Control System: Homework 09 for Units 6D-6G: Bode Plot

Assigned: Dec 2, 2022

Due: Dec 8, 2022 (11:59pm)

1. (Nyquist plot)

22. (a) For $\omega = 0.1$ to 100 rad/sec, sketch the phase of the minimum-phase system

$$G(s) = \left.\frac{s+1}{s+10}\right|_{s=j\omega}$$

and the nonminimum-phase system

$$G(s) = -\left.\frac{s-1}{s+10}\right|_{s=j\omega},$$

noting that $\angle(j\omega - 1)$ decreases with ω rather than increasing.

(b) Does a RHP zero affect the relationship between the -1 encirclements on a polar plot and the number of unstable closed-loop roots in Eq. (6.28)?

(c) Sketch the phase of the following unstable system for $\omega = 0.1$ to 100 rad/sec:

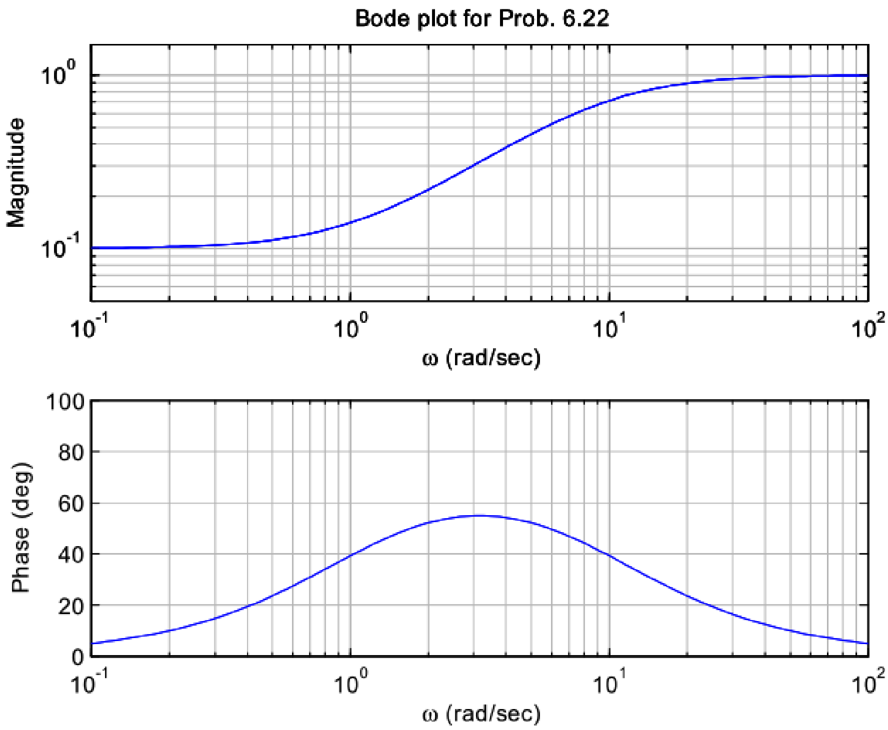
$$G(s) = \left.\frac{s+1}{s-10}\right|_{s=j\omega}.$$

(d) Check the stability of the systems in (a) and (c) using the Nyquist criterion on $KG(s)$. Determine the range of K for which the closed-loop system is stable, and check your results qualitatively using a rough root-locus sketch.

Solution :

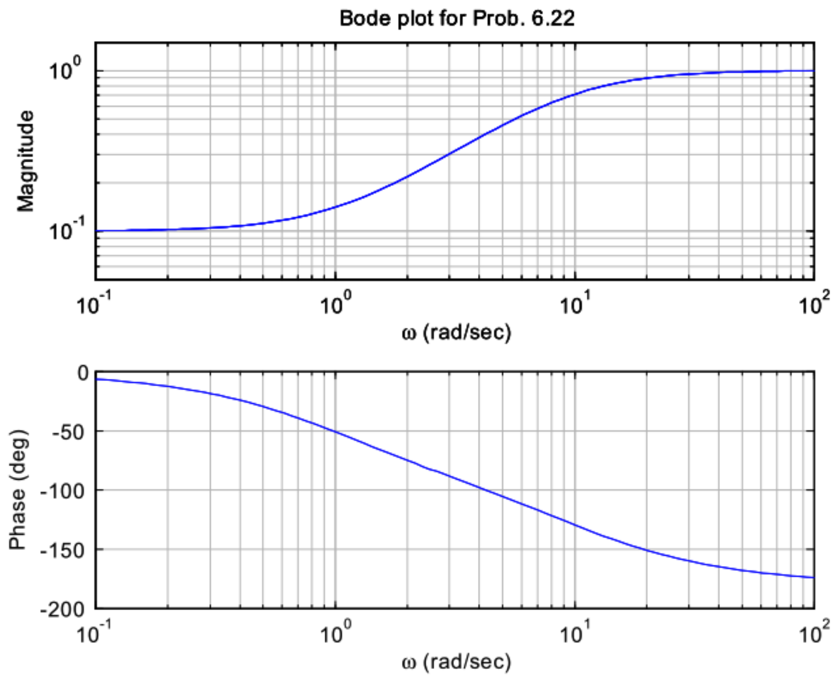
(a) Minimum phase system,

$$G_1(j\omega) = \frac{s+1}{s+10}|_{s=j\omega}$$



Non-minimum phase system,

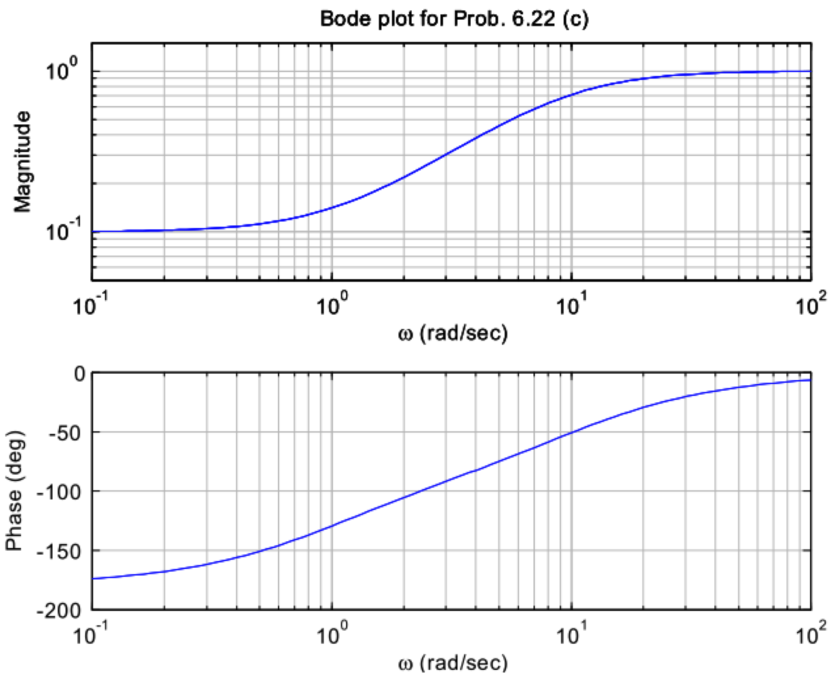
$$G_2(j\omega) = -\frac{s-1}{s+10}|_{s=j\omega}$$



(b) No, a RHP zero doesn't affect the relationship between the -1 encirclements on the Nyquist plot and the number of unstable closed-loop roots in Eq. (6.28).

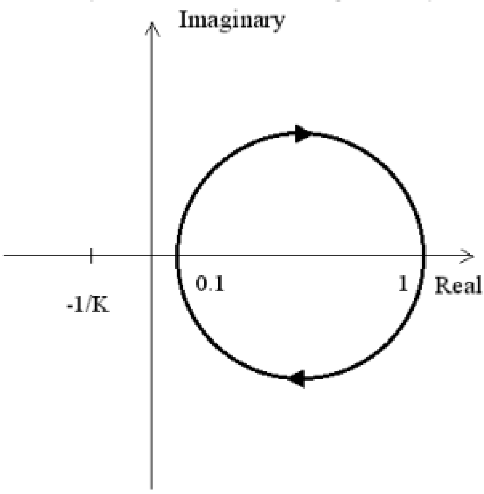
(c) Unstable system:

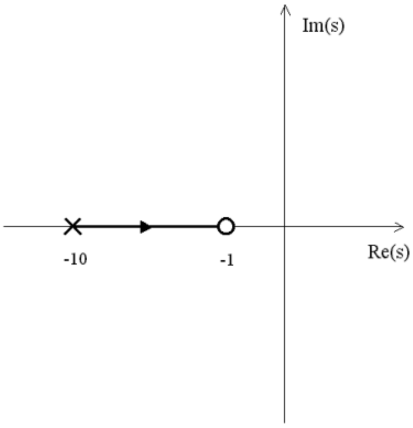
$$G_3(j\omega) = \frac{s+1}{s-10}\Big|_{s=j\omega}$$



(d) Minimum phase system $G_1(j\omega)$:

i. For any $K > 0$, $N = 0$, $P = 0 \Longrightarrow Z = 0 \Longrightarrow$ The system is stable, as verified by the root locus being entirely in the LHP.

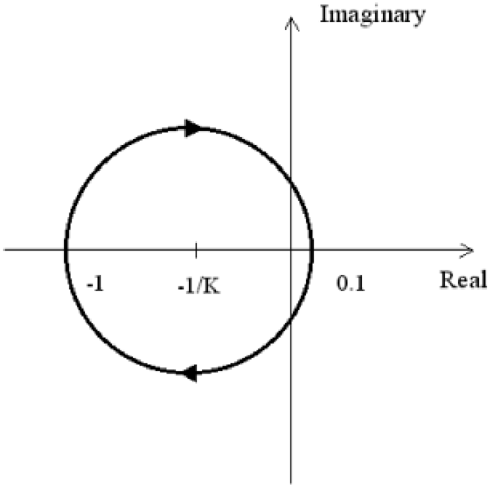


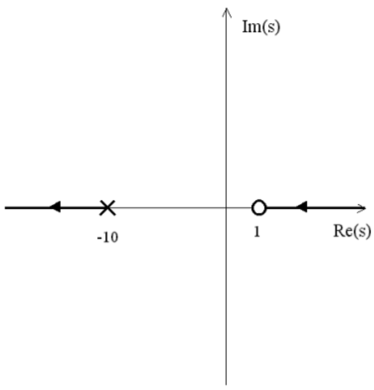


Non-minimum phase system $G_2(j\omega)$: the $-1/K$ point will not be encircled if $K < 1$.

$$\begin{array}{ll} 0 < K < 1 & N = 0,\ P = 0 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \\ 1 < K & N = 1,\ P = 0 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \end{array}$$

This is verified by the Root Locus shown below where the branch of the locus to the left of the pole is from $K < 1$.

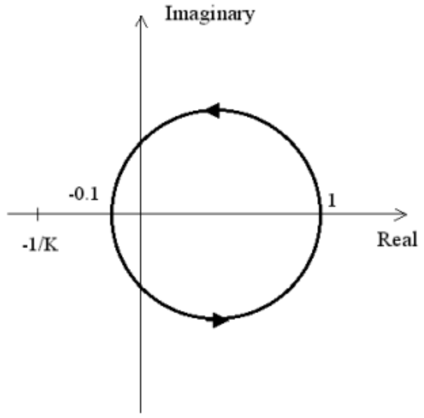


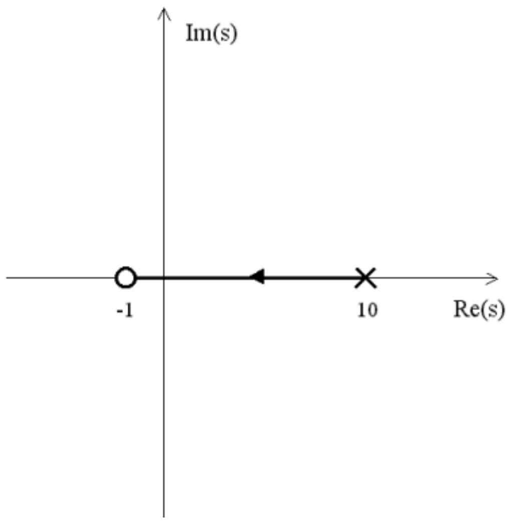


Unstable system $G_3(j\omega)$: The $-1/K$ point will be encircled if $K > 10$, however, $P = 1$, so

$$\begin{aligned} 0 < K < 10: &\quad N = 0,\ P = 1 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \\ 10 < K\ : &\quad N = -1,\ P = 1 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \end{aligned}$$

This is verified by the Root Locus shown below right, where the locus crosses the imaginary axis when $K = 10$, and stays in the LHP for $K > 10$.





2. (Stability margin)

27. Consider the system given in Fig. 6.91.

(a) Use MATLAB to obtain Bode plots for $K = 1$ and use the plots to estimate the range of K for which the system will be stable.

(b) Verify the stable range of K by using margin to determine PM for selected values of K.

(c) Use rlocus and rlocfind to determine the values of K at the stability boundaries.

(d) Sketch the Nyquist plot of the system, and use it to verify the number of unstable roots for the unstable ranges of K.

(e) Using Routh's criterion, determine the ranges of K for closed-loop stability of this system.

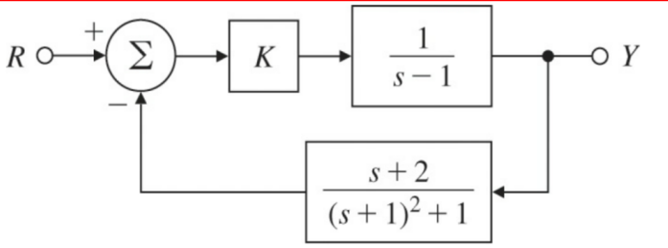
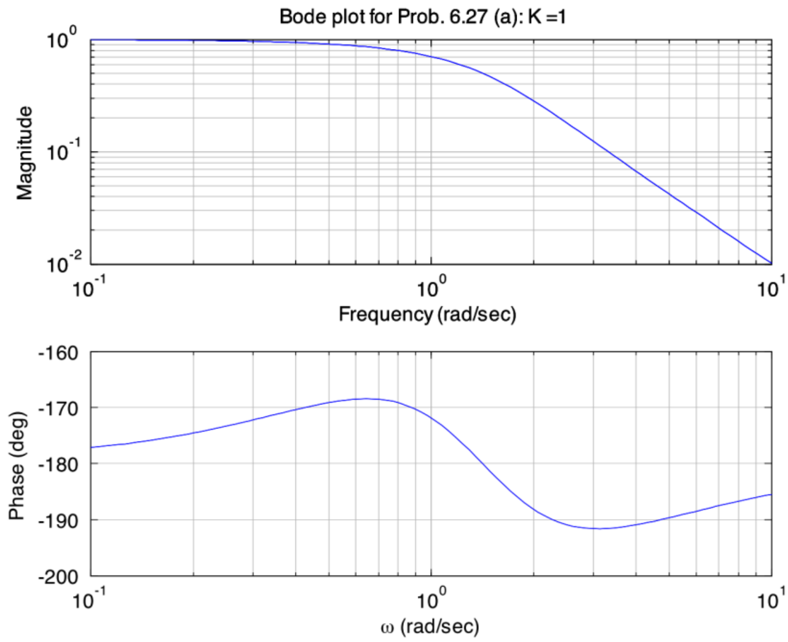


Fig. 6.91 Control system for Problem 6.27

Solution :

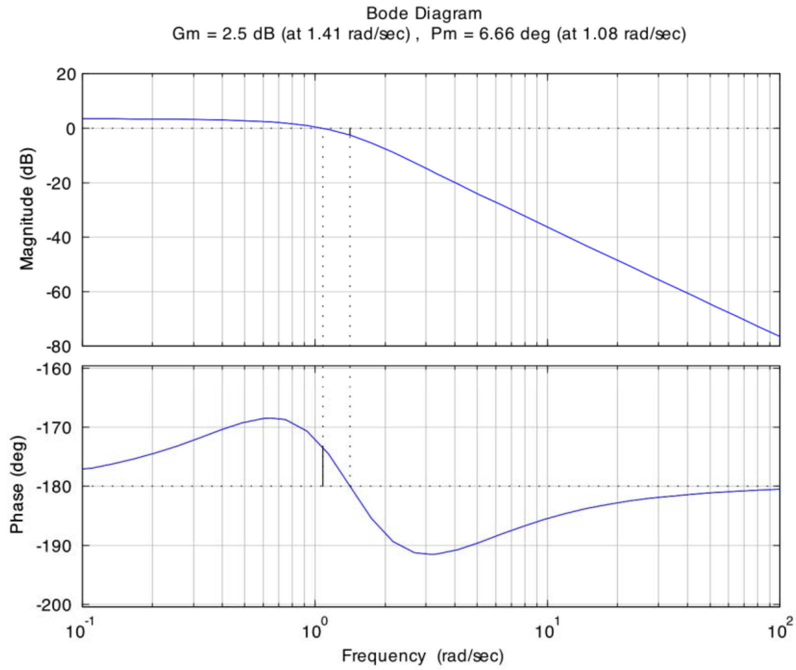
(a) The Bode plot for $K = 1$ is :



From the Bode plot, the closed-loop system is unstable for $K = 1$. But we can make the closed-system stable with positive GM by increasing the gain K up to where the crossover frequencyis at $\omega = 1.414$ rad/sec ($K = 2$), where the phase plot crosses the $-180°$ line. Therefore :

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$

(b) For example, $PM = 6.66\,\text{deg}$ for $K = 1.5$.



Margin determination for K=1.5

(c) Root locus is :
and it shows that the $j\omega$-crossing information is $K = 2$ and $\omega = \pm\sqrt{2}$, or $K = 1$ at the origin. It can also be calculated by:

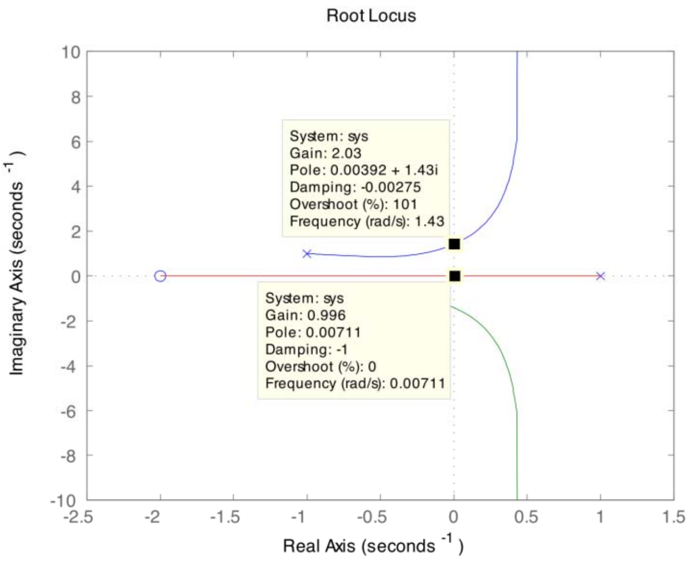
$$1 + K\frac{j\omega + 2}{(j\omega)^3 + (j\omega)^2 - 2} = 0$$

$$\begin{aligned} \omega^2 - 2K + 2 &= 0 \\ \omega(\omega^2 - K) &= 0 \end{aligned}$$

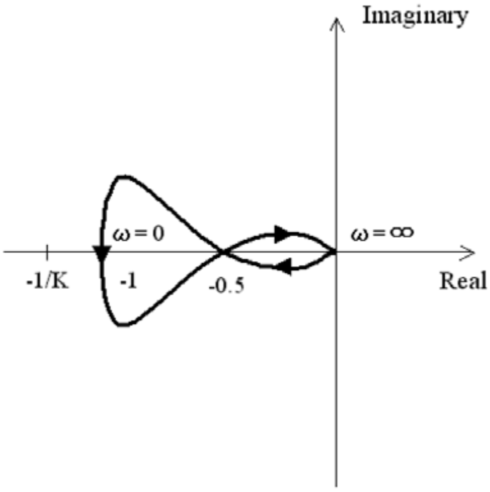
$$K = 2,\ \omega = \pm\sqrt{2},\ \text{or}\ K = 1,\ \omega = 0$$

Therefore,

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$



(d)



i. $0 < K < 1$
   $N = 0,\ P = 1 \implies Z = 1$
   One unstable closed-loop root.
ii. $1 < K < 2$
   $N = -1,\ P = 1 \implies Z = 0$
   Stable.
iii. $2 < K$
   $N = 1,\ P = 1 \implies Z = 2$
   Two unstable closed-loop roots.

(e) The closed-loop transfer function of this system is :

$$\begin{aligned} \frac{y(s)}{r(s)} &= \frac{k\frac{1}{s-1}}{1+k\frac{1}{s-1}\times\frac{s+2}{(s+1)^2+1}} \\ &= \frac{K(s^2+2s+2)}{s^3+s^2+Ks+2K-2} \end{aligned}$$

So the characteristic equation is :

$$\Longrightarrow s^3+s^2+Ks+2K-2=0$$

Using the Routh's criterion,

$$\begin{array}{ccc} s^3: & 1 & K \\ s^2: & 1 & 2K-2 \\ s^1: & 2-K & 0 \\ s^0: & 2k=2 & \end{array}$$

For stability,

$$\begin{aligned} 2-K &> 0 \\ 2K-2 &> 0 \end{aligned}$$

$$\Longrightarrow 2>K>1$$

$0<K<1$	Unstable
$1<K<2$	Stable
$2<K$	Unstable

3. (Gain margin and phase margin)

36. For the system shown in Fig. 6.96, determine the Nyquist plot and apply the Nyquist criterion.

(a) to determine the range of values of K (positive and negative) for which the system will be stable, and

(b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

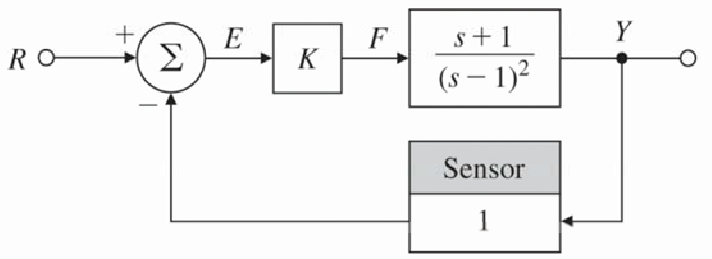


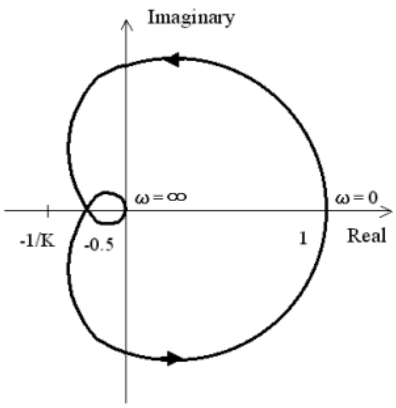
Fig. 6.96 Control system for Problem 6.36

Solution :

(a) & b.

$$KG(s) = K\frac{s+1}{(s-1)^2}$$

Use of the nyquist routine in Matlab yields the plot below, where the contour crosses the real axis at -0.5 and +1.



From the Nyquist plot we see that:

i.

$$-\infty < -\frac{1}{K} < -\frac{1}{2} \implies 0 < K < 2$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

ii.

$$-\frac{1}{2} < -\frac{1}{K} < 0 \implies 2 < K$$

$$N = -2,\ P = 2 \implies Z = 0$$

The closed-loop system is stable.

iii.

$$0 < -\frac{1}{K} < 1 \implies K < -1$$

$$N = -1,\ P = 2 \implies Z = 1$$

One closed-loop root in RHP.

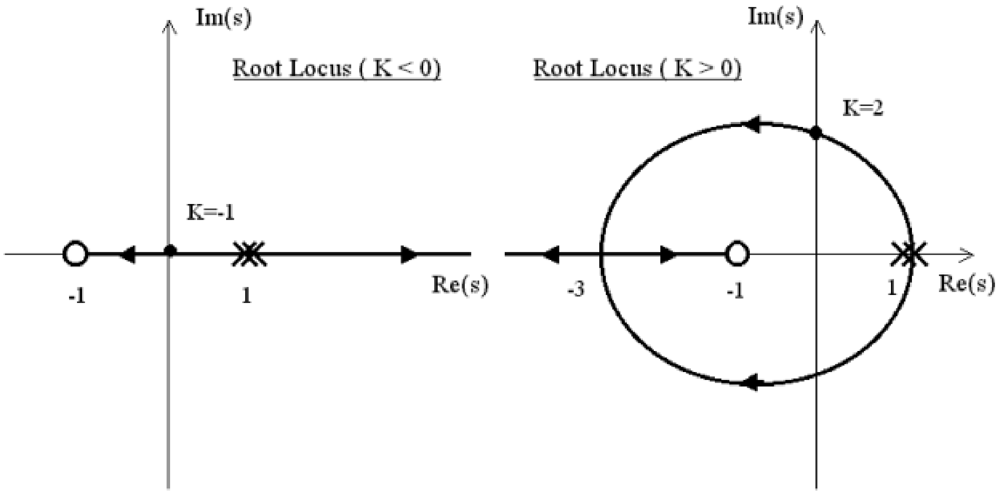
iv.

$$1 < -\frac{1}{K} < \infty \implies -1 < K < 0$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

These results are confirmed by looking at the root loci below:



4. (Gain-Phase relation)

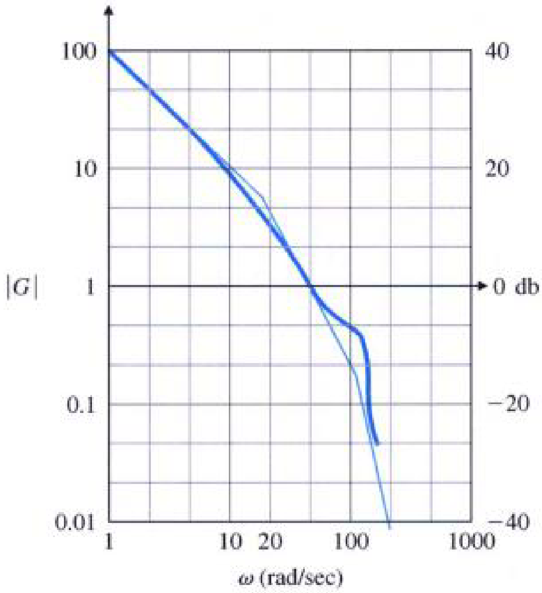
41. The frequency response of a plant in a unity feedback configuration is sketched in Fig. 6.99. Assume the plant is open-loop stable and minimum phase.

(a) What is the velocity constant K_v for the system as drawn?

(b) What is the damping ratio of the complex poles at $\omega = 100$?

(c) What is the PM of the system as drawn? (Estimate to within $\pm 10^o$.)

Figure 6.99: Magnitude frequency response for Prc



Solution :

(a) From Fig. 6.99,

$$K_v = \lim_{s \to 0} sG = |\text{Low frequency asymptote of } G(j\omega)|_{\omega=1} = 100)$$

(b) Let

$$G_1(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

For the second order system $G_1(s)$,

$$|G_1(j\omega)|_{\omega=1} = \frac{1}{2\zeta} \qquad (1)$$

From Fig. 6.99 :

$$|G_1(j\omega)|_{\omega=100} = \frac{|G(j\omega)|_{\omega=100}}{|\text{Asymptote of } G(j\omega)|_{\omega=100}} \cong \frac{0.4}{0.2} = 2 \qquad (2)$$

From (1) and (2) we have :

$$\frac{1}{2\zeta} = 2 \implies \zeta = 0.25$$

(c) Since the plant is a minimum phase system, we can apply the Bode's approximate gain-phase relationship.

When $|G| = 1$, the slope of $|G|$ curve is $\cong$ -2.

$$\implies \angle G(j\omega) \cong -2 \times 90^\circ \;= -180^\circ$$

$$PM \cong \angle G(j\omega) + 180^\circ \;= 0^\circ$$

Note : Actual PM by Matlab calculation is 6.4°, so this approximation is within the desired accuracy.

參考觀摩的作業 1. (Nyquist plot)
無

參考觀摩的作業 2. (Stability margin)
無

參考觀摩的作業

3. (Gain margin and phase margin)

作者： b08901085，施彥宇 **理由：** 用波德圖與 Routh's criterion 討論與驗證閉迴路系統穩定性與增益 K 的關係
作者： b08901176，陳育楷 **理由：** 討論閉迴路系統在不同增益 K 值的波德圖與穩定性，並用步階響應圖說明結果。建議統一整份作業使用語言。
作者： b09901015，王 瑋 **理由：** 建議解釋如何繪製與分析簡化後的 CMOS op amp 轉移函數的 Nyquist plot
作者： b10202032，卓 然 **理由：** 詳細討論不同求穩定性的方法以及其正確性

HW09 – Unit 6, Bode Plot

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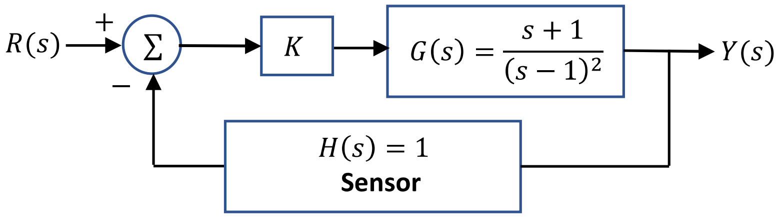
- **Question :**

For the following system, determine the Nyquist plot and apply the Nyquist criterion.



a). To determine the range of values of K (positive and negative) for which the system will be stable.

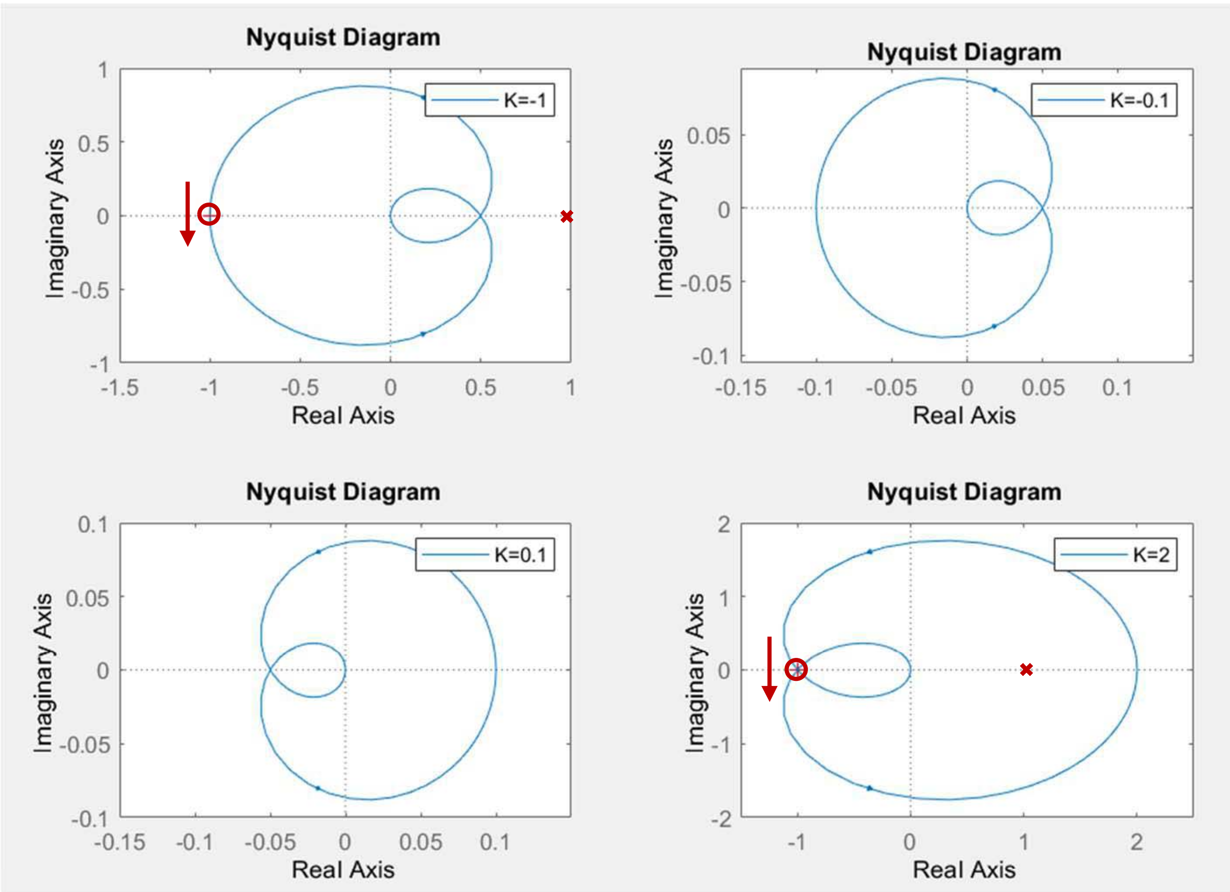
b). To determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

- **Solution :**

The close-loop transfer function of this system is

$$KG(s) = K\frac{s+1}{(s-1)^2} \qquad (1).$$

Then the Nyquist plot of this system with different K is :



- **Solution :**

From this plot we can briefly observe the trend of Nyquist plot with variation of K. And notice that P is the number of poles in RHP, Z is the number of zeros in RHP, and N is the number of clockwise encirclements of -1. Therefore,

$$Z = N + P \qquad (2).$$

1. For $K < -1$:

   $(N, P) = (-1, 2) \Rightarrow Z = 1$, one closed-loop root in RHF.

2. For $1 \prec K < 0$:

   $(N, P) = (0, 2) \Rightarrow Z = 2$, two closed-loop roots in RHF.

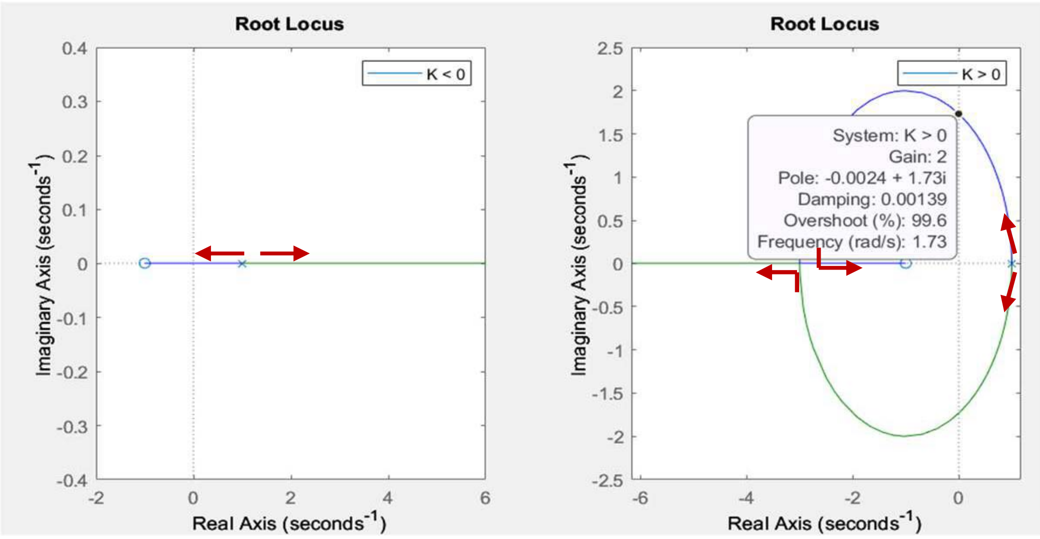
3. For $0 < K < 2$:

   $(N, P) = (0, 2) \Rightarrow Z = 2$, two closed-loop roots in RHF.

4. For $2 < K$:

   $(N, P) = (-2, 2) \Rightarrow Z = 0$, closed-loop system is stable.

Using root-locus to check the result :



One can observe that the conclusion is correct.

- **What I can do more:**

We can use another method for finding the stability of this system.

1. Using Routh's criterion :

   The closed-loop transfer function of this system is :

$$\frac{Y(s)}{X(s)} = \frac{KG(s)}{1+KG(s)} = \frac{K\frac{s+1}{(s-1)^2}}{1+K\frac{s+1}{(s-1)^2}} = \frac{K(s+1)}{s^2(K-2)s+(K+1)} \qquad (3).$$

$$s^2 : 1 \qquad (K+1) \qquad \Rightarrow 1 > 0$$

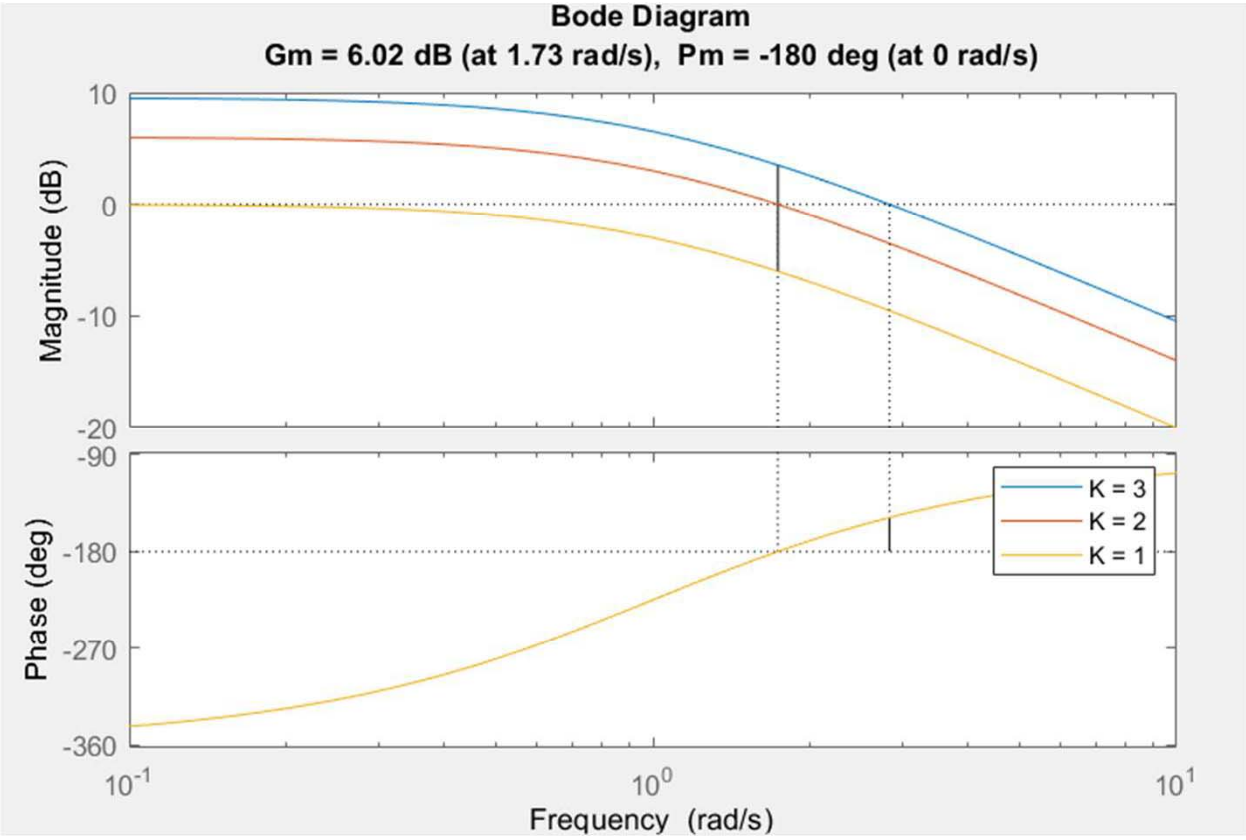
$$s^1 : (K-2) \qquad 0 \qquad \Rightarrow K - 2 > 0 \;\Rightarrow\; K > 2$$

$$s^0 : \frac{(K+1)\times(K-2)-0}{K-2} = K + 1 \;\Rightarrow K + 1 > 0 \;\Rightarrow\; K > -1$$

To reach stable, the range of K is $K > 2$.

- **What I can do more:**

    2. Using Bode plot to verify the conclusion :



While $K = 1$, the phase at unit gain frequency is less than $-180°$. And for $K = 2$, the phase is equal to $-180°$ at unit gain frequency. For $K = 3$, the phase at unit gain frequency is larger than $-180°$. From this trend, we can observe that this system goes to stability for $K > 2$.

B08901176
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3(Gain margin and phase margine)

Problem:

Plot the magnitude, phase of $\frac{Y}{R}(s)$ in Fig. 6.96 (from $\omega = 0.001$ to 1000 rad/sec)at $K = -1$and 4. And discuss the situations that match the results from the original problem.

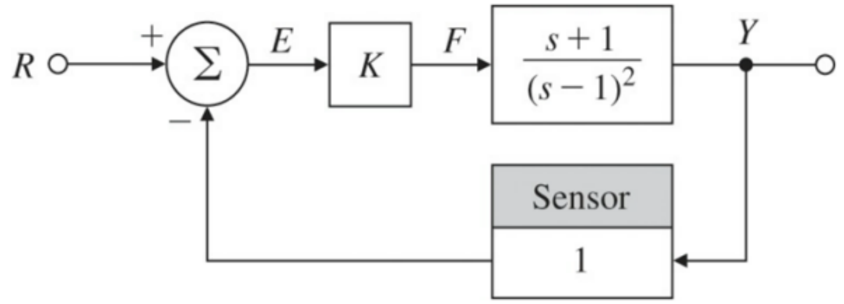


Fig. 6.96 Control system for Problem 6.36

Solution:

For $K = -1$,

$$\frac{Y(s)}{R(s)} = \frac{-\frac{s+1}{(s-1)^2}}{1-\frac{s+1}{(s-1)^2}} = \frac{-(s+1)}{s^2-3s} = \frac{s+1}{s(s-3)} \text{ has a pole in RHP} \Rightarrow \text{the system is NOT stable.}$$

$$\left|\frac{Y}{R}(j\omega)\right| = \frac{\sqrt{\omega^2+1}}{\omega\sqrt{\omega^2+9}}$$

$$\angle\frac{Y}{R}(j\omega) = -90 + \tan^{-1}\left(\frac{\omega}{1}\right) + \tan^{-1}\left(\frac{\omega}{3}\right)$$

For $K = 4$,

$$\frac{Y(s)}{R(s)} = \frac{4\frac{s+1}{(s-1)^2}}{1+4\frac{s+1}{(s-1)^2}} = \frac{4(s+1)}{s^2+2s+5} \text{ has all poles in LHP} \Rightarrow \text{the system is stable.}$$

$$\left|\frac{Y}{R}(j\omega)\right| = \frac{4\sqrt{\omega^2+1}}{\sqrt{(-\omega^2+5)^2+(2\omega)^2}}$$

$$\angle\frac{Y}{R}(j\omega) = \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{2\omega}{-\omega^2+5}\right)$$

Plot:

在 Fig. 1 中，左邊縱軸為 magnitude(dB)，右邊縱軸為 phase (degree)，橫軸為ω(rad/sec)，藍線和灰線分別為$K=-1$和$K=4$的 magnitude，橘線和黃線分別為$K=-1$和$K=4$的 phase。

- For $K=-1, PM=116°$ @0.35Hz
- For $K=4,\quad PM=107°$ @4.65Hz

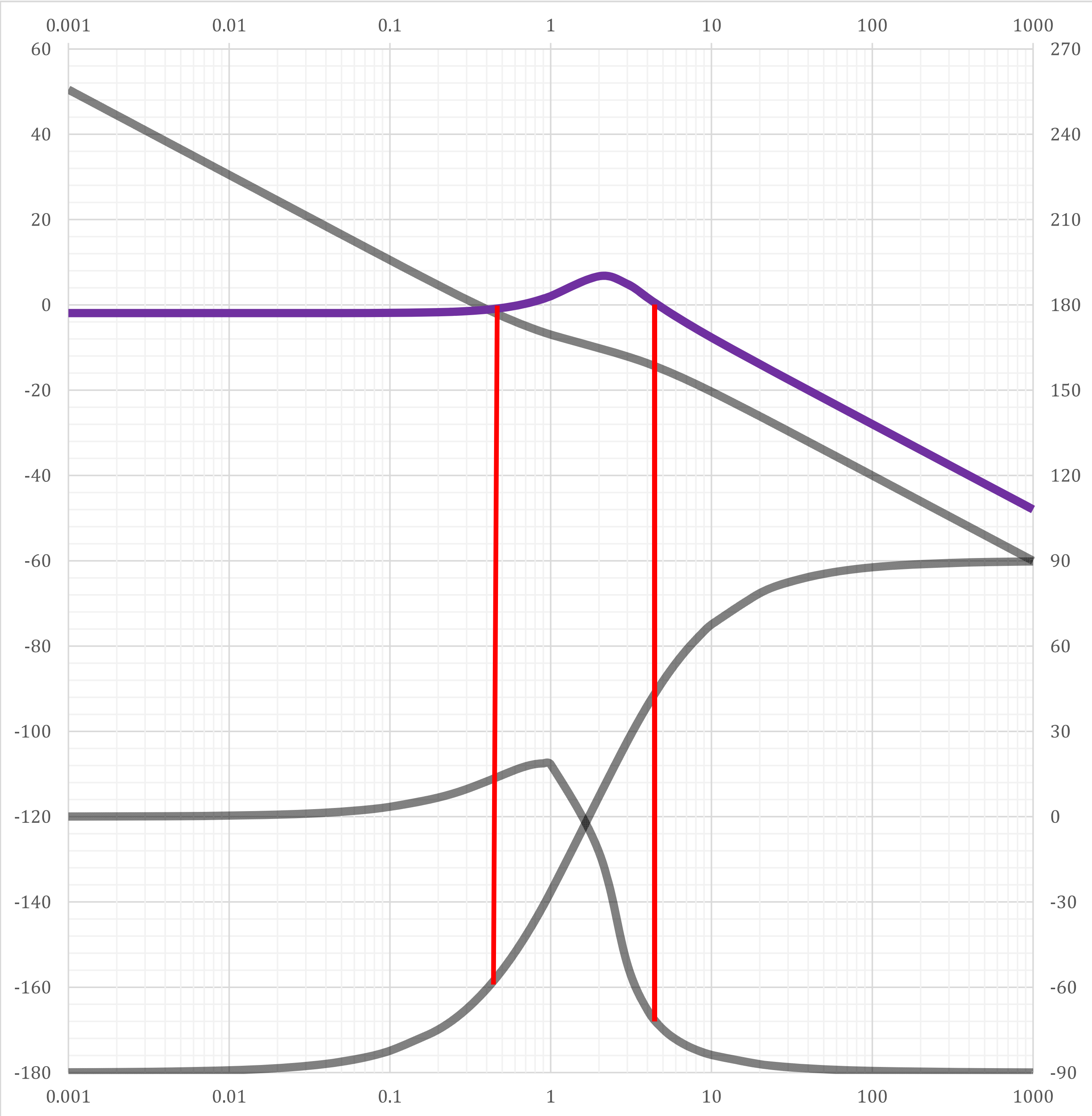


Fig.1

Observation:

- By the result of original problem, since 4>2, the system in Fig.6.96 is stable.

  Fig.2 是 K=4 時的 unit step 的 time response，可以看到此系統是穩定的。

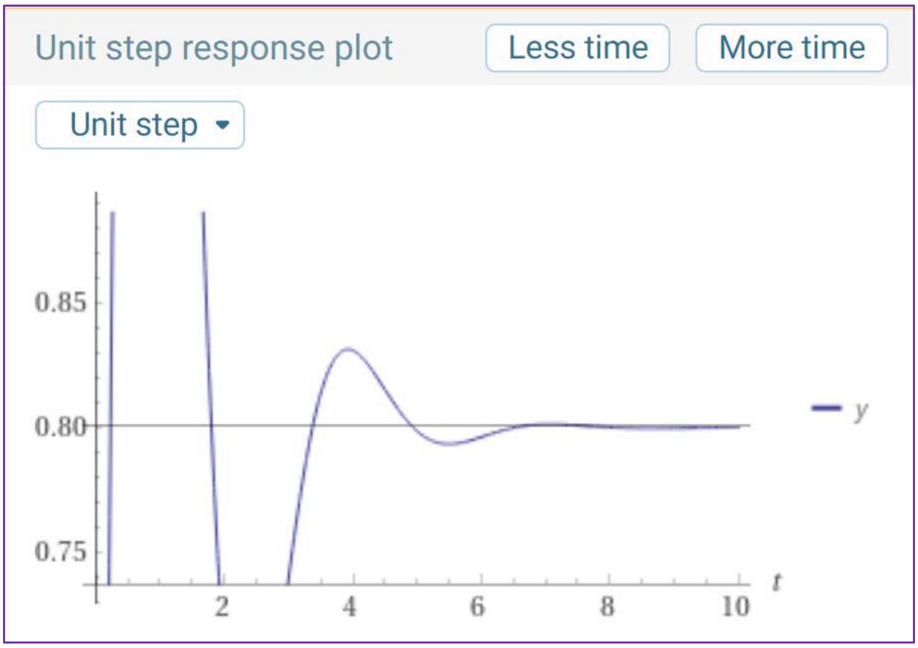


Fig.2

- By the result of original problem, since -1<2, the system in Fig.6.96 is NOT stable.

  Fig.2 是 K=-1 時的 unit step 的 time response，可以看到此系統是不穩定的。

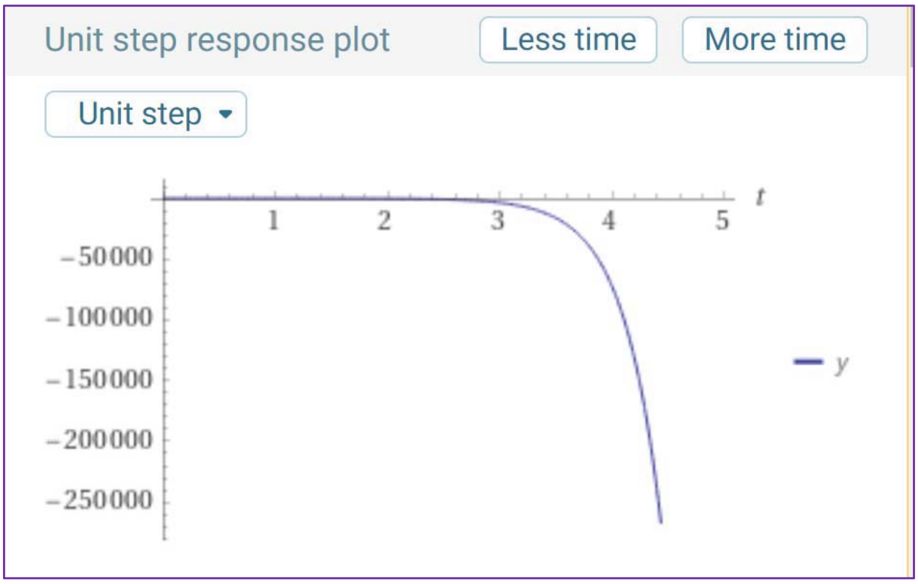
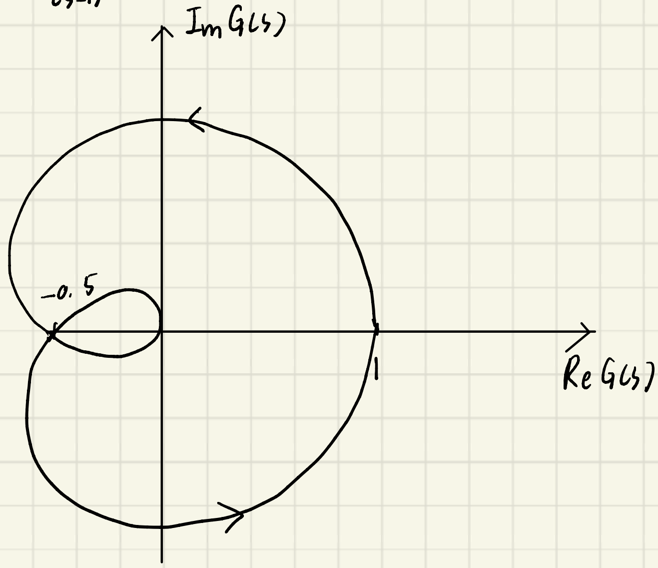


Fig.3

Problem 3.

$KG(s) = K\frac{s+1}{(s-1)^2}$



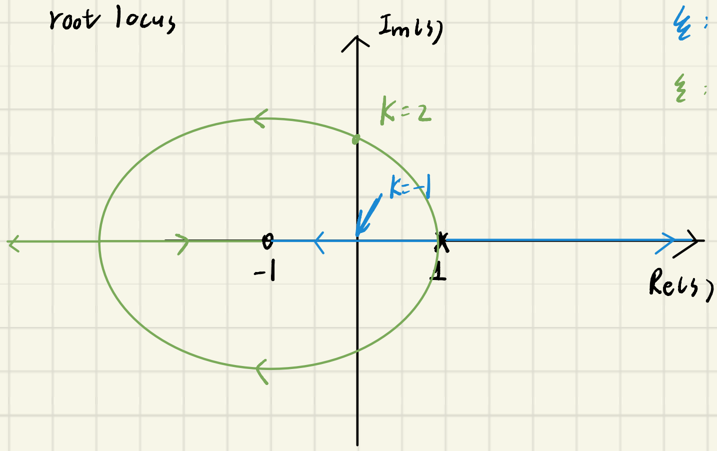
① $0<K<2$: $N=0$, $P=2$ ⇒ $Z=2$

② $K>2$: $N=-2$, $P=2$ ⇒ $Z=0$

③ $K<-1$: $N=-1$, $P=2$ ⇒ $Z=1$

③ $-1<K<0$: $N=0$, $P=2$ ⇒ $Z=2$

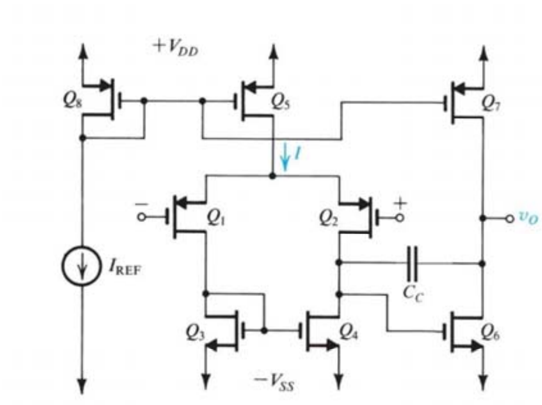
root locus



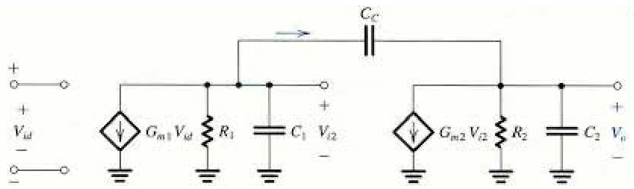
ξ : for K<0

ξ : for K>0

B09901015 王瑋



For low-frequency small signal analysis, a 2-stage CMOS op amp can be approximated by the following circuit:
Consider a particular implementation with



Gm1 = 1ma/V
Gm2 = 2mA/V
ro2=ro4=100kOhm (R1=ro2||ro4=50kOhm)
ro6=ro7=40kOhm (R2=ro6||ro7=20kOhm)
C2=1pF
Cc=1.6pF

We can obtain the transfer function (Vo/Vi)

$$\frac{G_{m1}(G_{m2}-sC_C)R_1R_2}{1+s[C_1R_1+C_2R_2+C_C(G_{m2}R_1R_2+R_1+R_2)]+s^2(C_1C_2+C_CC_1+C_CC_2)R_1R_2}$$

The transfer function can be further approximated to:

$$\frac{A_v(1-s/\omega_Z)}{(1+s/\omega_{P1})(1+s/\omega_{P2})}$$

With

$$A_v = G_{m1}G_{m2}R_1R_2 \qquad \omega_Z = \frac{G_{m2}}{C_C}$$

$$\omega_{P1} \approx \frac{1}{C_1R_1+C_2R_2+C_C(G_{m2}R_1R_2+R_1+R_2)} \approx \frac{1}{R_1G_{m2}R_2C_C}$$

$$\omega_{P2} \approx \frac{C_1R_1+C_2R_2+C_C(G_{m2}R_1R_2+R_1+R_2)}{(C_1C_2+C_CC_1+C_CC_2)R_1R_2} \approx \frac{G_{m2}}{C_2} \text{ for } C_C \gg C_2 \gg C_1$$

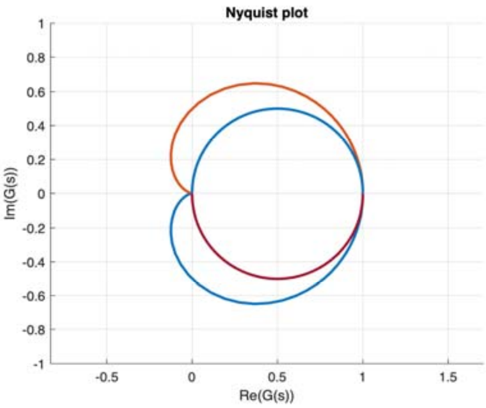
Then we can obtain a simpler transfer function:

$$\frac{2000(1-\frac{s}{1.25*10^9})}{(1+\frac{s}{312500})(1+\frac{s}{2*10^9})}$$

After normalizing and ignoring the gain factor, we obtain:

$$\frac{(1 - \frac{s}{1.25*10^9})}{(1 + \frac{s}{312500})(1 + \frac{s}{2*10^9})}$$

The Nyquist plot for this system:



We can observe that

(1) for all K > 0, the system is stable
(2) for -1 < K < 0, the system is stable
(3) for K < -1, the system is unstable, with Z = N + P = 2 + 0 = 2

Control Systems HW 9
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3. (Gain margin and phase margin)

36. For the system shown in Fig. 6.96, determine the Nyquist plot and apply the Nyquist criterion.

(a) to determine the range of values of K (positive and negative) for which the system will be stable, and

(b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

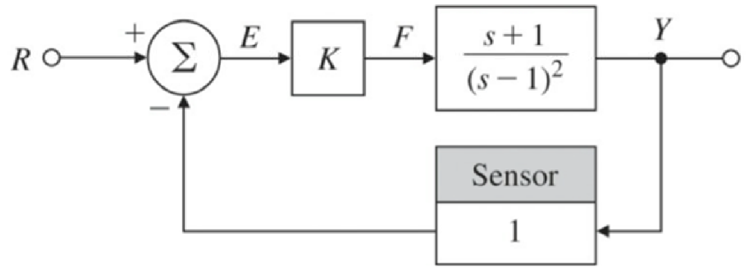


Fig. 6.96 Control system for Problem 6.36

The open-loop transfer function is $K\frac{s+1}{(s-1)^2}$, and the closed-loop transfer function is $\frac{K\frac{s+1}{(s-1)^2}}{1+\frac{K(s+1)}{(s-1)^2}} = \frac{K(s+1)}{(s-1)^2+K(s+1)} = \frac{Ks+K}{s^2+(K-2)s+(K+1)}$. Define $G(s) = \frac{s+1}{(s-1)^2}$.

I will use several different approaches learned in this course so far to attack the stability criterion part (part (a)):

1. Nyquist stability criterion. (part (a) and (b))
2. Root Locus (a portion of part (b) in the problem's statement).
3. Routh stability criterion.
4. Stability margins.

1. Nyquist Stability Criterion (Chapter 6E)

According to Nyquist Criterion, whether the point $s = -\frac{1}{K}$ is encircled by the Nyquist Plot $\{G(j\omega) \mid \omega \in \mathbb{R}\}$ determines the number of right hand poles (RHPs) and thus determines the stability of the closed-loop system.

$$G(j\omega) = \frac{s+1}{(s-1)^2}\Big|_{s=j\omega} = \frac{j\omega+1}{(j\omega-1)^2} = \frac{(j\omega+1)^3}{\omega^2+1}$$

$\omega = 0 \Rightarrow G(j0) = \frac{1}{(-1)^2} = 1 + j0$

$\omega = 0.5 \Rightarrow G(j0.5) = 0.16 + j0.88$

$\omega = 1 \Rightarrow G(j1) = -0.5 + j0.5$

$\omega = 1.5 \Rightarrow G(j1.5) = -0.54 + j0.11$

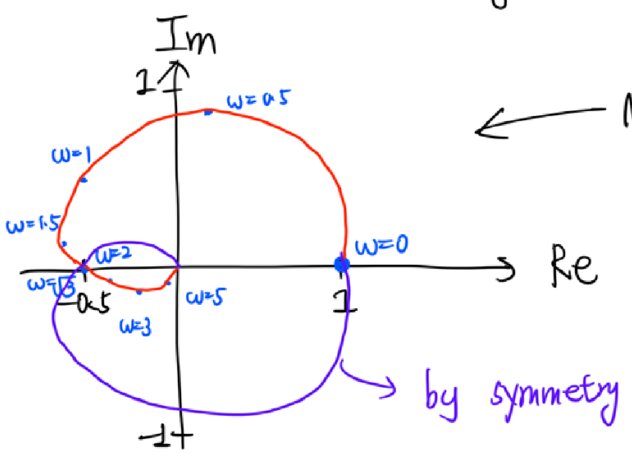
$\omega = \sqrt{3} \Rightarrow G(j\sqrt{3}) = -0.5$

$\omega = 2 \Rightarrow G(j2) = -0.44 - j0.08$

$\omega = 3 \Rightarrow G(j3) = -0.26 - j0.18$

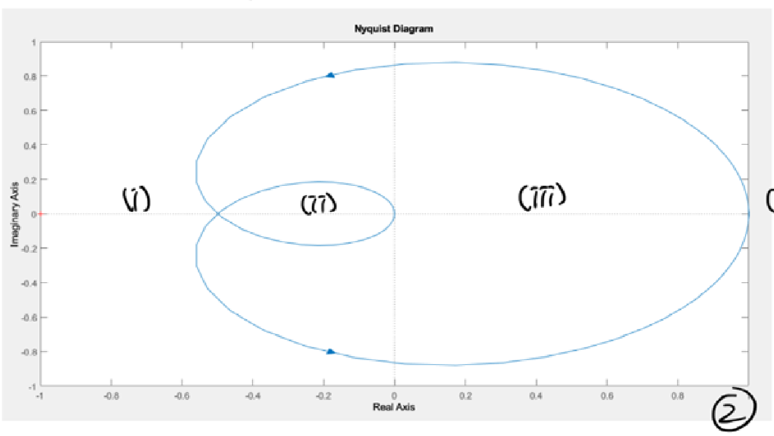
$\omega = 5 \Rightarrow G(j5) = -0.11 - j0.16$

```
w=
z=(i*w+1)/(i*w-1)^2;
disp('real')
disp(real(z))
disp("imag")
disp(imag(z))
```



← Nyquist Plot

Matlab gives:




```
sys=tf([1 1],[1 -2 1])
nyquist(sys)
```

The Nyquist plot cuts the Real axis into 4 segments.

(i) $(-\infty, -0.5)$ (ii) $(-0.5, 0)$ (iii) $(0,1)$ (iv) $(1,\infty)$

Now, let's discuss the case when the value $\frac{-1}{K}$ is in each segment. There are two poles $s=1$ and $s=1$ for the plant $G(s)$, so $P=2$ in all circumstances. To make the closed-loop transfer function stable, we require $Z=0$, i.e. $N=-2$ (by the criterion, $Z=N+P$), where Z is the number of RHP.

(i) $\frac{-1}{K} \in (-\infty, -0.5) \Rightarrow \frac{-1}{K} < \frac{-1}{2} \Rightarrow 0<K<2$

$\frac{-1}{K}$ not encircled $\Rightarrow N=0 \Rightarrow Z=N+P=0+2=2$

$\Rightarrow$ 2 RHPs for CL; unstable CL.

(ii) $\frac{-1}{K} \in (-0.5, 0) \Rightarrow \frac{-1}{2} < \frac{-1}{K} < 0 \Rightarrow K>2$

2 counterclockwise encirclement $\Rightarrow N=-2$

$\Rightarrow Z=N+P=-2+2=0 \Rightarrow$ 0 RHP for CL; stable CL.

(iii) $\frac{-1}{K} \in (0,1) \Rightarrow 0 < \frac{-1}{K} < 1 \Rightarrow K<-1$

1 counterclockwise encirclement $\Rightarrow N=-1$

$\Rightarrow Z=N+P=-1+2=1 \Rightarrow$ 1 RHP for CL; unstable CL.

(iv) $\frac{-1}{K} \in (1,\infty) \Rightarrow 1 < \frac{-1}{K} \Rightarrow -1<K<0$

$\frac{-1}{K}$ not encircled. $\Rightarrow N=0 \Rightarrow Z=N+P=0+2=2$

$\Rightarrow$ 2RHPs for CL; unstable CL.

Summary :

K	$\frac{-1}{K}$	number of RHP	stable?
$K<-1$	$0<\frac{-1}{K}<1$	1	No
$-1<K<0$	$\frac{-1}{K}>1$	2	No
$0<K<2$	$\frac{-1}{K}<\frac{-1}{2}$	2	No
$K>2$	$-\frac{1}{2}<\frac{-1}{K}<0$	0	Yes

Part (a) and the first half of part (b) is included in this chart.

2. Root Locus

characteristic equation: $1+KG(s)=0 \Rightarrow s^2+(k-2)s+(k+1)=0$

$$\Rightarrow s=\frac{2-k}{2}\pm\frac{\sqrt{(k-2)^2-4(k+1)}}{2}=(1-\frac{k}{2})\pm\frac{\sqrt{k^2-8k}}{2}$$

- For $K<0$, the roots are real. $K<0 \Rightarrow \frac{k}{2}<0 \Rightarrow 1-\frac{k}{2}>1$; $\frac{\sqrt{k^2-8k}}{2}\geq 0 \Rightarrow$ at least one root lies in RHP. $\Rightarrow$ unstable.
- For $k\geq 8$, the roots are real. $k\geq 8 \Rightarrow \frac{k}{2}\geq 4 \Rightarrow 1-\frac{k}{2}\leq -3$

$$S_-\equiv(1-\frac{k}{2})-\frac{\sqrt{k^2-8k}}{2}\leq -3 \text{ for all } k\geq 8$$

$$S_+\equiv(1-\frac{k}{2})+\frac{\sqrt{k^2-8k}}{2},$$

$$\lim_{k\to\infty}S_+=1+\frac{1}{2}\left(\lim_{k\to\infty}\sqrt{k^2-8k}-k\right)=1+\frac{1}{2}\lim_{k\to\infty}\frac{-8k}{(\sqrt{k^2-8k}+k)}$$

$$=1+\frac{1}{2}\lim_{k\to\infty}\frac{-8}{\sqrt{1+\frac{8}{k}}+1}=1+\frac{1}{2}(-4)=-1$$

$\Rightarrow S_+<-1$ for all $K\in[8,\infty)$

$\therefore$ All poles are in LHP for $k>8 \Rightarrow$ stable.

- For $0\leq K\leq 8$: the real part is $1-\frac{k}{2}$.

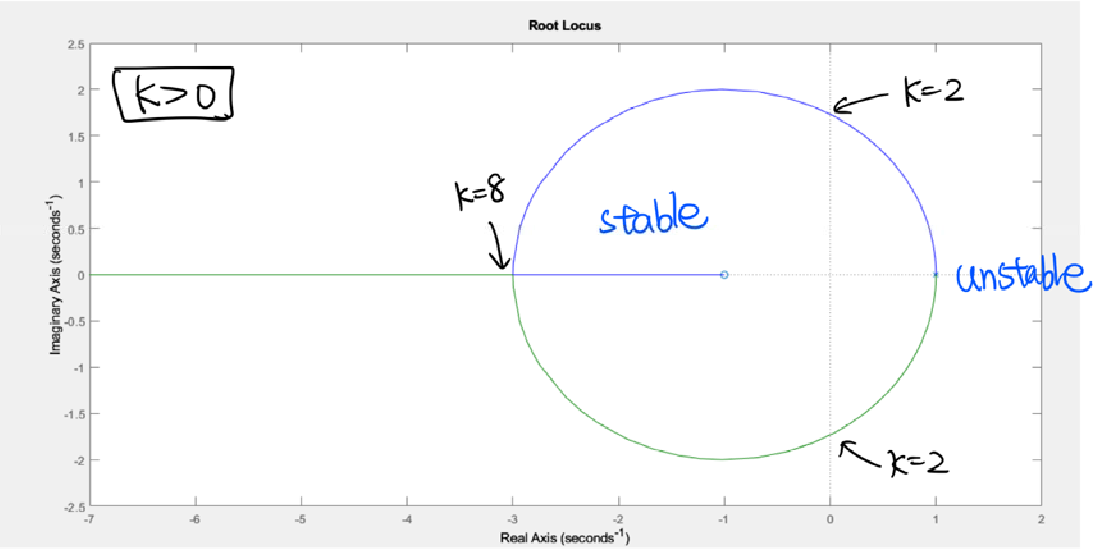
$0<k<2 \Rightarrow 1-\frac{k}{2}>0 \Rightarrow$ the poles are in RHP $\Rightarrow$ unstable

$2<k<8 \Rightarrow 1-\frac{k}{2}<0 \Rightarrow$ the poles are in LHP $\Rightarrow$ stable

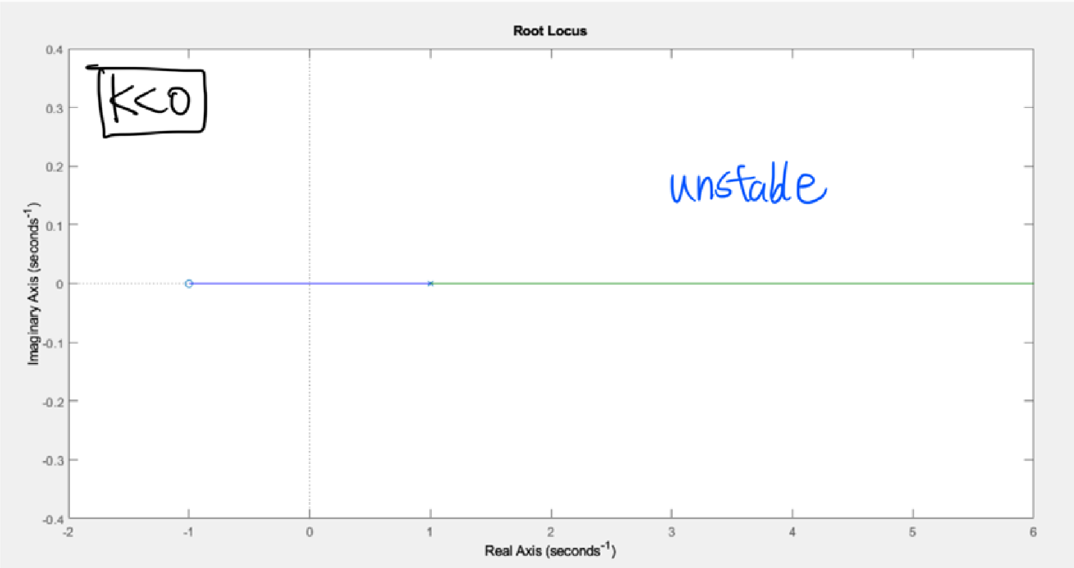
Therefore, $\begin{cases} K>2 \iff \text{the CL system is stable} \\ K<2 \iff \text{the CL system is unstable.} \end{cases}$

This result agree with the result of using the Nyquist stability criterion.

The root locus is drawn with Matlab in these figures, but if we only need to determine stability, the argument above is sufficient.




```
sys=tf([1 1],[1 -2 1])
rlocus(sys)
```




```
sys=tf(-[1 1],[1 -2 1])
rlocus(sys)
```

3. Routh stability criterion

The characteristic equation is $s^2+(k-2)s+(k+1)=0$.

The Routh array is

$$
\begin{array}{lll}
s^2 & 1 & k+1 \\
s^1 & k-2 & 0 \\
s^0 & -\dfrac{\begin{vmatrix} 1 & k+1 \\ k-2 & 0 \end{vmatrix}}{k-2} &
\end{array}
$$

Routh stability criterion requires that all the numbers in the first column to be positive to make the CL system stable.

That is, to make it stable, we have

$$
\begin{cases} 1>0 \\ k-2>0 \\ -\dfrac{-(k+1)(k-2)}{k-2}>0 \end{cases} \Rightarrow \begin{cases} k>2 \\ k>-1 \end{cases} \Rightarrow k>2
$$

Therefore, $\begin{cases} K>2 \iff \text{the CL system is stable} \\ K<2 \iff \text{the CL system is unstable.} \end{cases}$

This result is same as the previous two, as it should be.

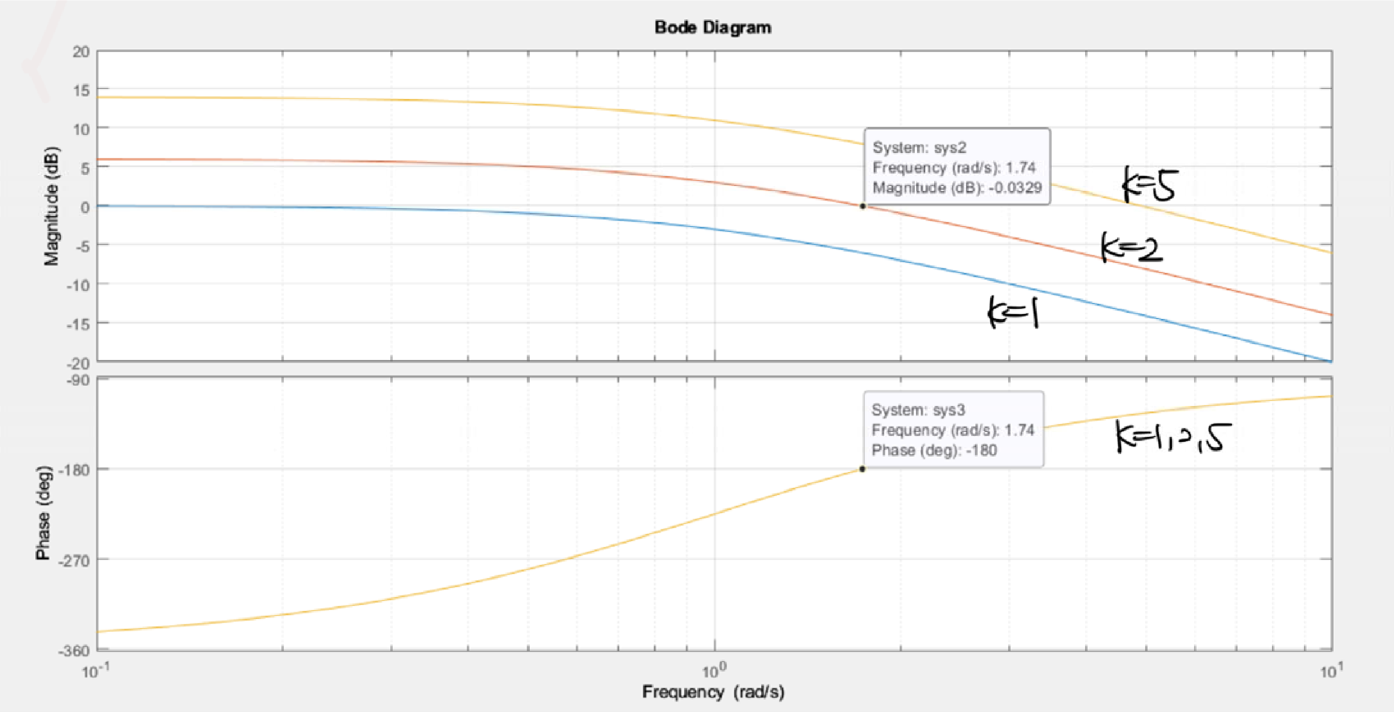
4. Stability Margins

Here, I will use the Bode plot for $K=1$, $K=2$ and $K=5$ to show the result.

```
K1=1
K2=2
K3=5

G=tf([1 1],[1 -2 1]);
sys1=K1*G;
sys2=K2*G;
sys3=K3*G;

bode(sys1),grid
hold on
bode(sys2),grid
hold on
bode(sys3),grid
hold off
```



(i) Gain Margin

The frequency such that the phase is 180° is approximately $\omega_c = 1.74$ rad/s, as seen in the figure. Therefore, the gain margin is the magnitude at $\omega = \omega_c$.

K=1 (K<2): $|KG(j\omega_c)| < 1$ (since $20\log_{10}|KG(j\omega_c)| < 0$, as shown in the magnitude plot) $\Rightarrow GM = \frac{1}{|KG(j\omega_c)|} > 1 \Rightarrow$ stable

K=2: $|KG(j\omega_c)| \approx 1$ (since $20\log_{10}|KG(j\omega_c)| \approx 0$)

$\Rightarrow GM = \frac{1}{|KG(j\omega_c)|} \approx 1$ (critical condition)

K=5 (K>2): $|KG(j\omega_c)| > 1$ (since $20\log_{10}|KG(j\omega_c)| > 0$)

$\Rightarrow GM = \frac{1}{|KG(j\omega_c)|} < 1 \Rightarrow$ unstable.

(ii) Phase Margin

K=1 (K<2): $|KG(j\omega)| = 1$ when $\omega \to 0^+$.

Phase at really small ω is below $-180° \Rightarrow PM < 0°$

$\Rightarrow$ unstable

$K=2$: $|KG(j\omega)|=1$ when $\omega=\omega_c=1.74$ rad/s

$\Rightarrow$ Phase at $\omega=\omega_c$ is almost exactly $-180°$. $\Rightarrow PM \doteqdot 0°$

(critical condition)

$K=5$ $(K>2)$: $|KG(j\omega)|=1$ when $\omega \doteqdot 4.9$ rad/s

$\Rightarrow$ Phase at $\omega=4.9$ rad/s is above $-180°$ $\Rightarrow PM>0°$

$\Rightarrow$ stable.

We can see that the results obtained by analyzing the gain margin conflicts with that obtained by analyzing the phase margin and by using the previous 3 stability criteria. If I have not done it wrong, this case may be an example of the conflicting results obtained by GM and PM (perhaps the double RHP poles of G(s) is the main reason). Since the criteria of GM and PM on stability is not that rigorous as the previous 3 criteria, it still seems that the system is stable if $K>2$.

參考觀摩的作業 4. (Gain-Phase relation)
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