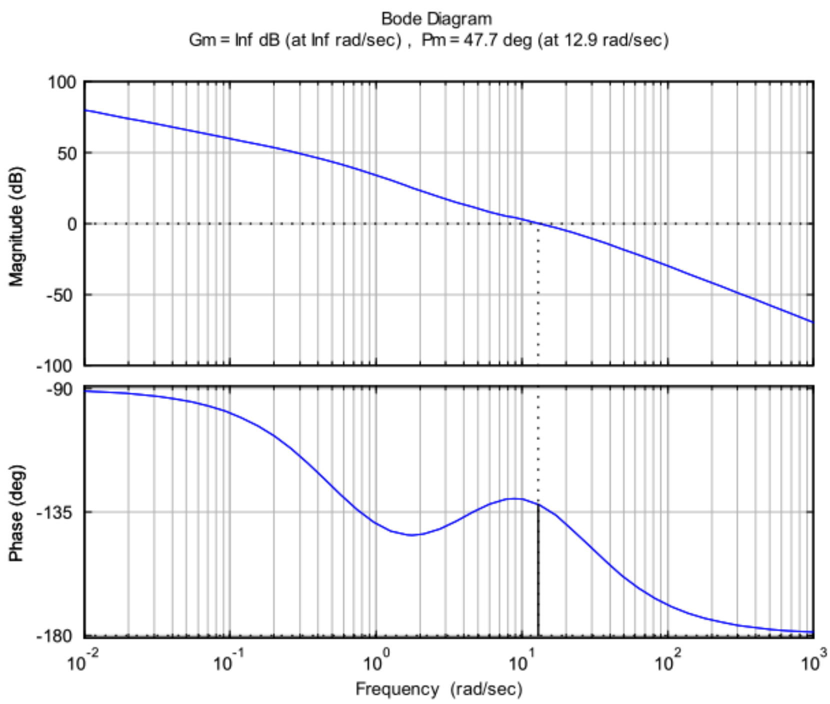
(f) One possible lead-lag compensator is :

$$D_c(s) = 100 \frac{\frac{s}{8.52}+1}{\frac{s}{22.36}+1} \frac{\frac{s}{4.47}+1}{\frac{s}{0.568}+1}$$

which meets all the specification :

$$\begin{aligned} K_v &= 100 \\ PM &= 47.7^\circ \text{ (at } \omega_c = 12.9 \text{ rad/sec)} \\ |KG| &= 50.45 \text{ (at } \omega = 1 \text{ rad/sec)} > 49 \\ |KG| &= 0.032 \text{ (at } \omega = 100 \text{ rad/sec)} < 0.0526 \end{aligned}$$

The Bode plot of the compensated open-loop system $D_c(s)G(s)$ is



參考觀摩的作業 1. (Lead compensation)
作者：b08901085，施彥宇 **理由**：用波德圖觀察不同 PM 的閉迴路系統設計與頻寬的關係
作者：b08901176，陳育楷 **理由**：用波德圖討論不同 lead compensator 設計得到的 PM 與實際預估值的差異
作者：b10202032，卓 然 **理由**：用波德圖、根軌跡圖以及步階響應觀察不同補償器造成的閉迴路表現

HW10 – Unit 6, Bode Plot

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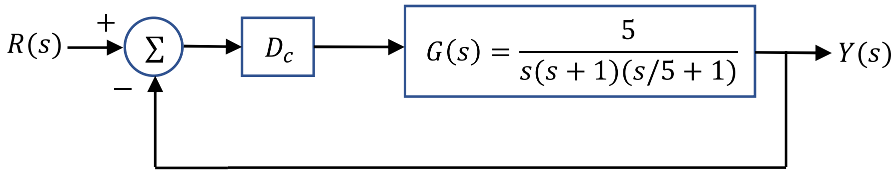
- **Question :**

For the system shown below, suppose that

$$G(s) = \frac{5}{s(s+1)(s/5+1)}$$

Design a lead compensation $D(s)$ with unity DC gain so that $PM > 40°$ using Bode plot sketches. What is the approximate bandwidth of the system?



- **Solution :**

Starting with a lead compensator with its transfer function :

$$D_c(s) = \frac{T_D s+1}{\alpha T_D s+1} \qquad (1).$$

with its DC gain is unity and $\alpha < 1$.

Using MATLAB to demonstrate the Bode plot of this system :

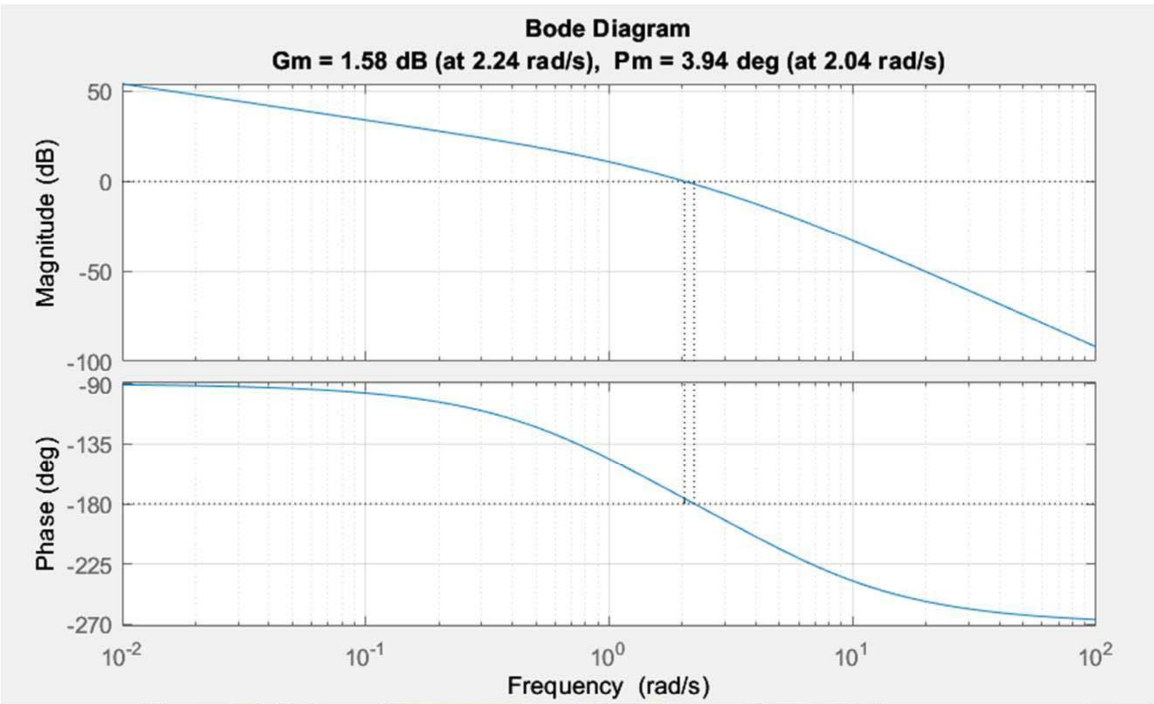


Fig. 1, Bode plot of opened-loop system in this question

One can find out that the phase margin of this system is only $3.9°$ at $\omega = 2.04\ \text{rad/s}$. To design a lead compensator such that phase margin becomes larger than $40°$, the maximum angle must larger than $36.1°$.

- **Solution :**

For safety, let's set the added phase to be about $60°$. Thus, α becomes :

$$\alpha = \frac{1-\sin 60°}{1+\sin 60°} = \frac{1-\sqrt{3}/2}{1+\sqrt{3}/2} = 7 - 4\sqrt{3} \cong 0.07 \qquad (2).$$

Choosing $\alpha = 0.05$. Another method of finding α is to use $Phase\ v.s.\alpha$ diagram, which is shown below :

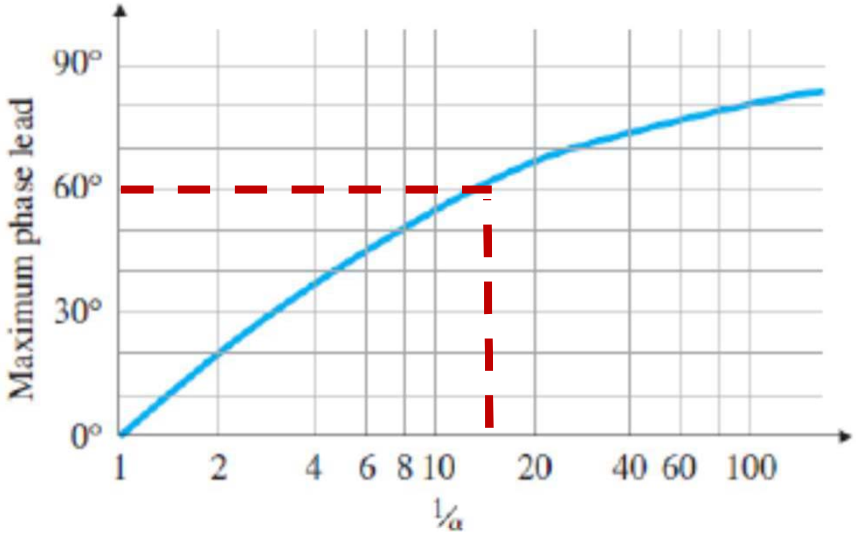


Fig. 2, $1/\alpha$ v.s. $phase$ diagram

From the diagram, $1/\alpha$ is about 15 to 16. Choose $1/\alpha$ to be 20, then $\alpha = 0.05$.

Supposing we want to increase phase at $\omega_c = 10\, rad/s$,

$$\omega_c = 10 = \frac{1}{\sqrt{\alpha T}} \Rightarrow \frac{1}{T} = 10\sqrt{\alpha} \cong 2.2\,, \frac{1}{\alpha T} \cong 45 \qquad (3).$$

Therefore, the lead compensator now becomes :

$$G_c(s) = \frac{T_D+1}{\alpha T_D s+1} = \frac{s/2.2+1}{s/45+1} \qquad (4).$$

As a result, the Bode plot of this system is :

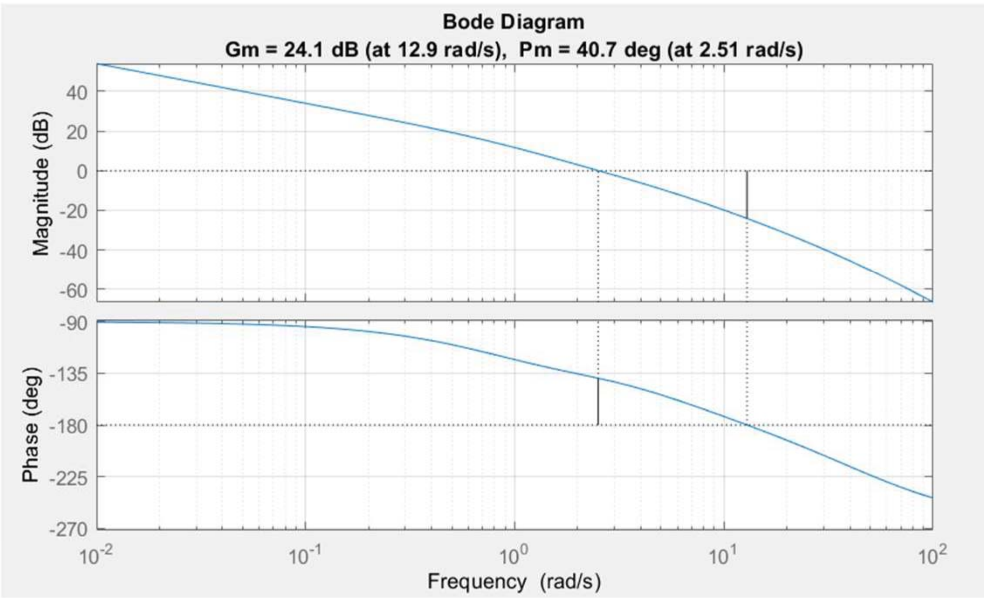


Fig. 3, Bode plot of this system with lead compensator is added

- **Solution :**

The phase margin is $40.7°$ at $\omega = 2.5\,rad/s$. One can find out that the maximum compensating degree must be larger than the target degree by $10°\sim20°$.

And we can approximate the bandwidth as :

$$\omega_{BW} \cong 2 \times \omega_c = 2 \times 2.5 = 5\,rad/s \quad (5).$$

- **What I can do more :**

We calculate the bandwidth by approximating. However, how come this approximation holds? Typically, a closed-loop system transfer function can be written as :

$$\frac{Y(j\omega)}{R(j\omega)} = \left|\frac{KG(j\omega)}{1+KG(j\omega)}\right|, K = 1;\ G(j\omega) = G(j\omega)D_C(j\omega) \quad (6).$$

One can observe that the typical behavior of $KG(j\omega)$ and crossover frequency is :

$$|KG(j\omega)| \gg 1 \quad \text{for} \quad \omega \ll \omega_c \quad (7).$$

$$|KG(j\omega)| \ll 1 \quad \text{for} \quad \omega \gg \omega_c \quad (8).$$

where ω_c is the crossover frequency. As a result, the closed-loop frequency-response magnitude is approximated by :

$$\left|\frac{Y(j\omega)}{R(j\omega)}\right| = |\mathcal{T}(j\omega)| = \left|\frac{KG(j\omega)}{1+KG(j\omega)}\right| \cong \begin{cases} 1 & ,\omega \ll \omega_c \\ |KG(j\omega)| & ,\omega \gg \omega_c \end{cases} \quad (9).$$

In the vicinity of crossover frequency, the closed-loop response depends heavily on phase margin. A phase margin of $90°$ means at the crossover frequency, $\angle G(j\omega_c) = -90°$. And $|\mathcal{T}(j\omega_c)| = \left|\frac{-j}{1-j}\right| = 0.707$. on the other hand, if a phase margin of $45°$, $\angle G(j\omega_c) = -135°$ and $|\mathcal{T}(j\omega_c)| = 1.3$.

The relationship of bandwidth and phase margin is demonstrated below :

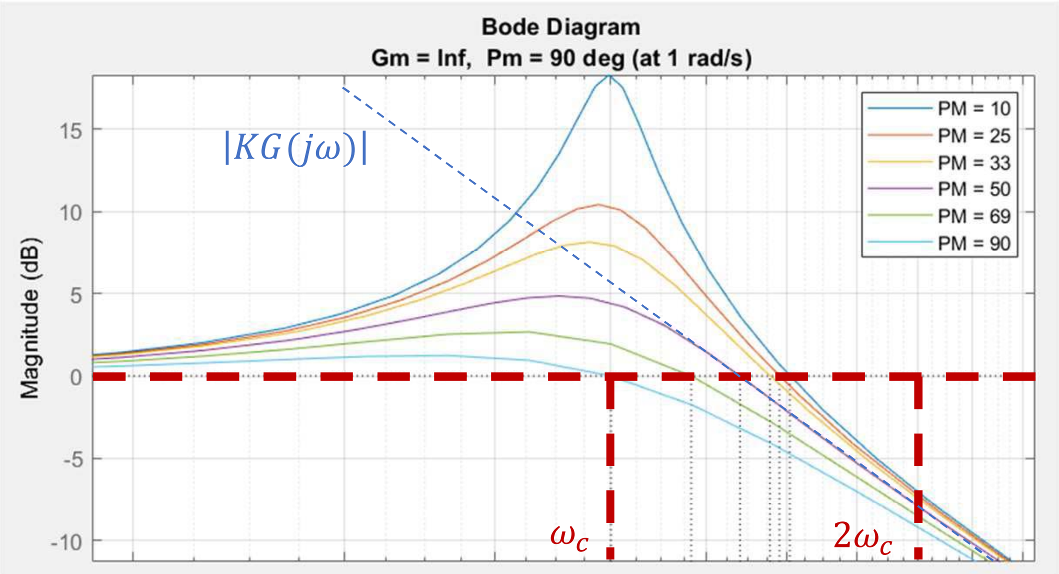


Fig. 4, Closed-loop bandwidth with respect to phase margin

- **What I can do more :**

The bandwidth is usually lower than $2\omega_c$. And it's inversely proportional to phase margin, which is, the lower the phase margin is, the closer the bandwidth to $2\omega_c$.

$$\omega_c \leq \omega_{BW} \leq 2\omega_c \qquad (10).$$

For the phase margin is $40°$ in this system, we can thus approximate the bandwidth to be $2\omega_c$.

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1 (Lead compensation)

Problem:

For the system shown in Fig. 1, suppose that $G(s) = \frac{5}{s(s+1)(s/5+1)}$ and $D(s) = \alpha\frac{s+T_D}{s+\alpha T_D}$.

Plot the Bode plot of open loop and calculate PM and GM for $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{100}$,

T_D = the rossover frequency of the system without lead compensation .

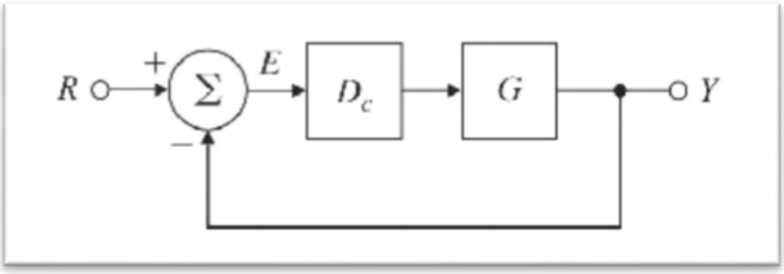

Fig. 1

Solution:

Table 1

PM and GM for $\alpha = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{100}$

$1/\alpha$	-	1/2	1/4	1/6	1/8	1/10	1/20	1/40	1/100
PM(deg)	3.94	17.1	28.9	34.0	36.7	38.4	41.9	43.7	44.8
$\omega_{cp}(\frac{rad}{s})$	2.04	2.35	2.51	2.55	2.56	2.57	2.58	2.59	2.59
GM(dB)	1.08	6.33	11.0	13.9	16.0	17.7	23.1	28.8	36.5
$\omega_{cg}(\frac{rad}{s})$	2.24	3.51	5.30	6.64	7.76	8.73	12.5	17.8	28.3

Plot:

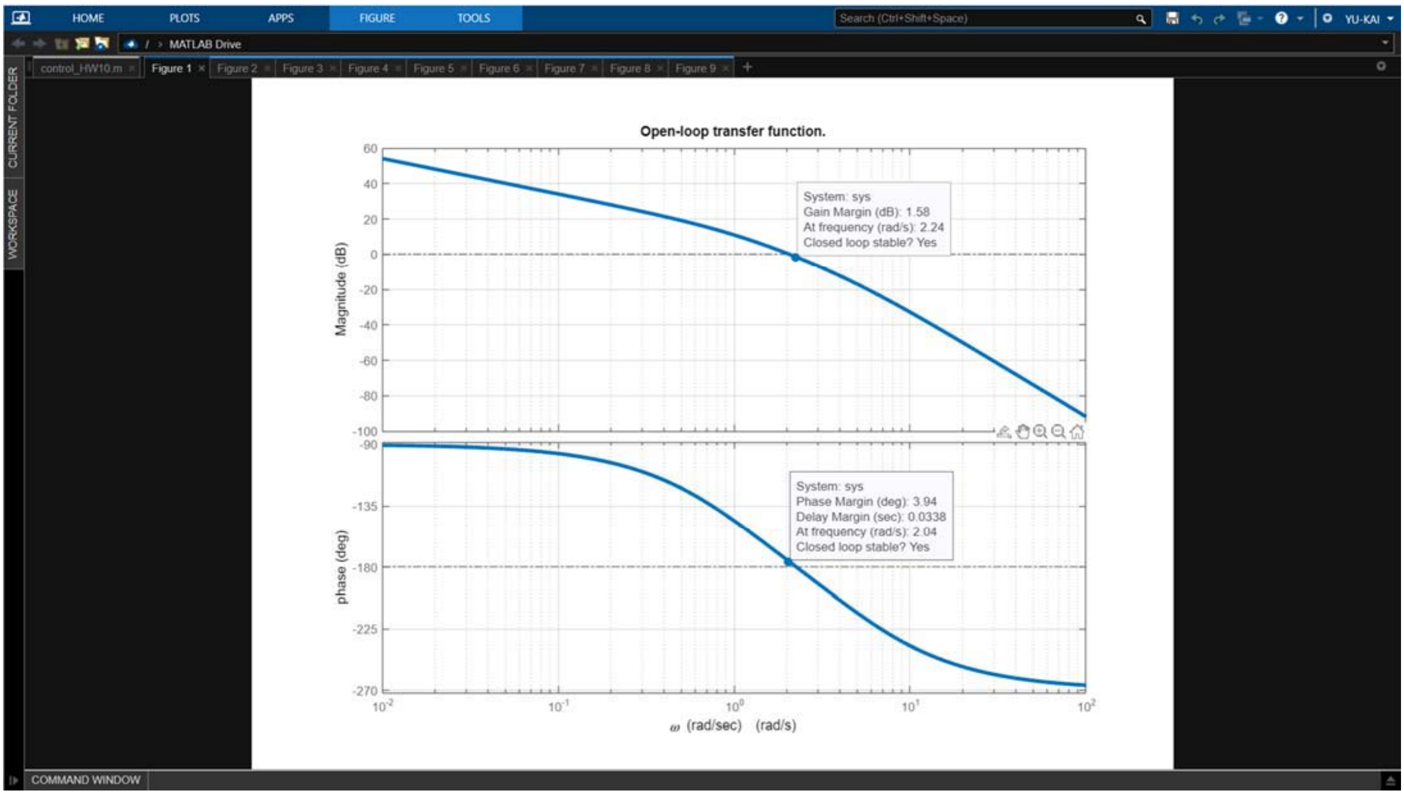

Fig. 2

PM and GM for the system without lead compensation

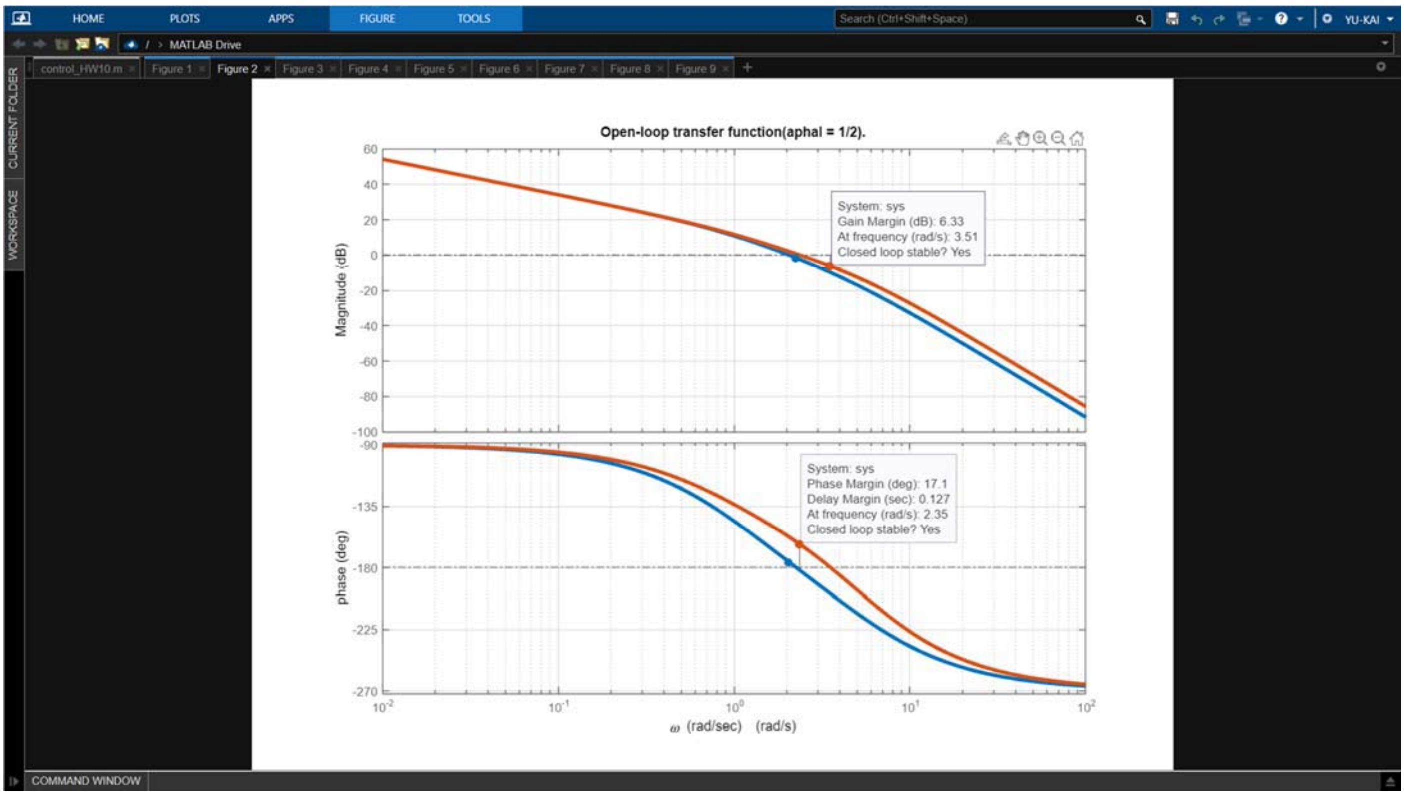

Fig. 3

PM and GM for $\alpha = \frac{1}{2}$

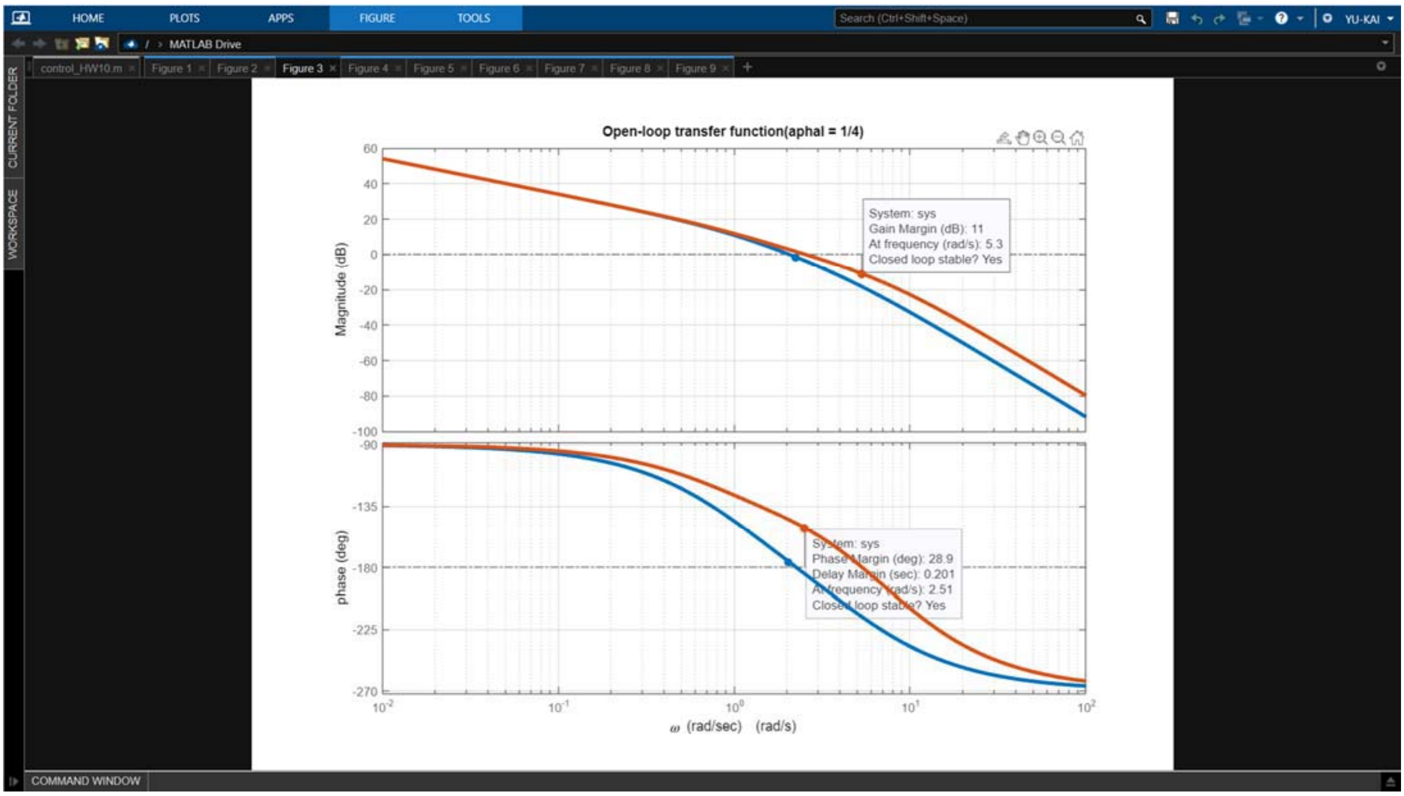

Fig. 4

PM and GM for $\alpha = \frac{1}{4}$

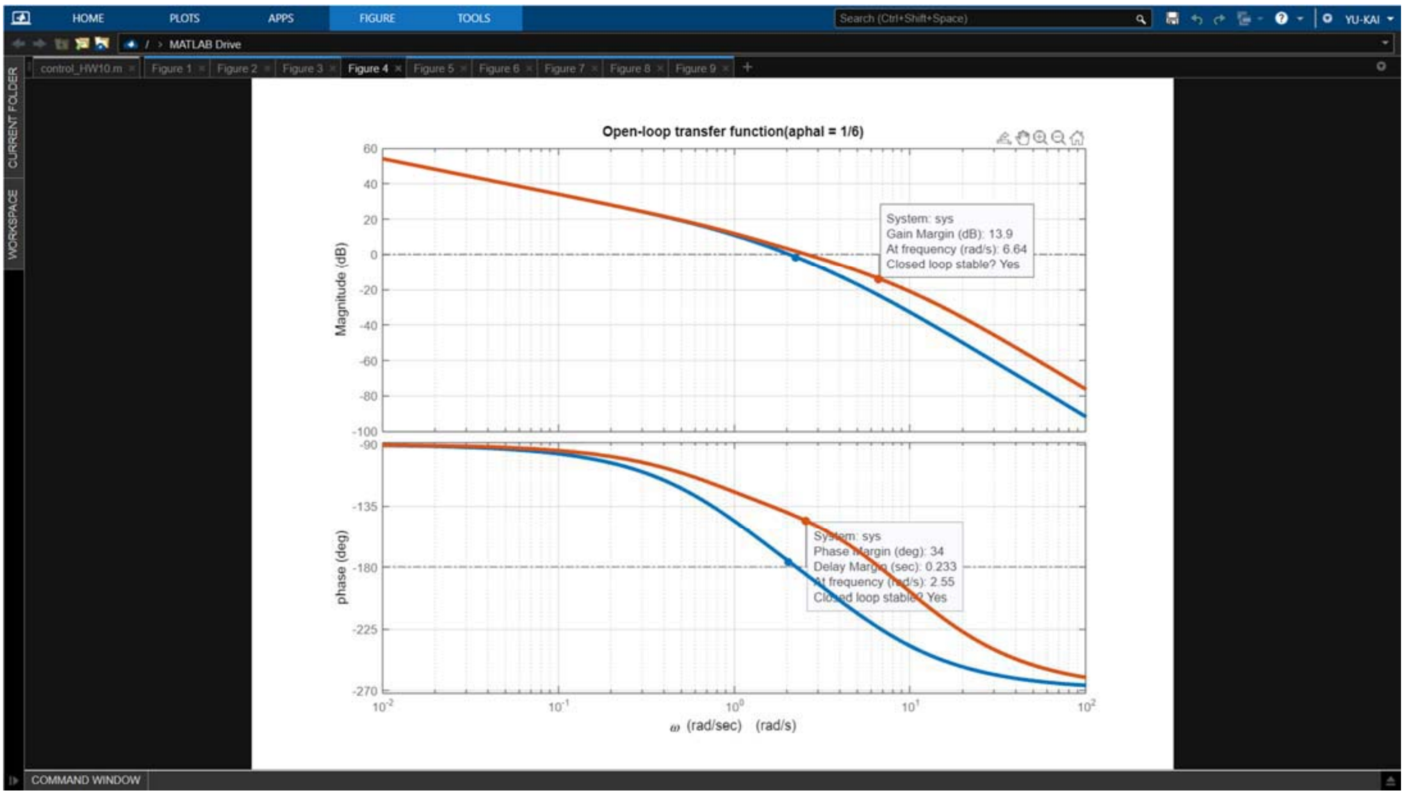

Fig. 5

PM and GM for $\alpha = \frac{1}{6}$

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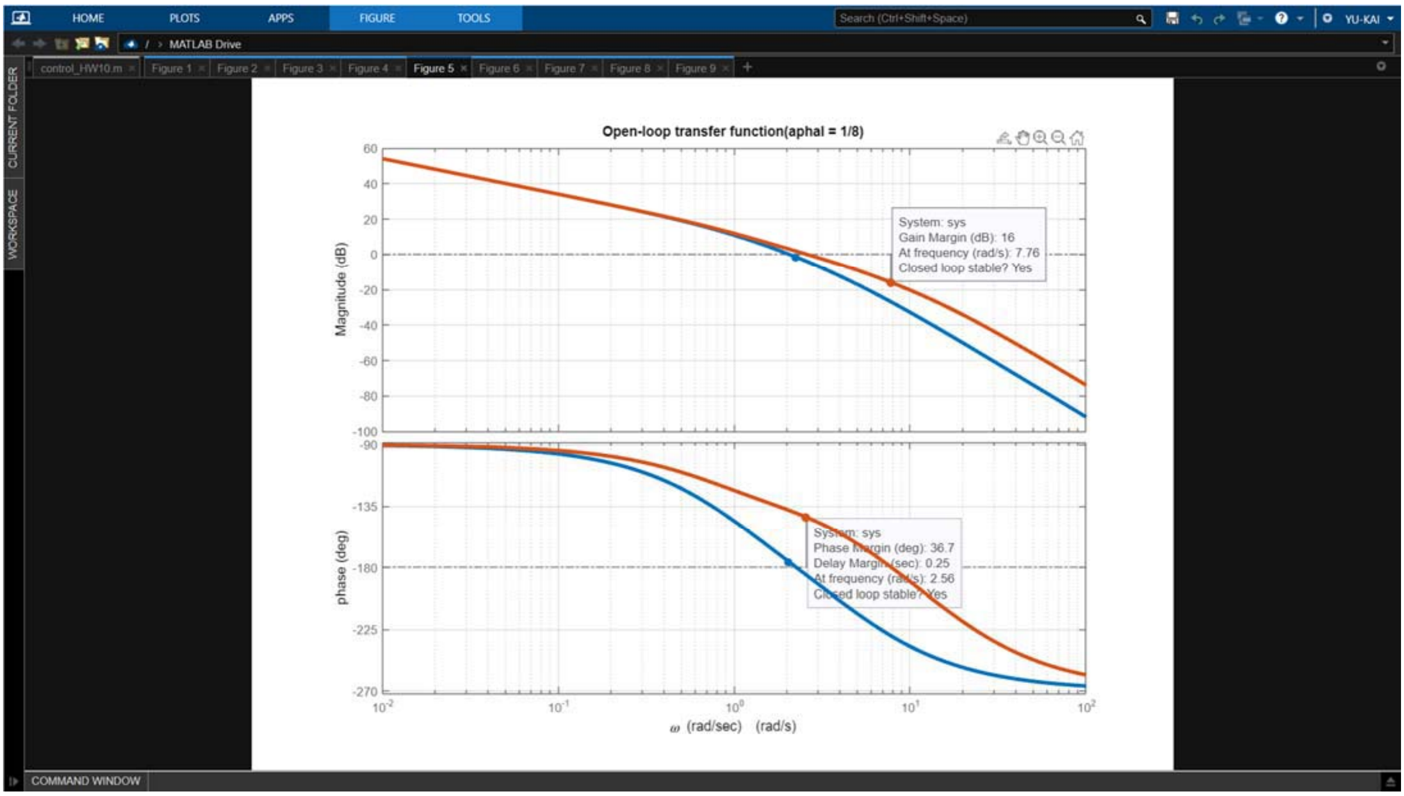


Fig. 6

PM and GM for $\alpha = \frac{1}{8}$

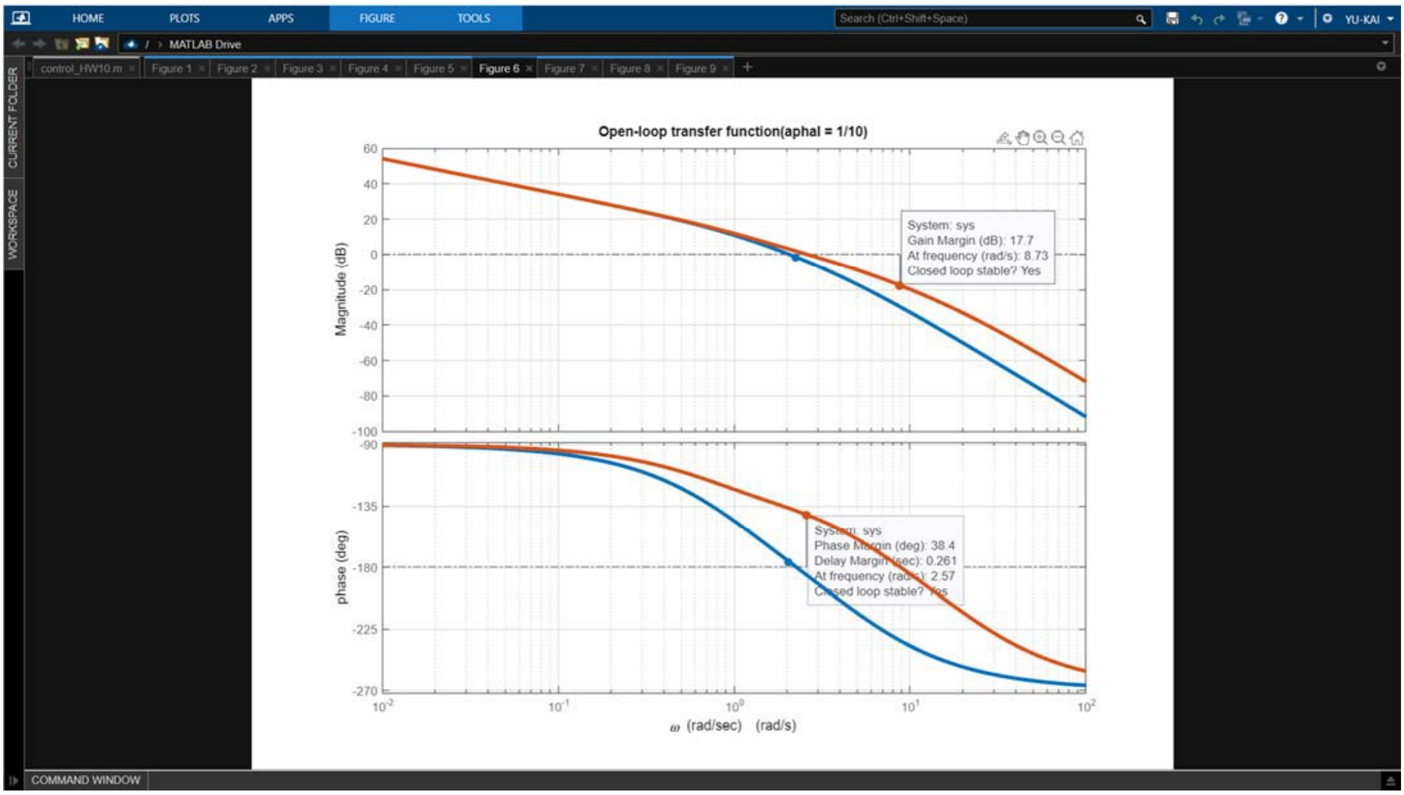


Fig. 7

PM and GM for $\alpha = \frac{1}{10}$

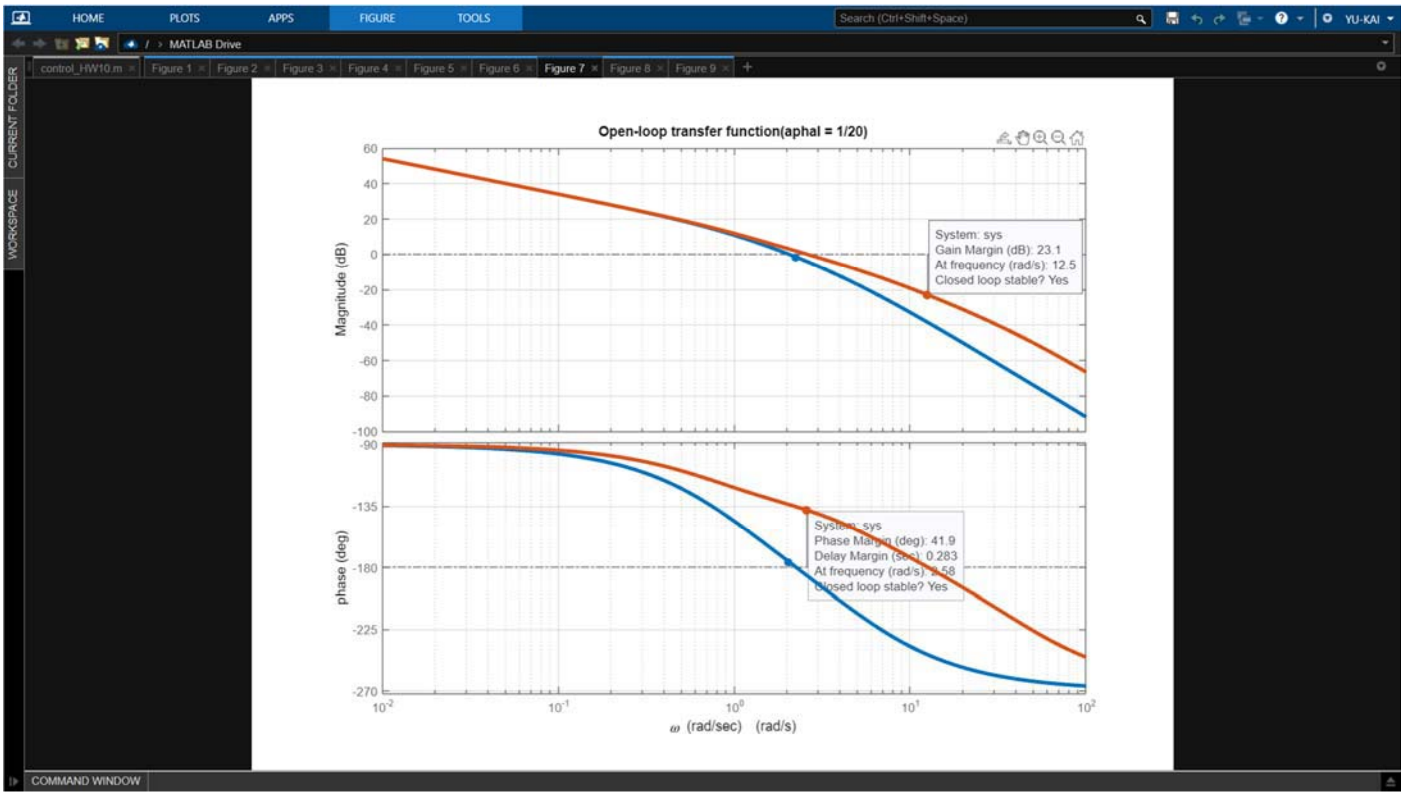


Fig. 8

PM and GM for $\alpha = \frac{1}{20}$

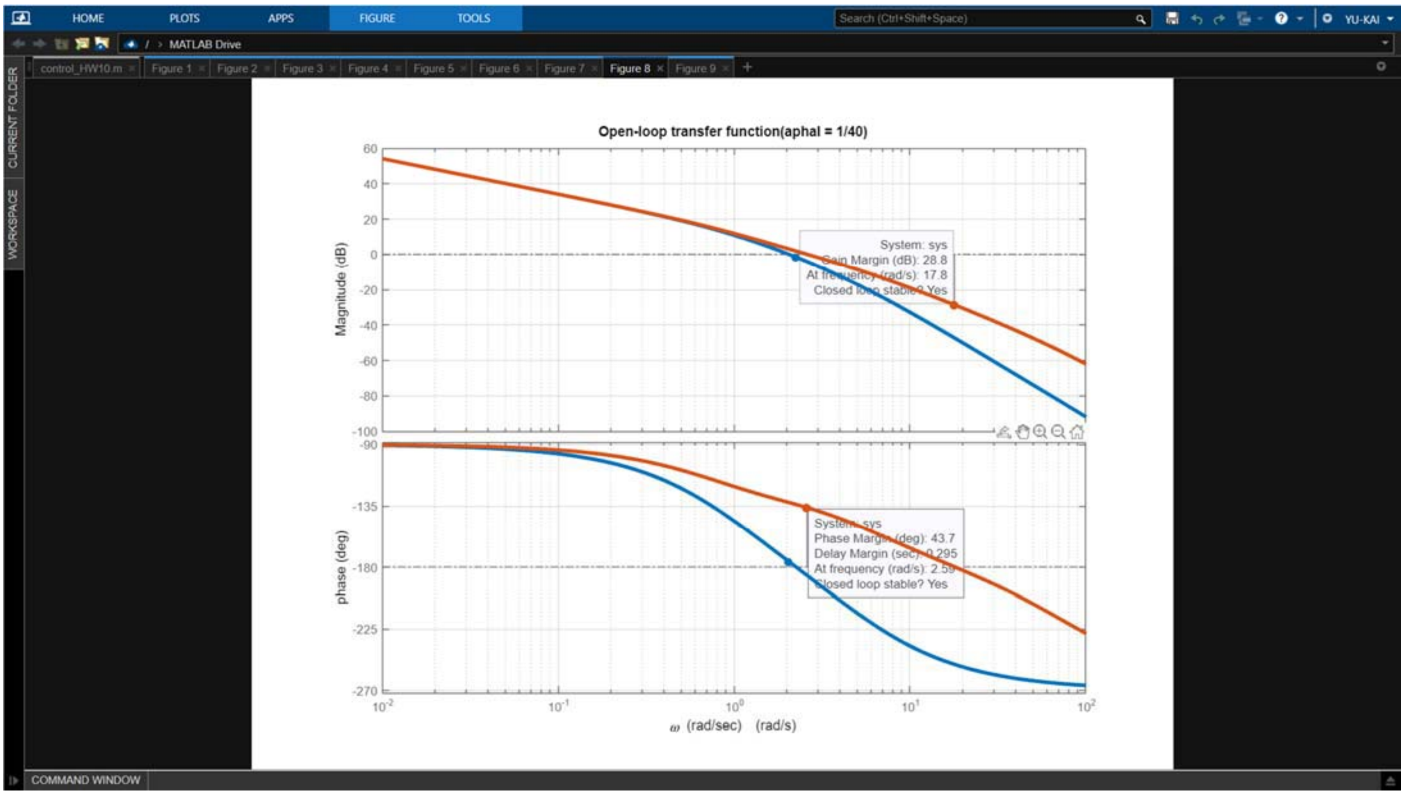


Fig. 9

PM and GM for $\alpha = \frac{1}{40}$

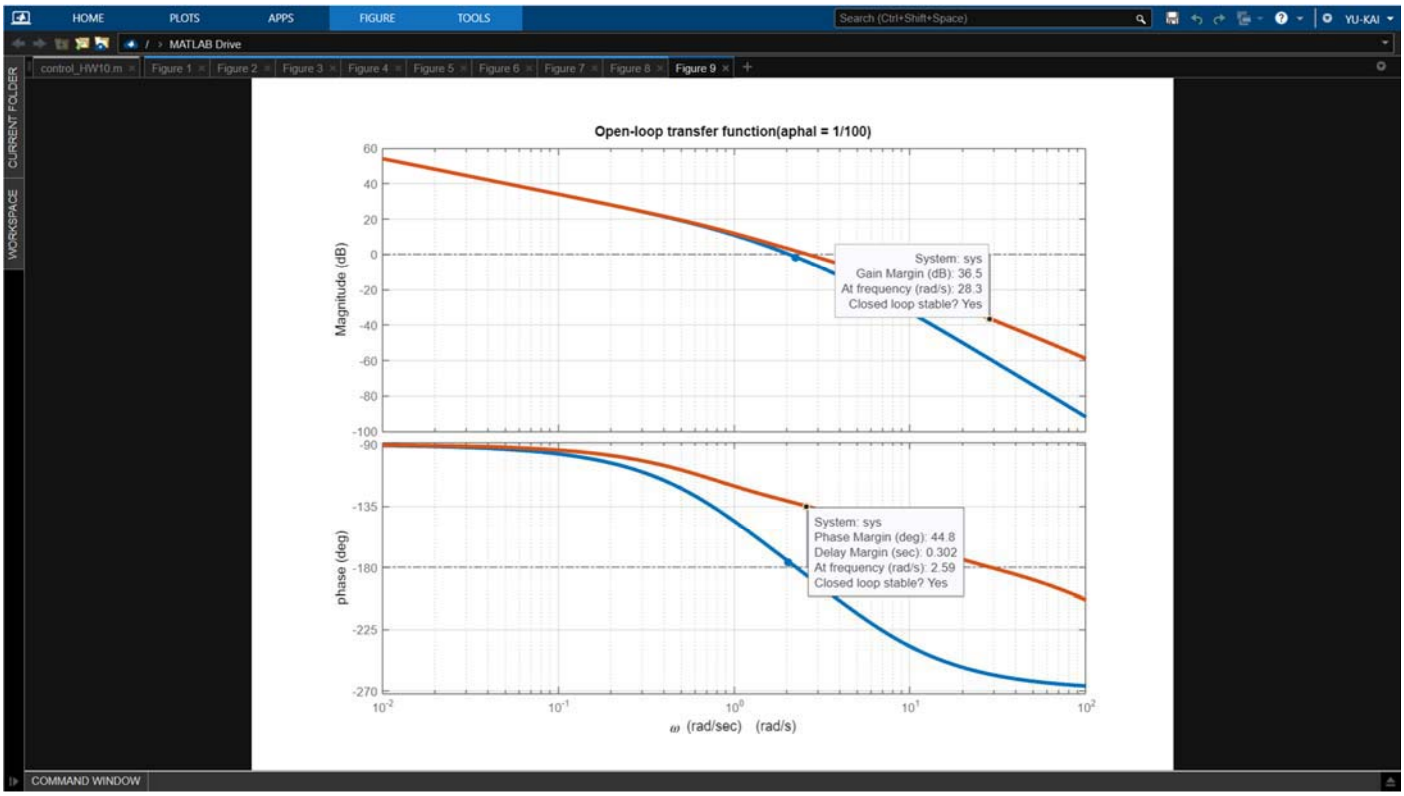


Fig. 10

PM and GM for $\alpha = \frac{1}{100}$

Observation:

- 按照留 extra margin 的方式估計出的 PM，在1/α越大，會和實際的 PM 相差越大。

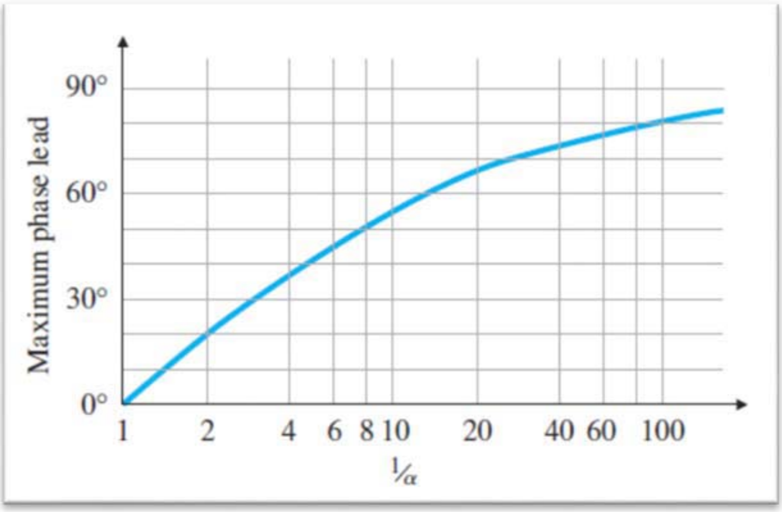


Fig. 11

Table 2

ϕ_{max}是照 Fig. 11 求出的，預估 $+ 10° - 3.94° = \phi_{max}$，相差 = 實際 − 預估，單位：度

1/α	-	2	4	6	8	10	20	40	100
實際	3.96	17.10	28.90	34.00	236.70	38.40	41.90	43.70	44.80
ϕ_{max}	-	20.00	35.00	45.00	50.00	55.00	65.00	75.00	80.00
預估	-	13.96	28.96	38.96	43.96	48.96	58.96	68.96	73.96
相差	-	3.14	-0.06	-4.96	-7.26	-10.56	-17.06	-25.26	-29.16

Control Systems HW10
B10202032 物理二 卓然

46. For the system shown in Fig. 6.100, suppose that

$$G(s) = \frac{5}{s(s+1)(s/5+1)}.$$

Design a lead compensation $D(s)$ with unity DC gain so that PM $\geq 40°$ using Bode plot sketches. What is the approximate bandwidth of the system?

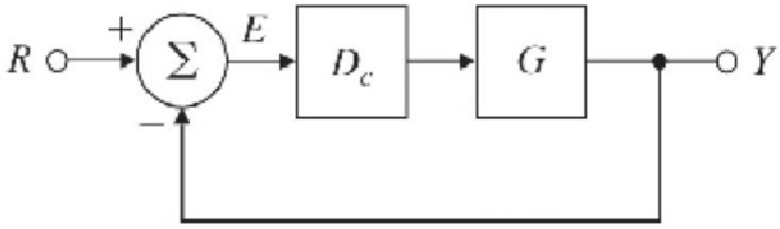


Figure 6.89: Fig. 6.100 Control system for Problem 6.46

The lead compensator should have the form $D_c(s) = \frac{T_D s + 1}{\alpha T_D s + 1}$ where $\alpha < 1$. To start the designing process, let's first take a look at the Bode plot of the open loop function $G(s)$.

$$G(j\omega) = \frac{5}{j\omega(j\omega+1)(\frac{1}{5}j\omega+1)}$$

$$\Rightarrow |G(j\omega)| = \frac{5}{|j\omega||j\omega+1||\frac{1}{5}j\omega+1|}$$

$$\Rightarrow dB = 20\log_{10}\frac{5}{|j\omega||j\omega+1||\frac{1}{5}j\omega+1|} = 20\left(\log_{10}5 - \log_{10}\omega - \log_{10}\sqrt{\omega^2+1} - \log_{10}\sqrt{\frac{\omega^2}{25}+1}\right)$$

$$\fallingdotseq 14 - 20\log_{10}\omega - 10\log_{10}(\omega^2+1) - 10\log_{10}\left(\frac{\omega^2}{25}+1\right)$$

① $\omega \ll 1$: $dB \fallingdotseq 14 - 20\log_{10}\omega$

② $\omega \cong 1$: $dB \fallingdotseq 14 - 10\log_{10}(1^2+1) \fallingdotseq 14 - 10 \cdot 0.3 = 11$

③ $\omega \cong 5$: $dB \fallingdotseq 14 - 20\log_{10}5 - 10\log_{10}25 - 10\log_{10}(2)$

$$\fallingdotseq 14 - 14 - 14 - 3 = -17$$

④ $\omega \gg 5$: $dB \fallingdotseq 14 - 10\log_{10}\frac{\omega^2}{25} = 14 - 20\log\omega + 14 = 28 - 20\log_{10}\omega$

$$\measuredangle G(j\omega) = -\measuredangle(j\omega) - \measuredangle(j\omega+1) - \measuredangle(\tfrac{1}{5}j\omega+1)$$
$$= -\frac{\pi}{2} - \tan^{-1}\omega - \tan^{-1}\left(\frac{\omega}{5}\right)$$

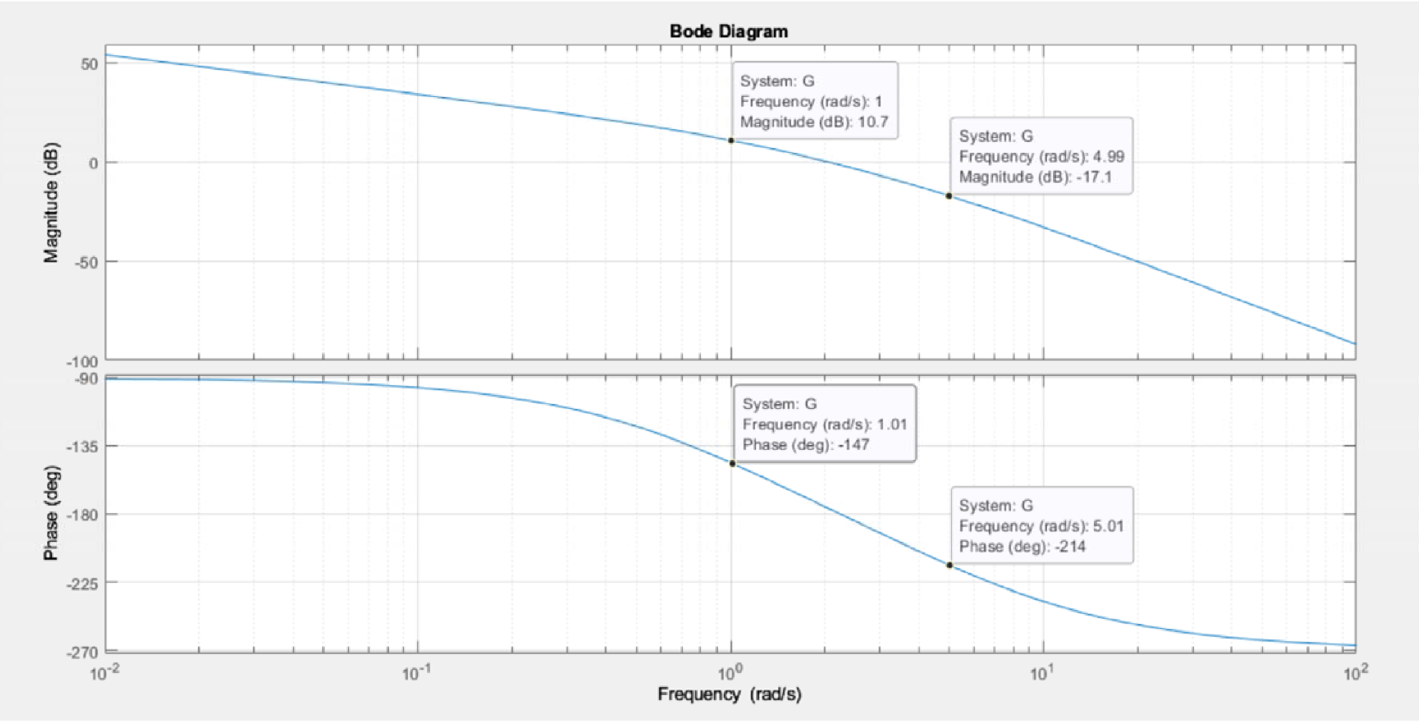
① $\omega \ll 1$: $\measuredangle G(j\omega) \doteq -\frac{\pi}{2} = -90^\circ$

② $\omega = 1$: $\measuredangle G(j\omega) = -\frac{\pi}{2} - \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{5}\right) = -\frac{3\pi}{4} - 0.063\pi = -0.813\pi$
$= -146.4^\circ$

③ $\omega = 5$: $\measuredangle G(j\omega) = -\frac{\pi}{2} - \text{"}\tan^{-1}5\text{"} - \frac{\pi}{4} = -\frac{3\pi}{4} - 0.437\pi = -1.187\pi$
$= -213.7^\circ$

④ $\omega \gg 5$: $\measuredangle G(j\omega) \doteq -\frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{3\pi}{2} = -270^\circ$

The MatLab Plots agree with these approximations:



Now we focus on the core part of this problem: PM.

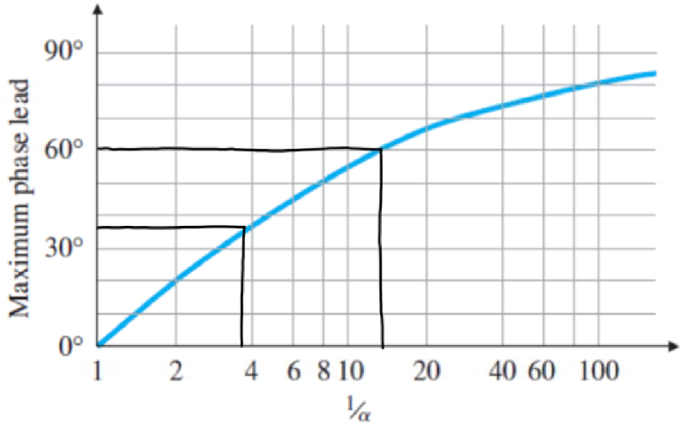
For G(s) along, the frequency at which $|G(j\omega)| = 0$ is $\omega_c = 2.04$, and $\measuredangle G(j\omega)\big|_{\omega=2.04} = -176^\circ \Rightarrow PM = -176^\circ - (-180^\circ) = 4^\circ$

$\Rightarrow$ stable, but far from the required PM $= 40^\circ$.

Now, I will determine the value of α and T_0 such that PM $= 40^\circ$. The minimum needed phase lead is $\phi = 40^\circ - 4^\circ = 36^\circ$

According to the phase lead - $\frac{1}{\alpha}$ relationship graph (shown in right), to obtain $\phi = 36°$, we need approximately $\frac{1}{\alpha} = 3.7$

$\Rightarrow \alpha = 0.27$.

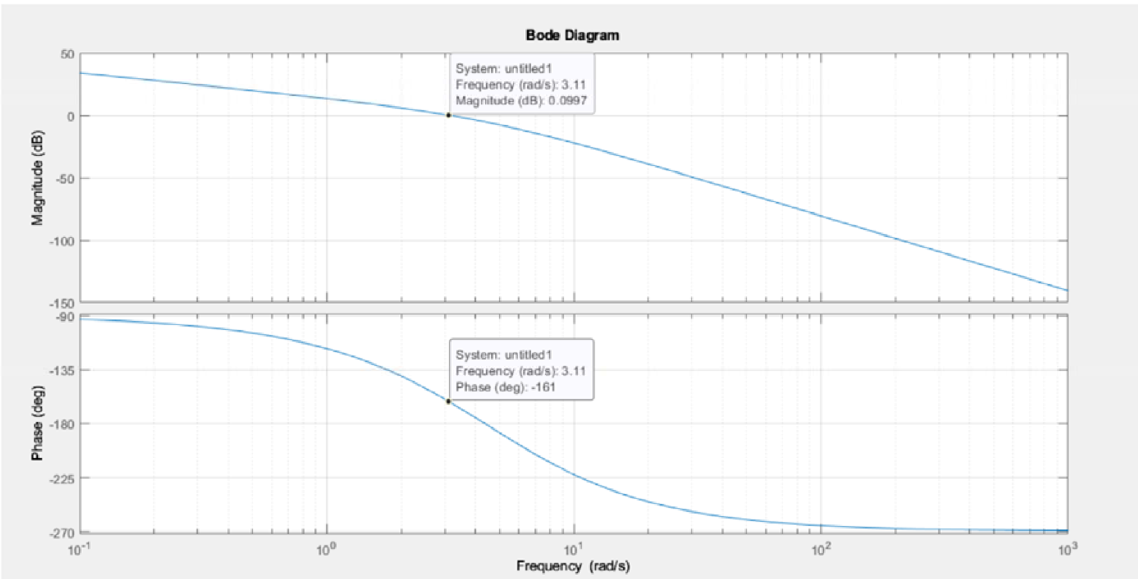


The crossover frequency is $\omega_c = 2.04$, and thus

$$\frac{1}{T_D} = 2.04\sqrt{\alpha} = 2.04\sqrt{0.27} = 1.06 \Rightarrow T_D = 0.943$$

$\therefore$ The lead compensator is $D_c(s) = \frac{(1.06)s+1}{(0.27)(1.06)s+1} = \frac{1.06s+1}{0.286s+1}$

However, it does not meet the requirement PM = 40°; here the PM is only 19° $(-161° - (-180°) = 19°)$.



Now, if we change the ϕ requirement for some room for error, say, $\phi = 64° \Rightarrow$ needed phase lead $= 60° \Rightarrow \frac{1}{\alpha} \doteq 15 \Rightarrow \alpha = 0.0667$;

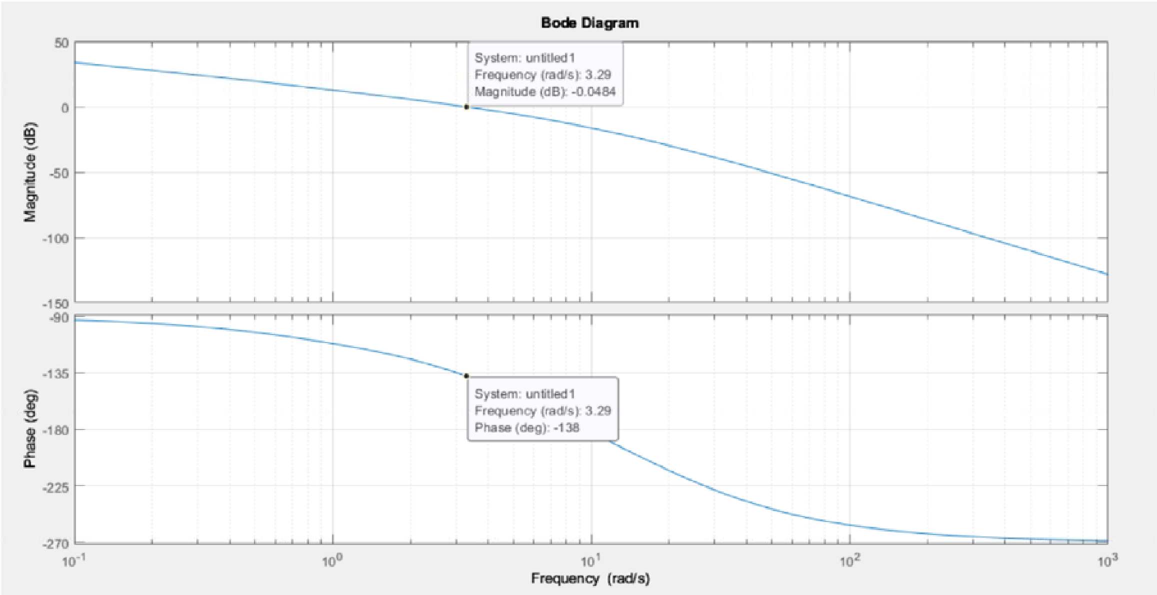
also, enlarge the frequency considered, say $\omega = 5$.

$\Rightarrow \frac{1}{T_D} = 5\sqrt{0.0667} \Rightarrow T_D = 0.7746$

(These numbers are the results of a series of trials and errors)

We arrive at the compensator $D_c(s) = \frac{(0.7746)s+1}{(0.0067)(0.7746)s+1} = \frac{0.7746s+1}{0.0052s+1}$.

The Bode Plot is



This time, the PM $= -138° - (-180°) = 42° > 40°$, which meets the design requirement.

Therefore, the design result is $\boxed{D_c(s) = \frac{0.7746s+1}{0.0052s+1}}$,

and the parameters are $\boxed{\alpha = 0.0667}$, $\boxed{T_D = 0.7746}$.

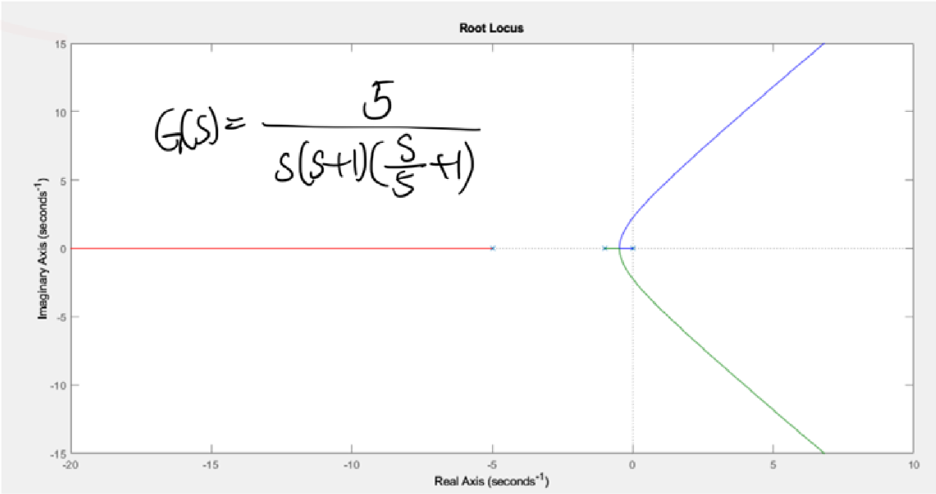
The crossover frequency of this design is $\omega_c = 3.29$ rad/sec, and hence the bandwidth is approximately

$\omega_{BW} \cong 2\omega_c = \boxed{6.58 \text{ rad/sec}}$ #.

Discussions

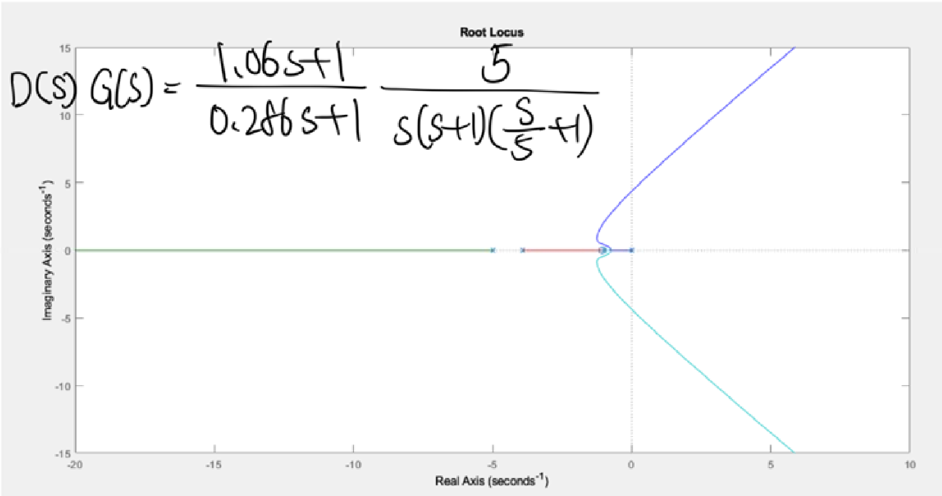
To demonstrate the effect of the D controller, let's take a look at the root loci and the step responses.

A. Root Loci



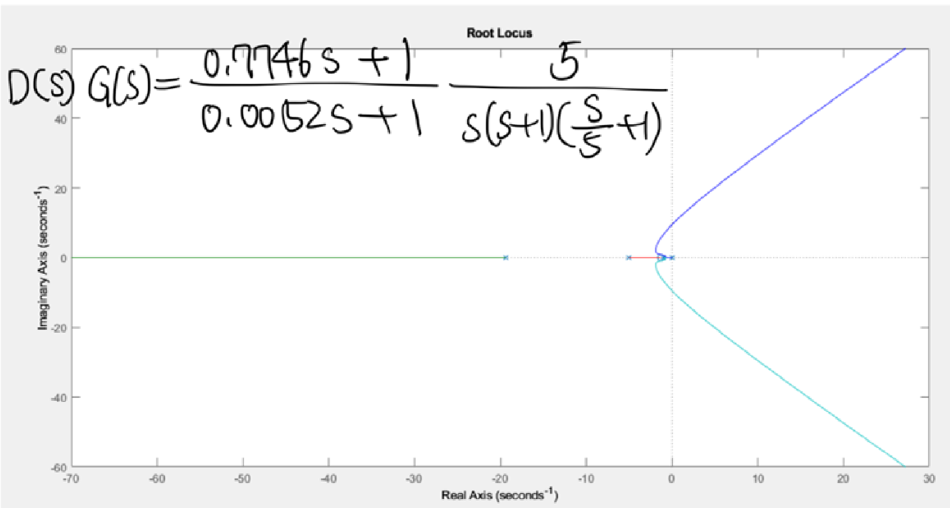
The initial system without the lead compensation D(s)

PM = 4°



The closed loop system with unsuccessful lead compensator design

$\alpha = 0.27$
$T_D = 0.943$ $\Rightarrow$ PM = 19°

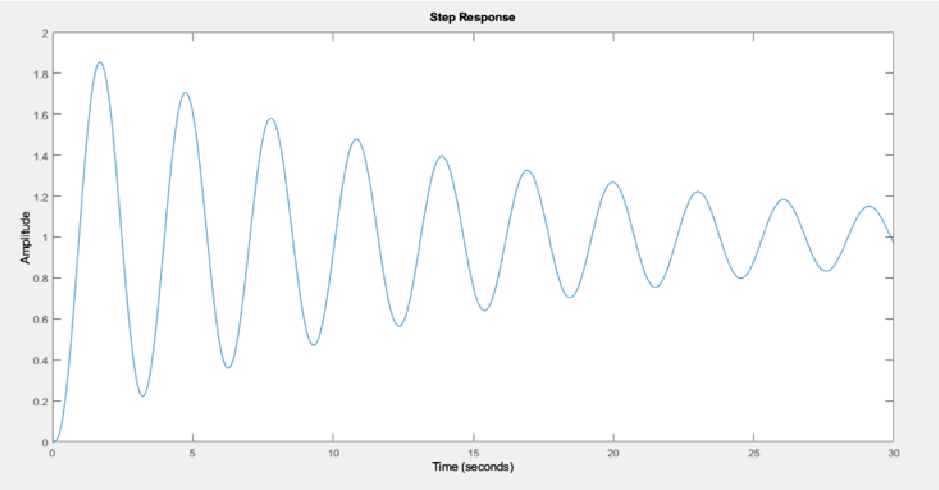


The closed loop system with successful lead compensator design

$\alpha = 0.0667$
$T_D = 0.7746$ $\Rightarrow$ PM = 42°

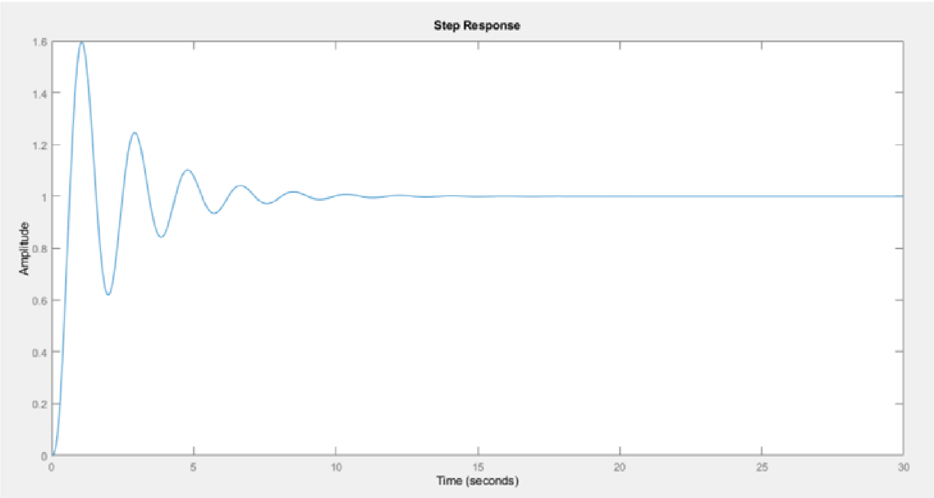
Comparing these loci, we can see that the length of the loci in the LHS increases after adding the lead compensators, and therefore the stability increases. However, the effect of the PD controller can be much more obviously seen with their step response.

B. The Step Responses



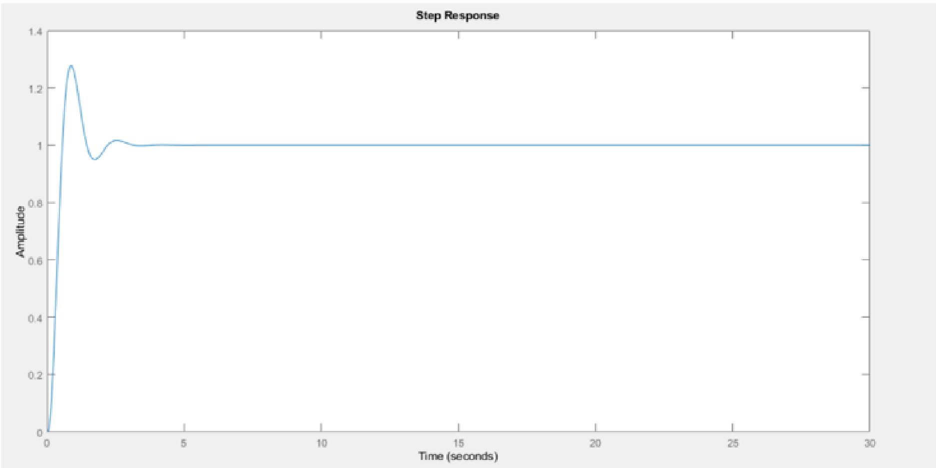
The Initial system without the lead compensation D(s)

$PM = 4^\circ$



The closed loop system with unsuccessful lead compensator design

$\alpha = 0.27$
$T_D = 0.943$ $\Rightarrow PM = 19^\circ$



The closed loop system with successful lead compensator design

$\alpha = 0.0667$
$T_D = 0.7746$ $\Rightarrow PM = 42^\circ$

Comparing the three step responses, we can notice that: (1) The number of apparent oscillation largely decreases with the increase of phase margin. Without the compensator (PM=4°), there are more than 10 oscillation cycles; with the unsuccessful compensator (PM=19°), there are approximately 6 apparent oscillation cycles; when it comes to the successful compensator (PM=42°), there are only about 2 oscillation cycles, which shows much more stability. (2) The overshoot ratio largely decreases with the increase of phase margin. Without the compensator (PM=4°), $M_p \cong \frac{1.86-1}{1} \times 100\% = 86\%$; with the unsuccessful compensator (PM=19°), $M_p \cong \frac{1.6-1}{1} \times 100\% = 60\%$; with the successful compensator (PM=42°), $M_p \cong \frac{1.28-1}{1} \times 100\% = 28\%$. The latter ones evidently show much better stability. (3) The time needed to reach approximate steady state decreases with the increase of phase margin. Without the compensator (PM=4°), it takes more than 30s to reach equilibrium; with the unsuccessful compensator (PM=19°), it takes 10s~15s to reach the steady state; with the successful compensator, it takes no more than 3s to do so.

Overall, the compensators (the increase of phase margin) have prominent positive effects on the stability of the resulting control system; this example excellently demonstrates the power of utilizing phase margin as a reference of control design.

參考觀摩的作業 2. (Lag compensation)
無

參考觀摩的作業

3. (Lead-Lag compensation)

無