Control System: Homework 08 for Units 6A, 6B, 6C: Bode Plot

Assigned: Nov 25, 2022

Due: Dec 1, 2022 (11:59pm)

1. (Frequency response and Bode plot)

2. (a) Calculate the magnitude and phase of

$$G(s) = \frac{1}{s+5}$$

by hand for $\omega = 1, 2, 5, 10, 20, 50,$ and 100 rad/sec.

(b) sketch the asymptotes for $G(s)$ according to the Bode plot rules, and compare these with your computed results from part (a).

For (a), you can use calculator to compute the exact numerical results.

In exam, we will provide the problem which can be easily calculated by pen.

Hence, you cannot use any calculators in exam.

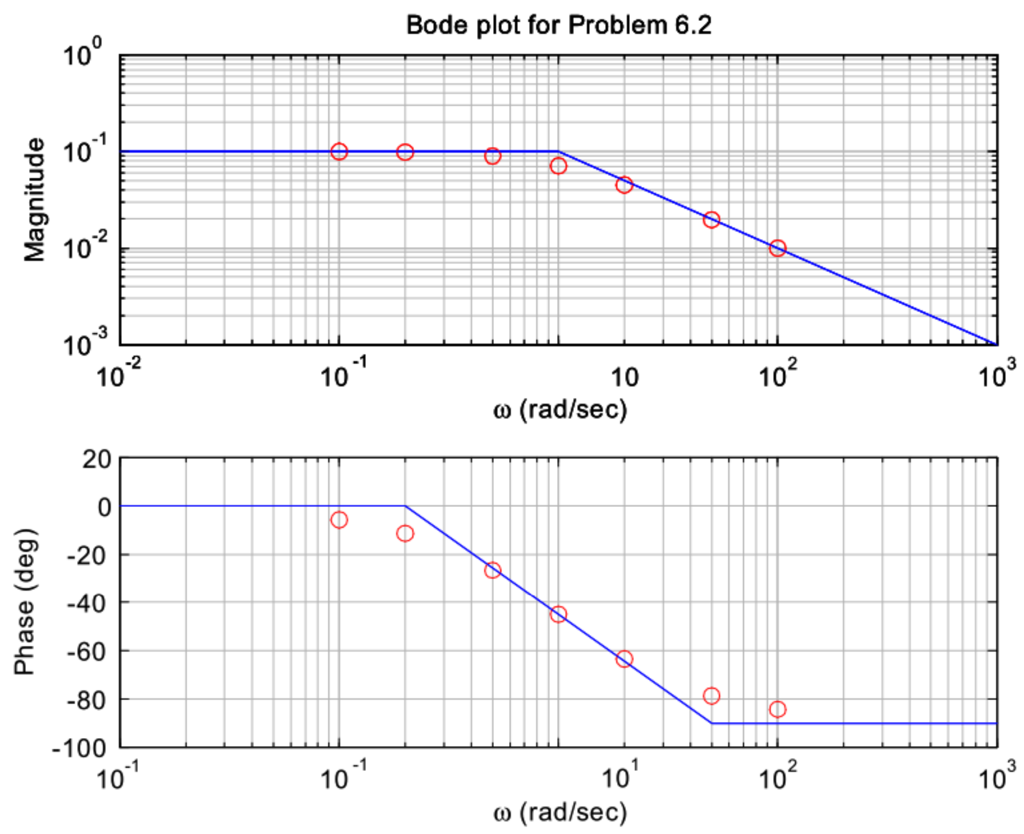
Solution:

(a)

$$\begin{aligned} G(s) &= \frac{1}{s+5},\ G(j\omega) = \frac{1}{5+j\omega} = \frac{5-j\omega}{25+\omega^2} \\ |G(j\omega)| &= \frac{1}{\sqrt{25+\omega^2}},\ \angle G(j\omega) = -\tan^{-1}\frac{\omega}{5} \end{aligned}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
1	0.1961	−11.309
2	0.18569	−21.801
5	0.1414	−45
10	0.08944	−63.43
20	0.0485	−75.96
50	0.0199	−84.3
100	0.00998	−87.137

(b) To plot the asymptotes, you first note that $n = 0$, as defined in Section 6.1.1. That signifies that the leftmost portion of the asymptotes will have zero slope. That portion of the asymptotes will be located at the DC gain of the transfer function, which, in this case it can be seen by inspection to be 0.1. So the asymptote starts with a straight horizontal line at 0.1 and that continues until the breakpoint at $\omega = 5$, at which point the asymptote has a slope of $n = -1$ that continues until forever, at least until the edge of the paper. The values computed above by "hand" (at least we hope you didn' tcheat) are plotted on the graph below and you see they match quite well except very near the breakpoint, a you should have expected. The Bode plot is :



2. (Bode Plot and Frequency Properties)

3. For the open-loop transfer functions of the unity feedback control systems. given below, sketch the Bode magnitude and phase plots. Find their gain margins, gain crossover frequencies, and phase crossover frequencies.

(a) $L(s) = \dfrac{10}{s\,[s+10]}$

(d) $L(s) = \dfrac{50}{s(0.5s+1)^2}$

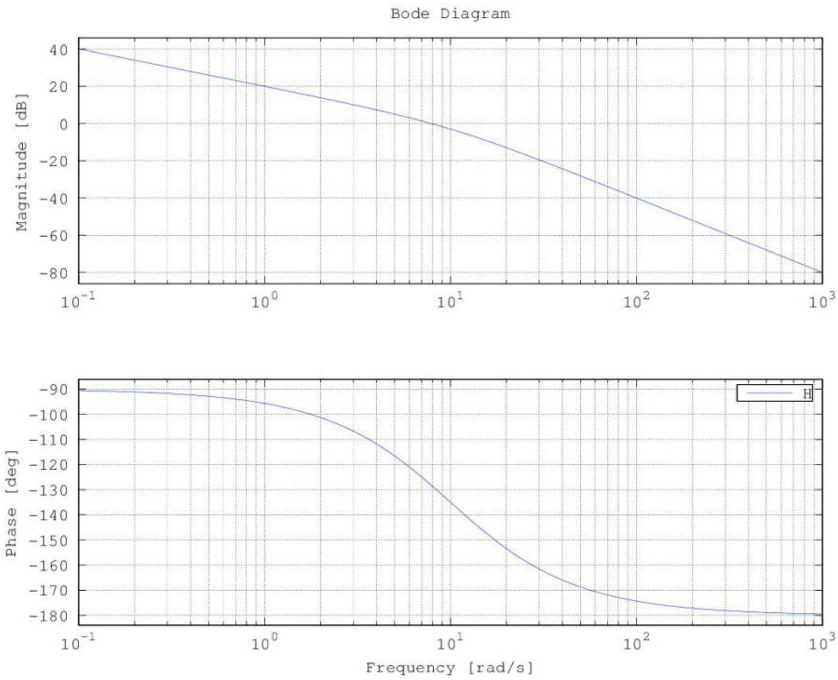
Try to sketch the plots by hand-pen and, then use Matlab code to verify your results.

Solution:

(a) $L(s) = \dfrac{10}{s\,[s+10]}$

In this case, Break frequency is 10.

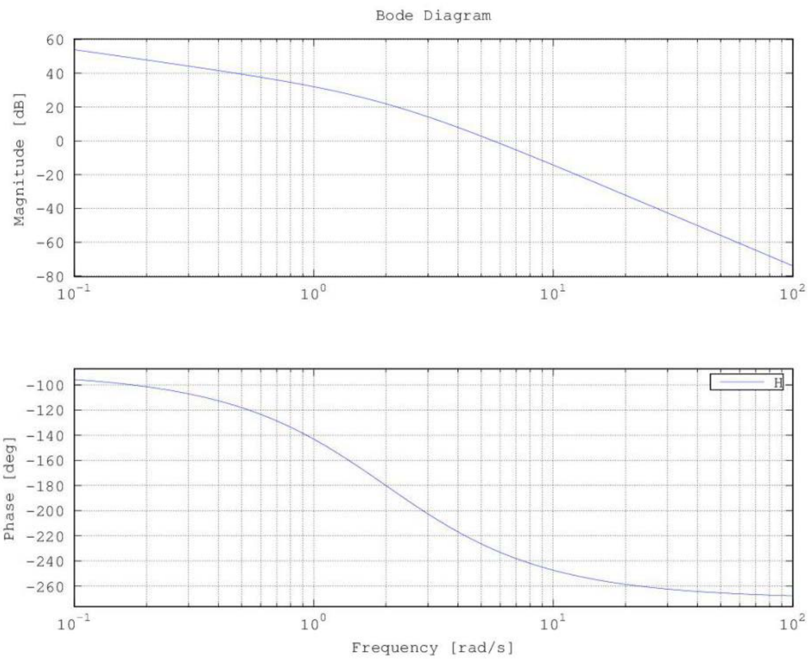
Bode plot for Prob. 6.3 (a)



(d) $L(s) = \dfrac{50}{s(0.5s+1)^2}$

Breaking frequency = 2.

Bode plot for Prob 6.3(d)



3. (Bode Plot and Timing Properties)

11. A normalized second-order system with a damping ratio $\zeta = 0.5$ and an additional zero is given by

$$G(s) = \frac{s/a + 1}{s^2 + s + 1}.$$

Use MATLAB to compare the M_p from the step response of the system for $a = 0.01, 0.1,$ 1, 10, and 100 with the M_r from the frequency response of each case. Is there a correlation between M_r and M_p?

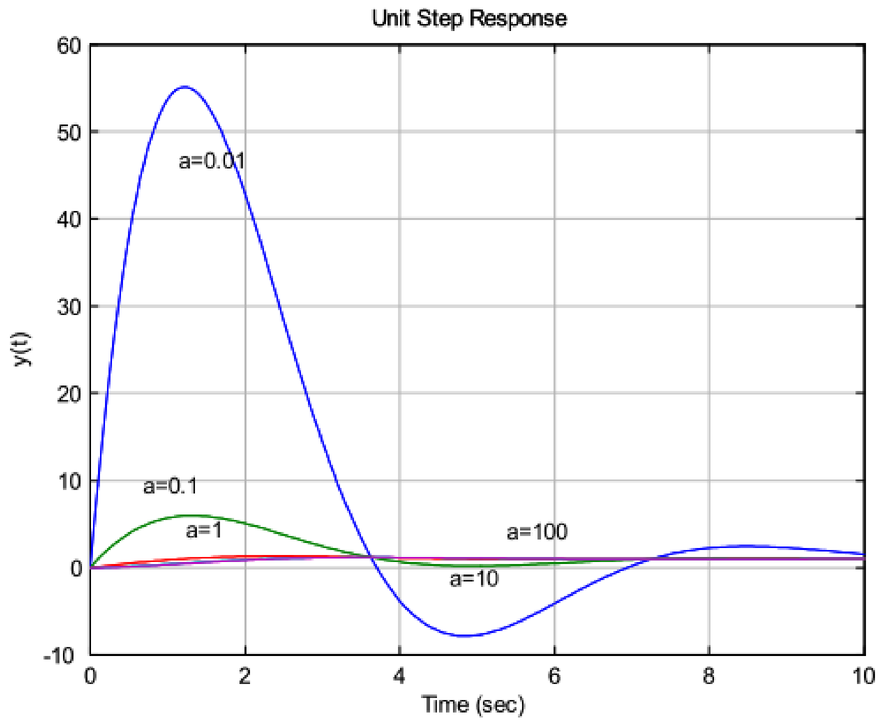
Solution:

α	Resonant peak, M_r	Overshoot, M_p
0.01	98.8	54.1
0.1	9.93	4.94
1	1.46	0.30
10	1.16	0.16
100	1.15	0.16

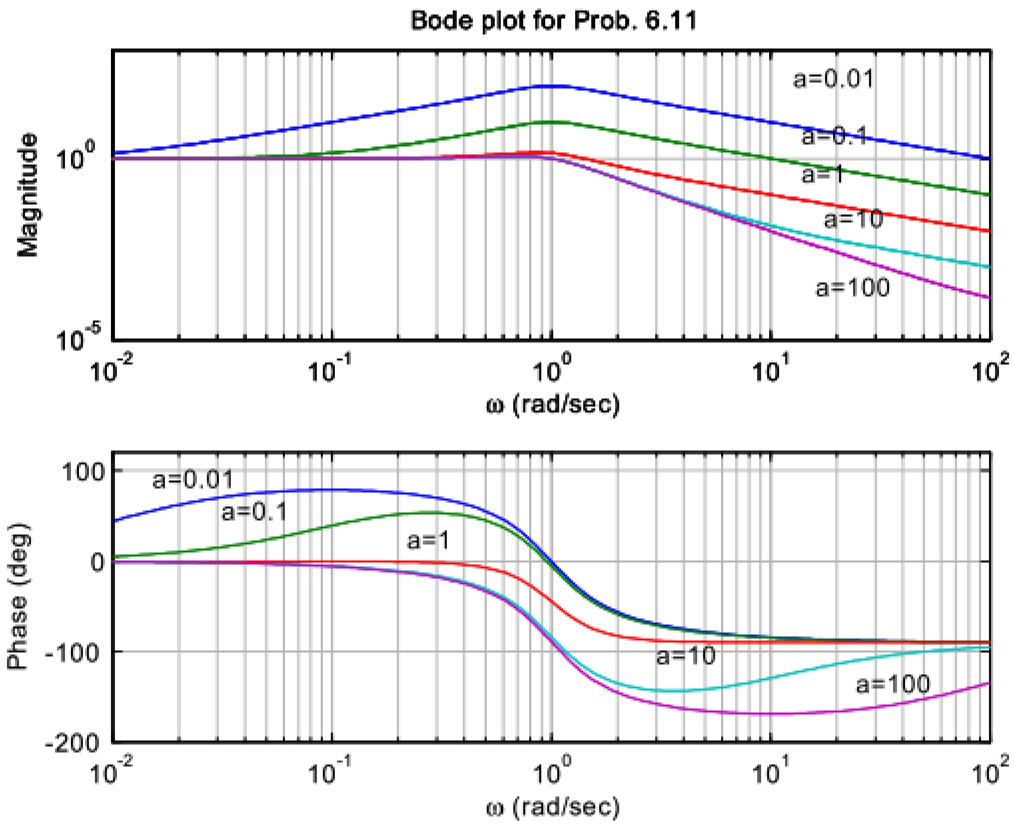
As α is reduced, the resonant peak in frequency response increases. This leads us to expect extra peak overshoot in transient response. This effect is significant in case of $\alpha = 0.01, 0.1, 1$, while the resonant peak in frequency response is hardly changed in case of $\alpha = 10$. Thus, we do not have considerable change in peak overshoot in transient response for $\alpha \geq 10$.

The response peak in frequency response and the peak overshoot in transient response are correlated.

The output derivative feedback is acting only when there is a change in the output. Therefore, for a ramp input, the derivative action will minimize the deviation from the reference because the input signal is continuously increasing.



Unit Step Response
60
50
40
30
20
10
0
-10
y(t)
a=0.01
a=0.1
a=1
a=100
a=10
0
2
4
6
8
10
Time (sec)



Bode plot for Prob. 6.11
Magnitude
10^0
10^{-5}
a=0.01
a=0.1
a=1
a=10
a=100
10^{-2}
10^{-1}
10^0
10^1
10^2
ω (rad/sec)
Phase (deg)
100
0
-100
-200
a=0.01
a=0.1
a=1
a=10
a=100
10^{-2}
10^{-1}
10^0
10^1
10^2
ω (rad/sec)

參考觀摩的作業

1. (Frequency response and Bode plot)

作者： b08901154，王煒騰 **理由：** 討論不同階數與極點位置的系統並從波德圖觀察穩定性
作者： b08901176，陳育楷 **理由：** 討論不同系統的波德圖並觀察手描與實際值的差異
作者： b09502033，朱本毅 **理由：** 針對一顆馬達進行系統識別並找出角度對電壓的二階轉移函數

Control System: Homework 08 for Units 6A, 6B, 6C: Bode Plot

Assigned: Nov 25, 2022

Due: Dec 1, 2022 (11:59pm)

B08901154

電機四

王煒騰

1. (Frequency response and Bode plot)

2. (a) Calculate the magnitude and phase of

$$G(s) = \frac{1}{s+5}$$

by hand for $\omega = 1, 2, 5, 10, 20, 50,$ and 100 rad/sec.

(b) sketch the asymptotes for $G(s)$ according to the Bode plot rules, and compare these with your computed results from part (a).

Solution:

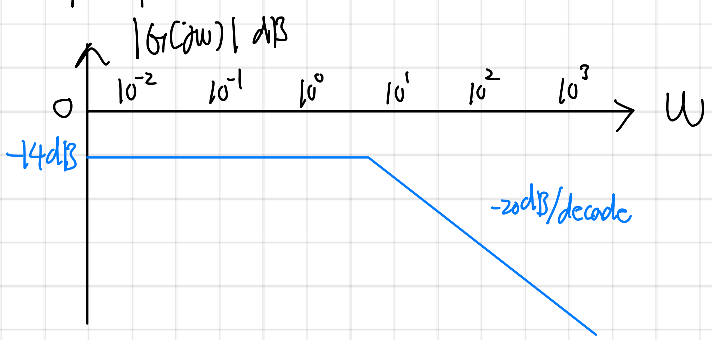
(a) $G(j\omega) = \frac{1}{j\omega+5} \Rightarrow |G(j\omega)| = \frac{1}{\sqrt{25+\omega^2}}$, $\angle G(j\omega) = -\tan^{-1}(\frac{\omega}{5})$

ω (rad/sec)	1	2	5	10	20	50	100
magnitude	0.1961	0.1857	0.1414	0.0894	0.0485	0.0199	0.01
phase	-11.3	-21.8	-45	-63.4	-76	-84.3	-87.1

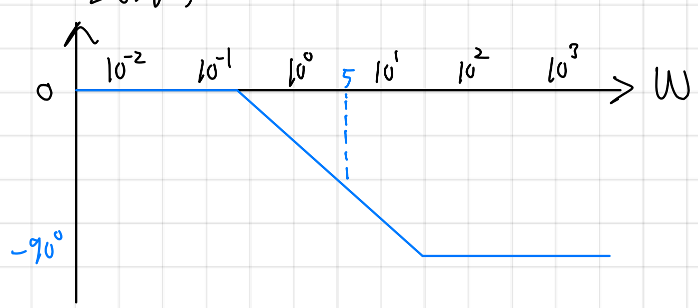
(b) asymptote:

$|G(j\cdot 0)| = \frac{1}{5} = -14\,dB$

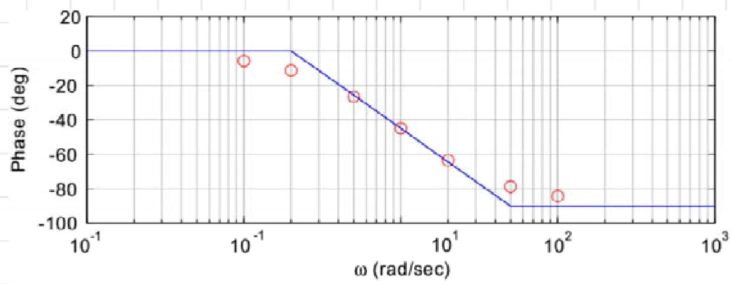
|G|



∠G



Comparison



討論：

(1) $\frac{1}{s+5}$ pole 位於 LHP，對於 single-pole system 將永遠 stable

(2) 修改此題為 $\frac{1}{(s+5)(s+\omega_p)}$，一樣將永遠 stable ($\omega_p > 0$)

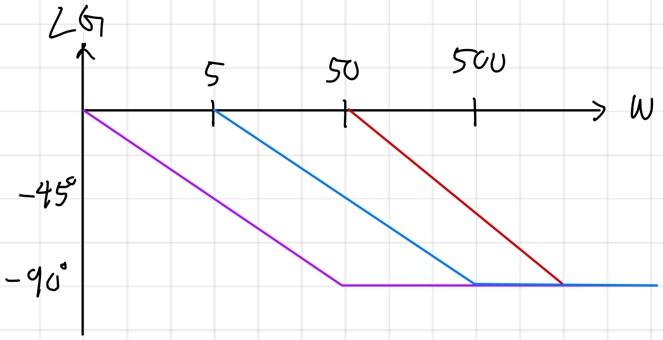
如 pole 共軛，$\frac{1}{(1+s/\omega_{p_1})(1+s/\omega_{p_2})}$ $\Rightarrow S_p = -\frac{\omega_{p_1}+\omega_{p_2}}{2} \pm \sqrt{(\omega_{p_1}+\omega_{p_2})^2 - 4\omega_{p_1}\omega_{p_2}}$

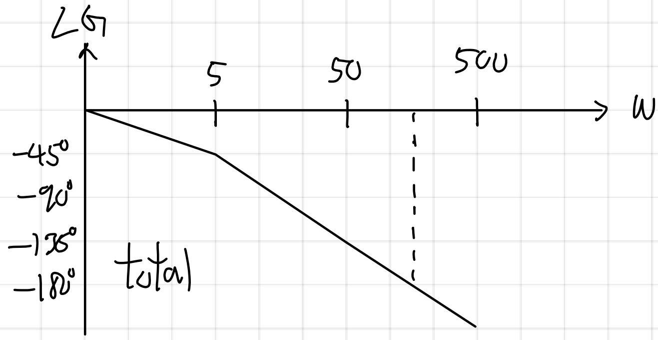
因此 two-pole system 將永遠 stable

(3) 對於 three-pole system，令為 $\frac{5\times 10^n}{(1+\frac{s}{5})(1+\frac{s}{50})(1+\frac{s}{500})}$

因為所有 pole 與相鄰之 pole 相差至少 10 倍

因此 $\omega_{-180^\circ} \approx \sqrt{50\times 500}$ (50, 500 中間) ≈ 158





$n \geq 2$: $|G(j\omega_{180^\circ})| = 20\log(5\times 10^n) - 40\text{ dB} > 0 \Rightarrow$ unstable

$n = 1$: $|G(j\omega_{180^\circ})| = 20\log 50 - 40\text{ dB} < 0 \Rightarrow$ stable

所以當 $n \geq 2$ 時 $\frac{5\times 10^n}{(1+\frac{s}{5})(1+\frac{s}{50})(1+\frac{s}{500})}$ 會 unstable

1(U6: Bode Plot)

Problem:

(1) Plot the magnitude, phase and of asymptotes of

$$G_1(s) = \frac{1}{\left(1+\frac{s}{100}\right)\left(1+\frac{s}{1000}\right)} \text{ and } G_2(s) = \frac{1+\frac{s}{1000}}{1+\frac{s}{100}} \text{ from } \omega = 1 \text{ to } 100\text{M rad/sec},$$

(2) Plot the magnitude, phase and of asymptotes of

$$G_3(s) = \frac{1}{\left(1+\frac{s}{100}\right)\left(1+\frac{s}{10000}\right)} \text{ and } G_4(s) = \frac{1+\frac{s}{10000}}{1+\frac{s}{100}} \text{ from } \omega = 1 \text{ to } 1\text{G rad/sec}.$$

Solution:

$$G_1(j\omega) = \frac{1}{\left(1+\frac{j\omega}{100}\right)\left(1+\frac{j\omega}{1000}\right)}, |G_1(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{100}\right)^2}\sqrt{1+\left(\frac{\omega}{1000}\right)^2}}, \angle G_1(j\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{1000}\right)$$

$$G_2(j\omega) = \frac{\left(1+\frac{j\omega}{1000}\right)}{\left(1+\frac{j\omega}{100}\right)}, |G_2(j\omega)| = \frac{\sqrt{1+\left(\frac{\omega}{1000}\right)^2}}{\sqrt{1+\left(\frac{\omega}{100}\right)^2}}, \angle G_2(j\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) + \tan^{-1}\left(\frac{\omega}{1000}\right)$$

$$G_3(j\omega) = \frac{1}{\left(1+\frac{j\omega}{100}\right)\left(1+\frac{j\omega}{10000}\right)}, |G_3(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{100}\right)^2}\sqrt{1+\left(\frac{\omega}{10000}\right)^2}}, \angle G_3(j\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{10000}\right)$$

$$G_4(j\omega) = \frac{\left(1+\frac{j\omega}{10000}\right)}{\left(1+\frac{j\omega}{100}\right)}, |G_4(j\omega)| = \frac{\sqrt{1+\left(\frac{\omega}{10000}\right)^2}}{\sqrt{1+\left(\frac{\omega}{100}\right)^2}}, \angle G_4(j\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) + \tan^{-1}\left(\frac{\omega}{10000}\right)$$

	$G_1(s)$				$G_2(s)$				$G_3(s)$				$G_4(s)$			
ω (rad/sec)	magnitude (dB)		phase(degree)		magnitude (dB)		phase(degree)		magnitude (dB)		phase(degree)		magnitude (dB)		phase(degree)	
	(calculated)	(estimate)	(calculate	(estimate)	(calculate	(estimate)	(calculate	(estimate)	(calculated	(estimate)	(calculate	(estimate)	(calculate	(estimate)	(calculate	(estimate)
1	-0.000439	0	-0.63023	0	-0.00043	0	-0.51564	0	-0.000434	0	-0.57867	0	-0.00043	0	-0.56721	0
2	-0.001754	0	-1.26035	0	-0.00172	0	-1.03117	0	-0.001737	0	-1.15722	0	-0.00174	0	-1.1343	0
5	-0.010952	0	-3.14888	0	-0.01074	0	-2.57593	0	-0.010845	0	-2.89105	0	-0.01084	0	-2.83376	0
10	-0.043648	0	-6.28353	0	-0.04278	0	-5.13765	0	-0.043218	0	-5.76789	0	-0.04321	0	-5.6533	0
20	-0.17207	0	-12.4557	0	-0.1686	0	-10.1642	0	-0.170351	0	-11.4245	0	-0.17032	0	-11.1953	0
50	-0.979944	0	-29.4275	-25.6196	-0.95826	0	-23.7026	-25.6196	-0.969209	0	-26.8515	-25.6196	-0.96899	0	-26.2786	-25.61955
100	-3.053514	0	-50.7106	-45	-2.96709	0	-39.2894	-45	-3.010734	0	-45.5729	-45	-3.00987	0	-44.4271	-45
200	-7.160033	-6.0206	-74.7449	-64.3804	-6.81937	-6.0206	-52.125	-64.3804	-6.991437	-6.0206	-64.5807	-64.3804	-6.98796	-6.0206	-62.2892	-64.38045
500	-15.11883	-13.9794	-105.255	-115.62	-13.1806	-13.9794	-52.125	-64.3804	-14.16058	-13.9794	-81.5525	-90	-14.1389	-13.9794	-75.8277	-90
1000	-23.05351	-20	-129.289	-135	-17.0329	-20	-39.2894	-45	-20.08643	-20	-90	-90	-20	-20	-78.5788	-90
2000	-33.02114	-32.0412	-150.573	-154.38	-19.0417	-20	-23.7026	-25.6196	-26.20178	-26.0206	-98.4475	-90	-25.8611	-26.0206	-75.8277	-90
5000	-48.13087	-47.9588	-167.544	-180	-19.8314	-20	-10.1642	0	-34.95024	-33.9794	-115.419	-115.62	-33.012	-33.9794	-62.2892	-64.38045
10000	-60.04365	-60	-173.716	-180	-19.9572	-20	-5.13765	0	-43.01073	-40	-134.427	-135	-36.9901	-40	-44.4271	-45
20000	-72.05215	-72.0412	-176.851	-180	-19.9893	-20	-2.57593	0	-53.01041	-52.0412	-153.148	-154.38	-39.031	-40	-26.2786	-25.61955
50000	-87.96055	-87.9588	-178.74	-180	-19.9983	-20	-1.03117	0	-68.12915	-67.9588	-168.575	-180	-39.8297	-40	-11.1953	0
100000	-100.0004	-100	-179.37	-180	-19.9996	-20	-0.51564	0	-80.04322	-80	-174.232	-180	-39.9568	-40	-5.6533	0
200000									-92.05204	-92.0412	-177.109	-180	-39.9892	-40	-2.83376	0
500000									-107.9605	-107.959	-178.843	-180	-39.9983	-40	-1.1343	0
1000000									-120.0004	-120	-179.421	-180	-39.9996	-40	-0.56721	0

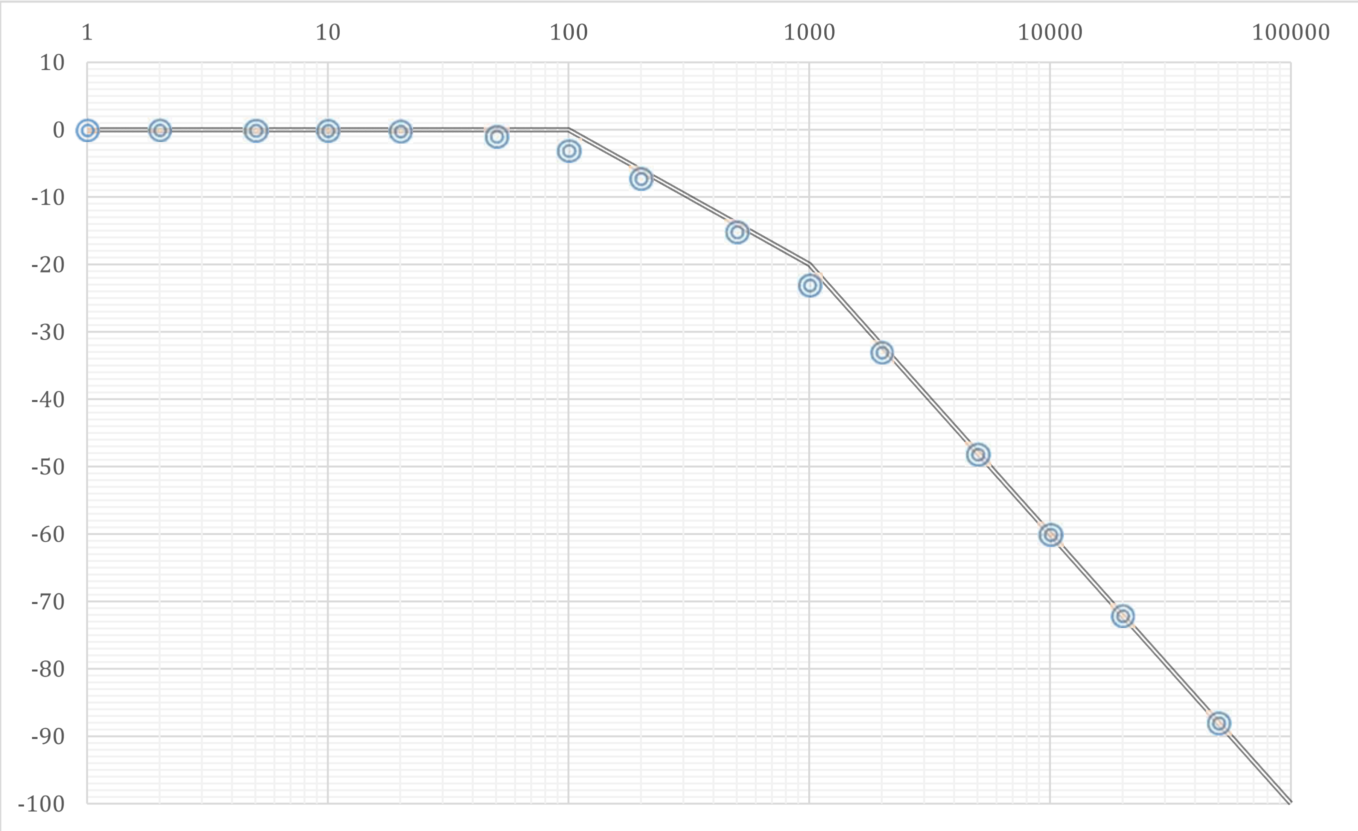
H W 8

B08901176
陳 育 楷

Plot:

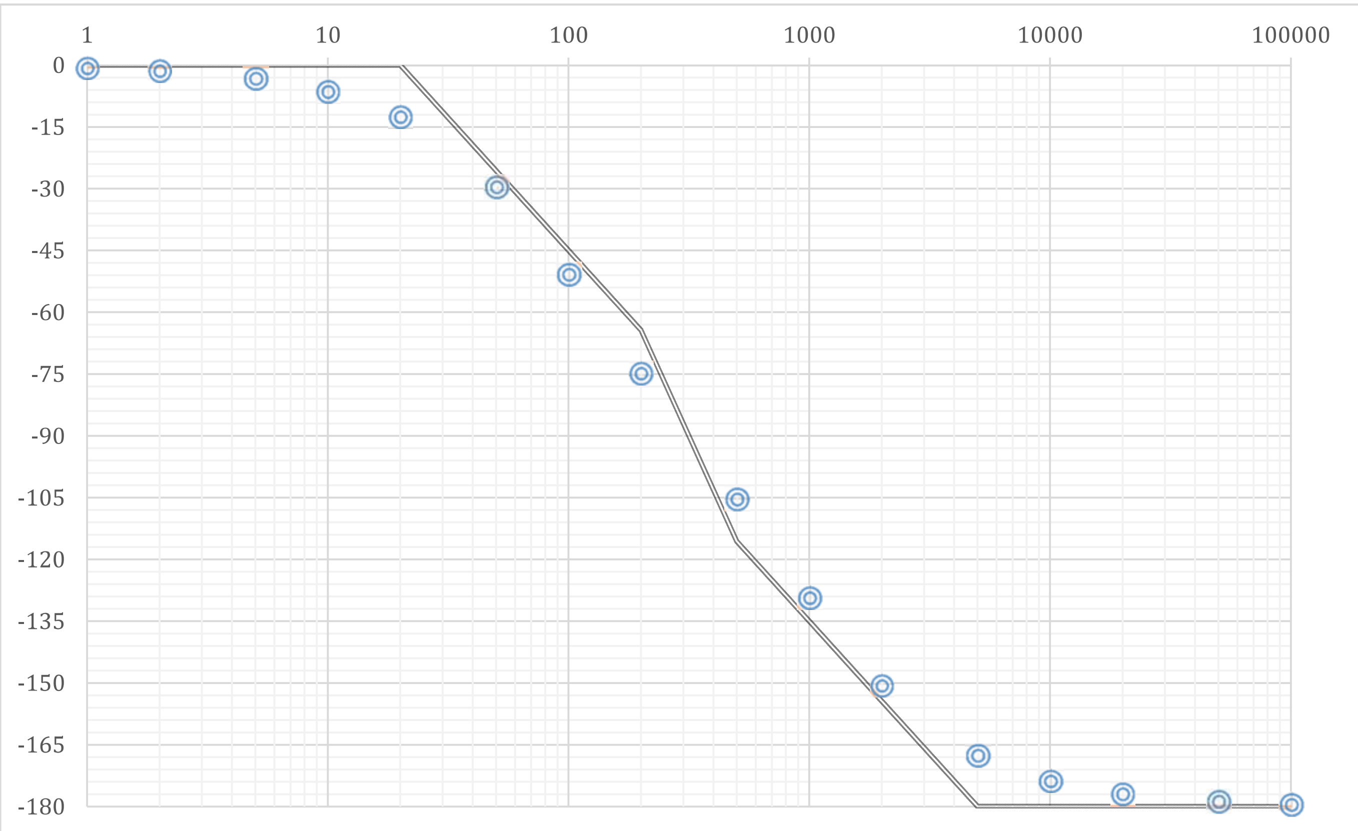
Magnitude of G_1

（縱軸為 magnitude(dB)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



Phase of G_1

（縱軸為 phase (degree)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）

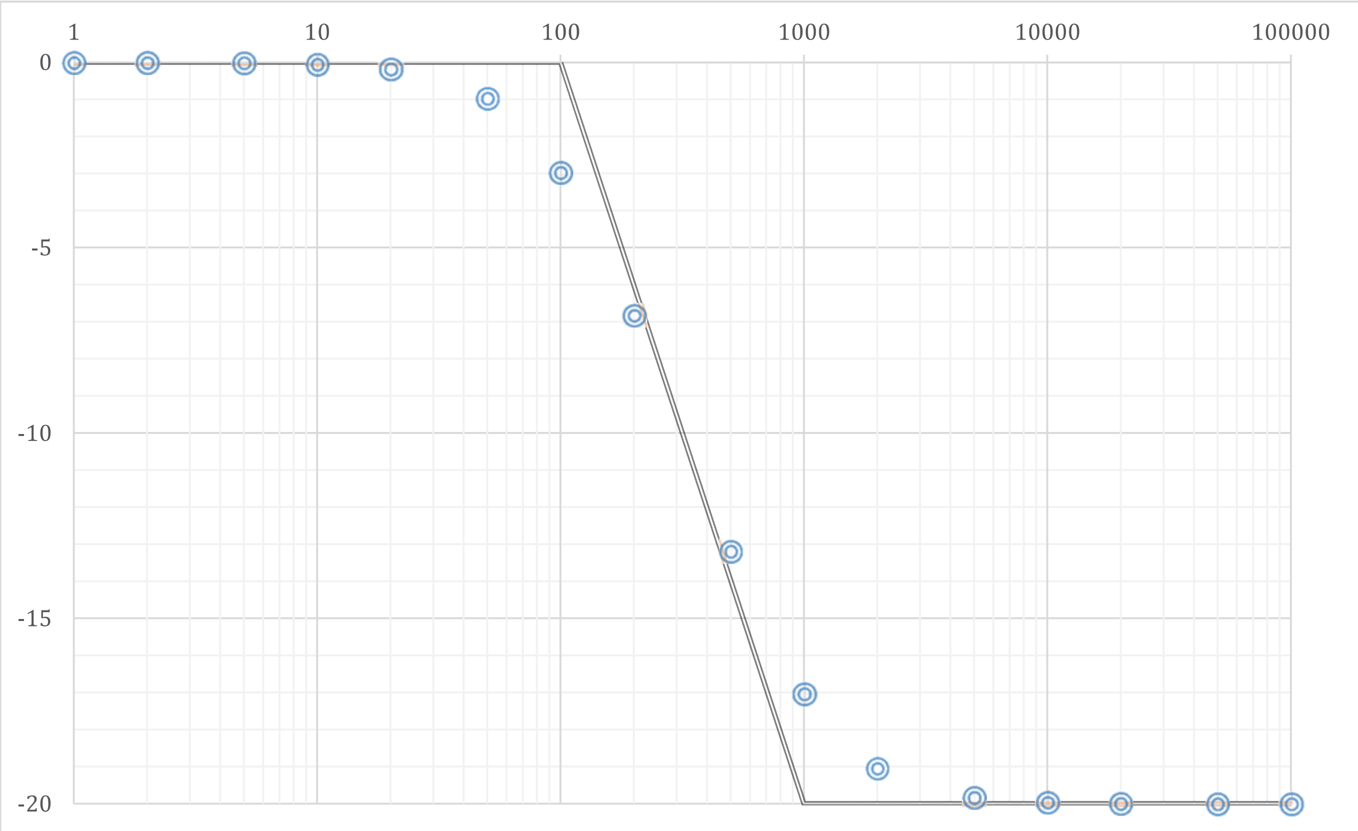


H W 8

B 0 8 9 0 1 1 7 6
陳 育 楷

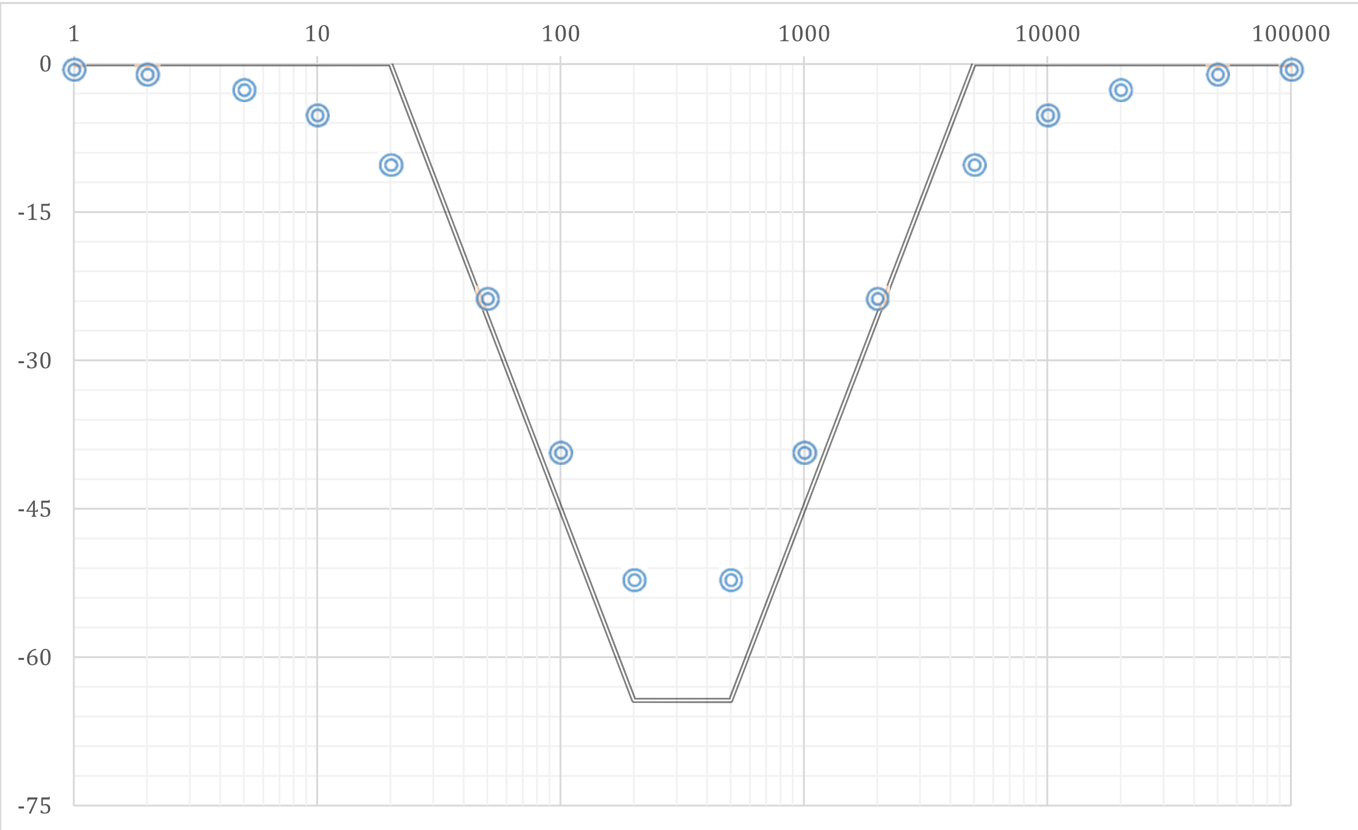
Magnitude of G_2

（縱軸為 magnitude(dB)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



Phase of G_2

（縱軸為 phase (degree)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



H W 8

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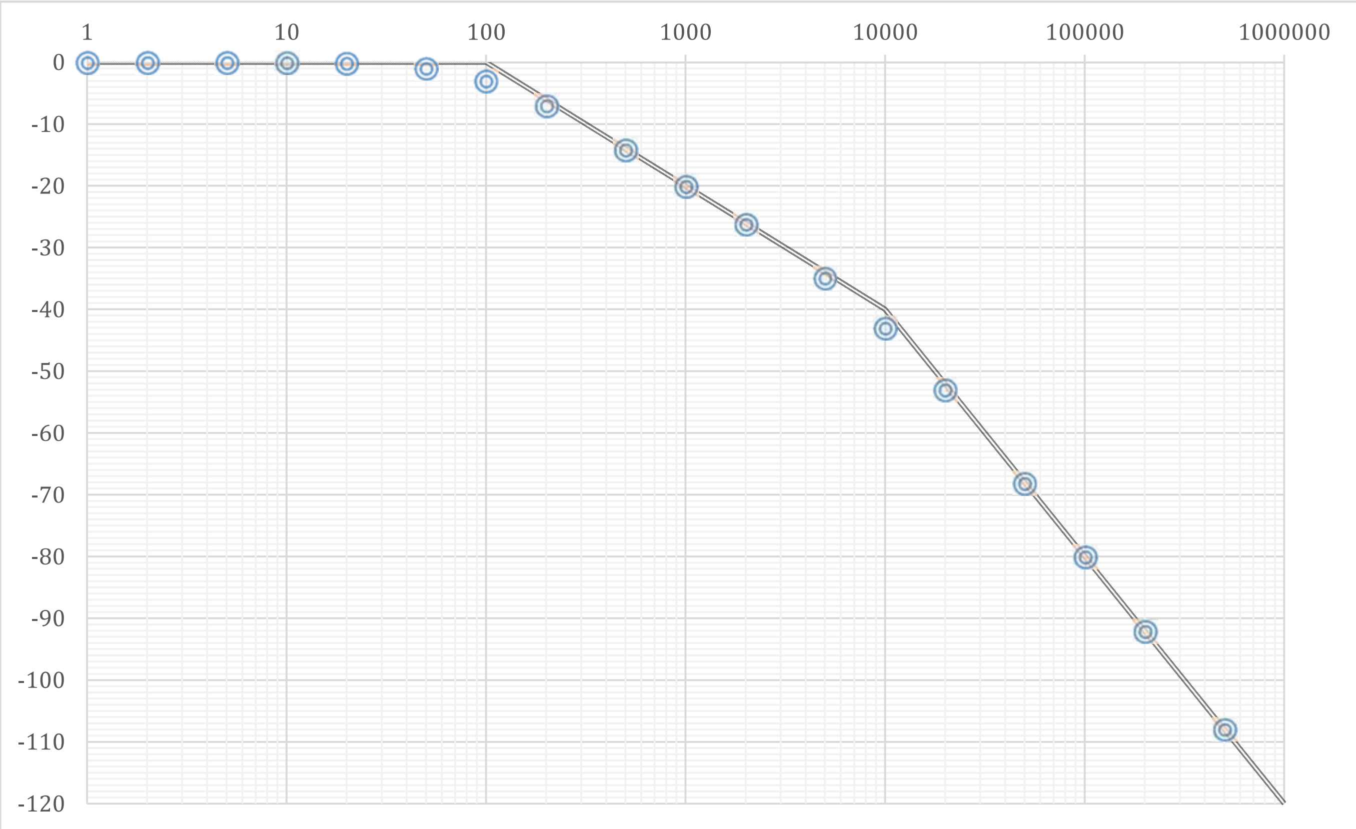
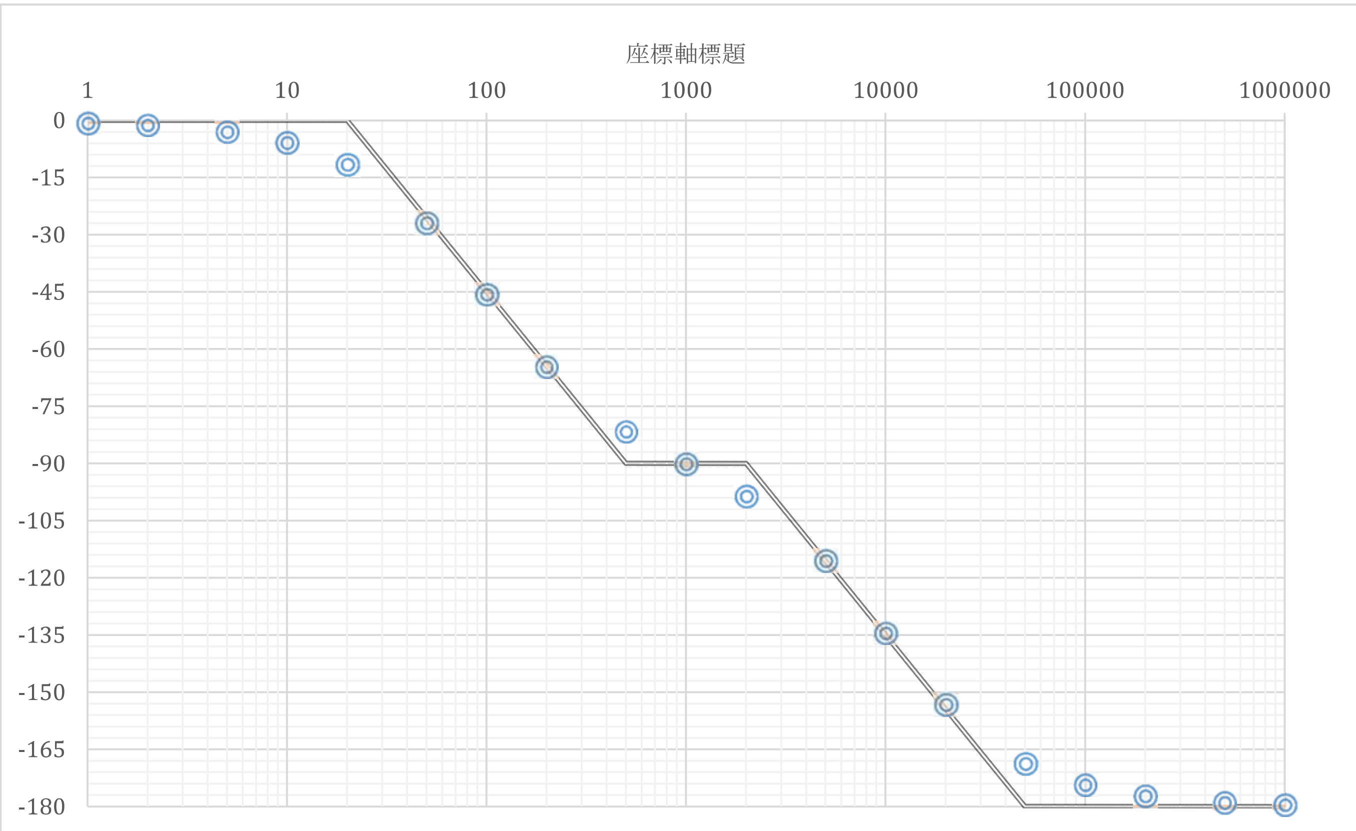
Magnitude of G_3

（縱軸為 magnitude(dB)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



Phase of G_3

（縱軸為 phase (degree)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



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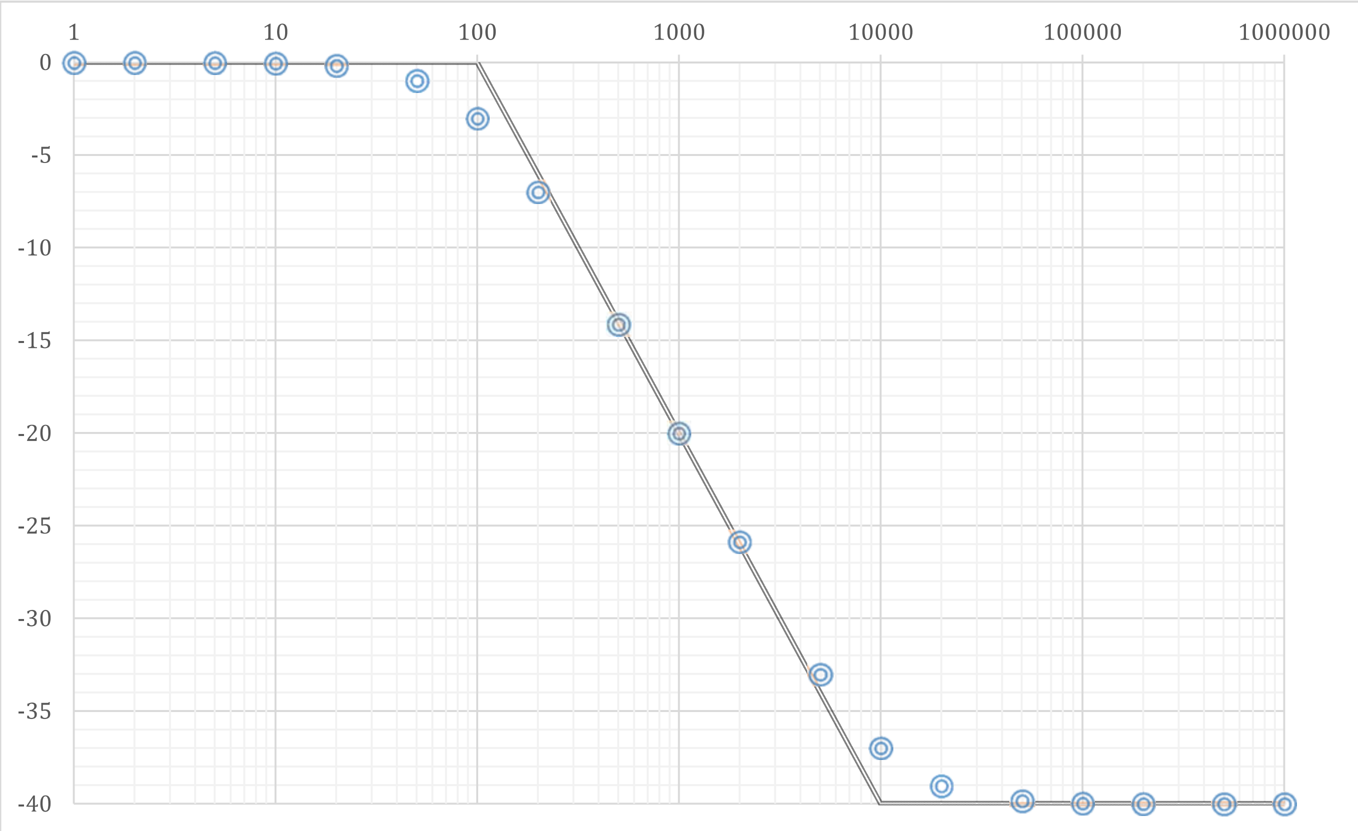
陳　育　楷

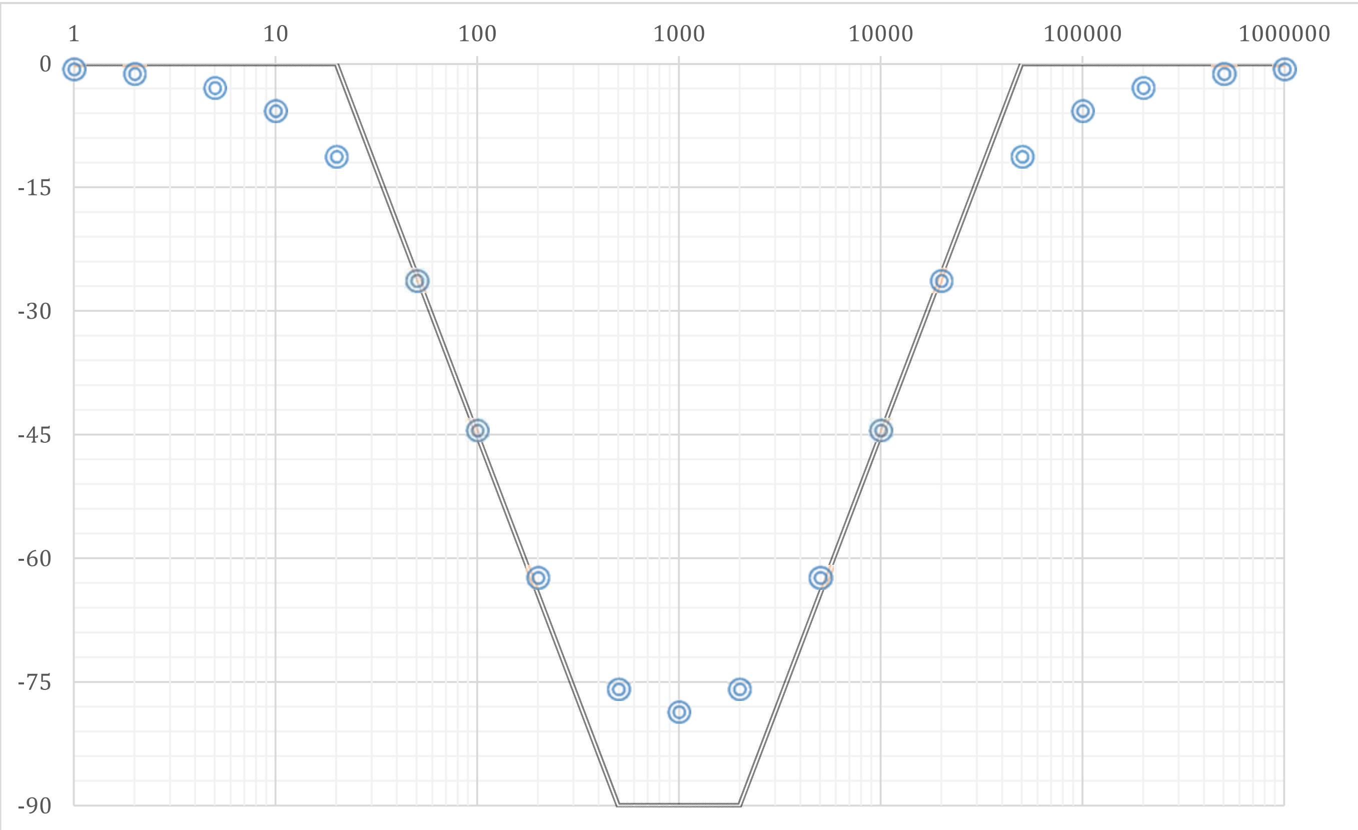
Magnitude of G_4

（縱軸為 magnitude(dB)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



Phase of G_4

（縱軸為 phase (degree)，橫軸為ω(rad/sec)，紅線為 asymptotes，藍點為實際計算的值。）



bser ation:

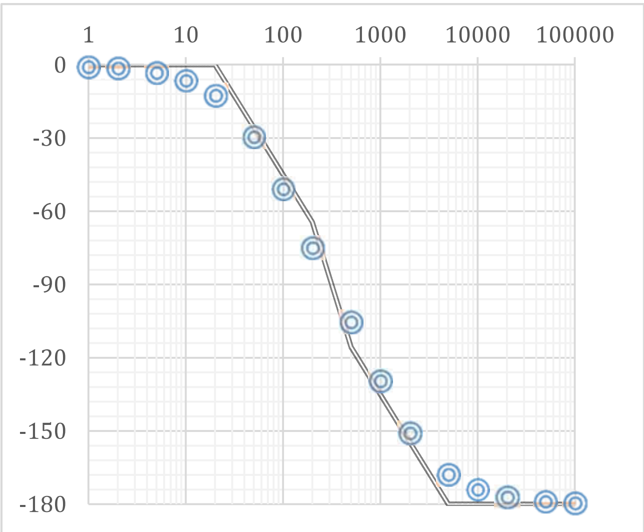
1 pole 1 ero ，實際 的 magnitude Bode plot rule 的 asymptote 3dB， phase 11 。

Magnitude asymptote 的 2 10 ， 實際的 magnitude asymptote 的 ， 實際的 phase asymptote 的 。

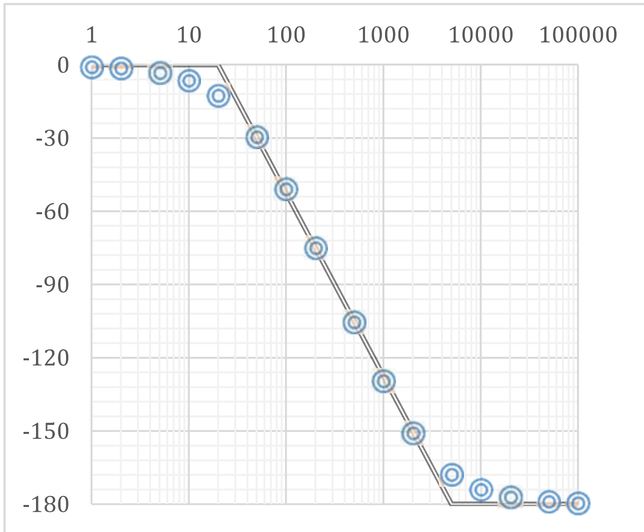
2 poles 100 ， phase 的 asymptote 實際的 。

ample:

Phase of G_1(original)



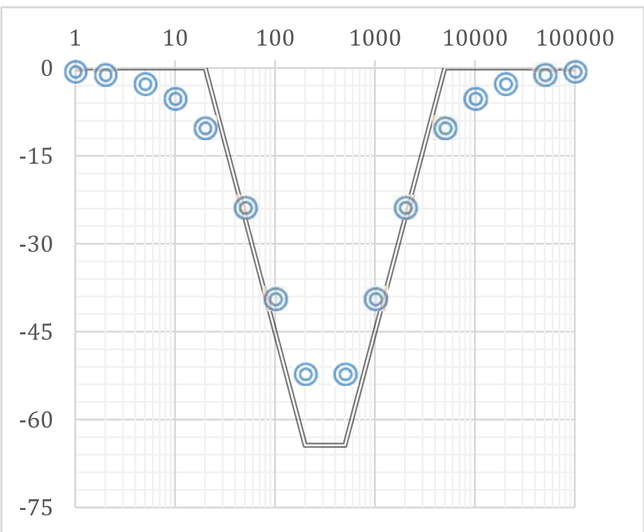
Phase of G_1(modified)



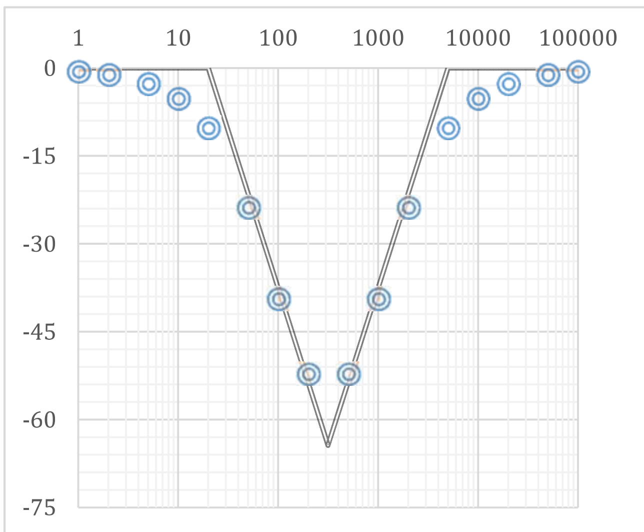
pole ero 100 ， phase 的 asymptote 的值 實際的 。

ample:

Phase of G_2(original)



Phase of G_2(modified)



HW 08: Bode Plot	Control Systems, Fall 2022, NTU-EE
Name: 朱本毅 B09502033	Date: 11/25, 2022

Problem

(a)針對一顆馬達，分析其 Transfer Function 以及繪製出 Bode Plot。

Answer

(a)

針對 IG220019X00015R，12V 直流馬達進行分析。

根據 Week 2 的課程，馬達的轉移函數如下：

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a}}{J_m s^2 + s\left(b + \frac{K_t K_e}{R_a}\right)} = \frac{K}{s(\tau s + 1)}$$

參照馬達的 Datasheet

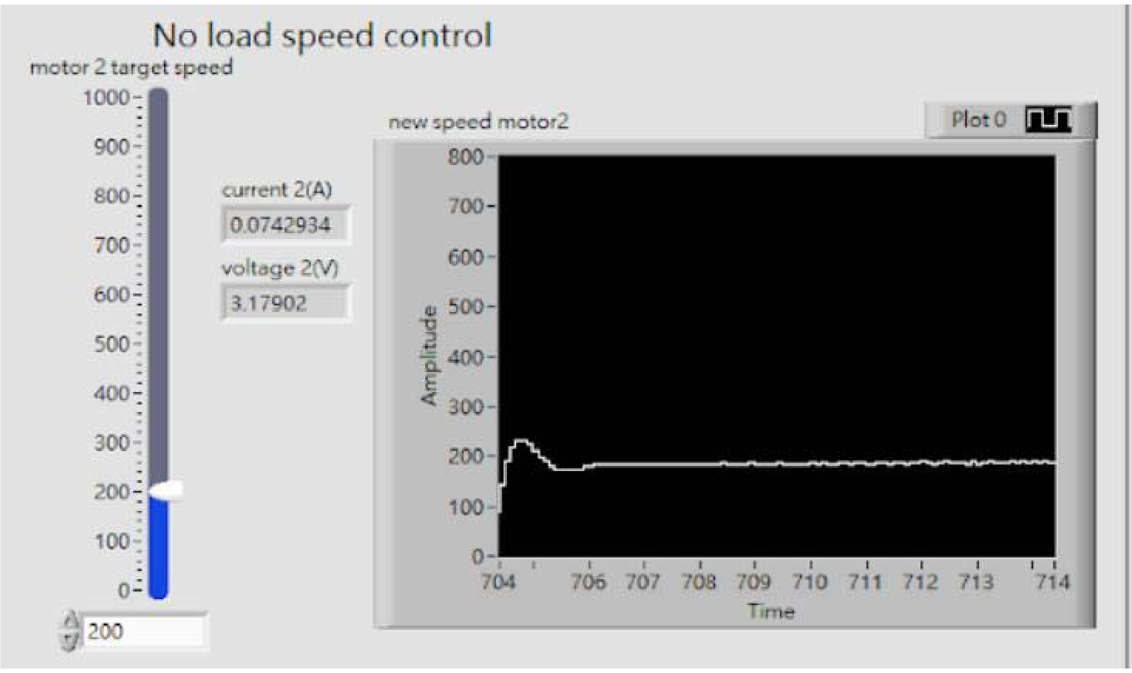
https://akizukidenshi.com/download/ds/shayangye/290-006_ig220019x00015r_ds.pdf

$$\mathrm{Kt} = 0.91\ \left(\frac{\mathrm{kgf} * \mathrm{cm}}{\mathrm{A}}\right) = 0.08918\ \left(\frac{\mathrm{N}}{\mathrm{A}}\right)$$

$\mathrm{Ke} = 9.68\ \left(\frac{\mathrm{mV}}{\mathrm{rpm}}\right) = 0.00924372\ \left(\frac{\mathrm{V}}{\mathrm{rad}}\right)$，內部電阻 Ra = 5 (Ω)

$$\mathrm{Jm} = 2.32226529 * 10^{-6}(kg * m^2)$$

viscous friction coefficient，也就是 b，比較難估計，當供應 12V 電壓時，實際測試的結果如下圖(有用 PI 控制)



HW 08: Bode Plot	Control Systems, Fall 2022, NTU-EE
Name: 朱本毅 B09502033	Date: 11/25, 2022

目標轉速為 200 rpm (20.944 rad/s)，此時的 Torque 應為$0.08918 * 0.0743 = b\omega$，因此 b 約 $= 0.00031637\ (N * m/rad)$

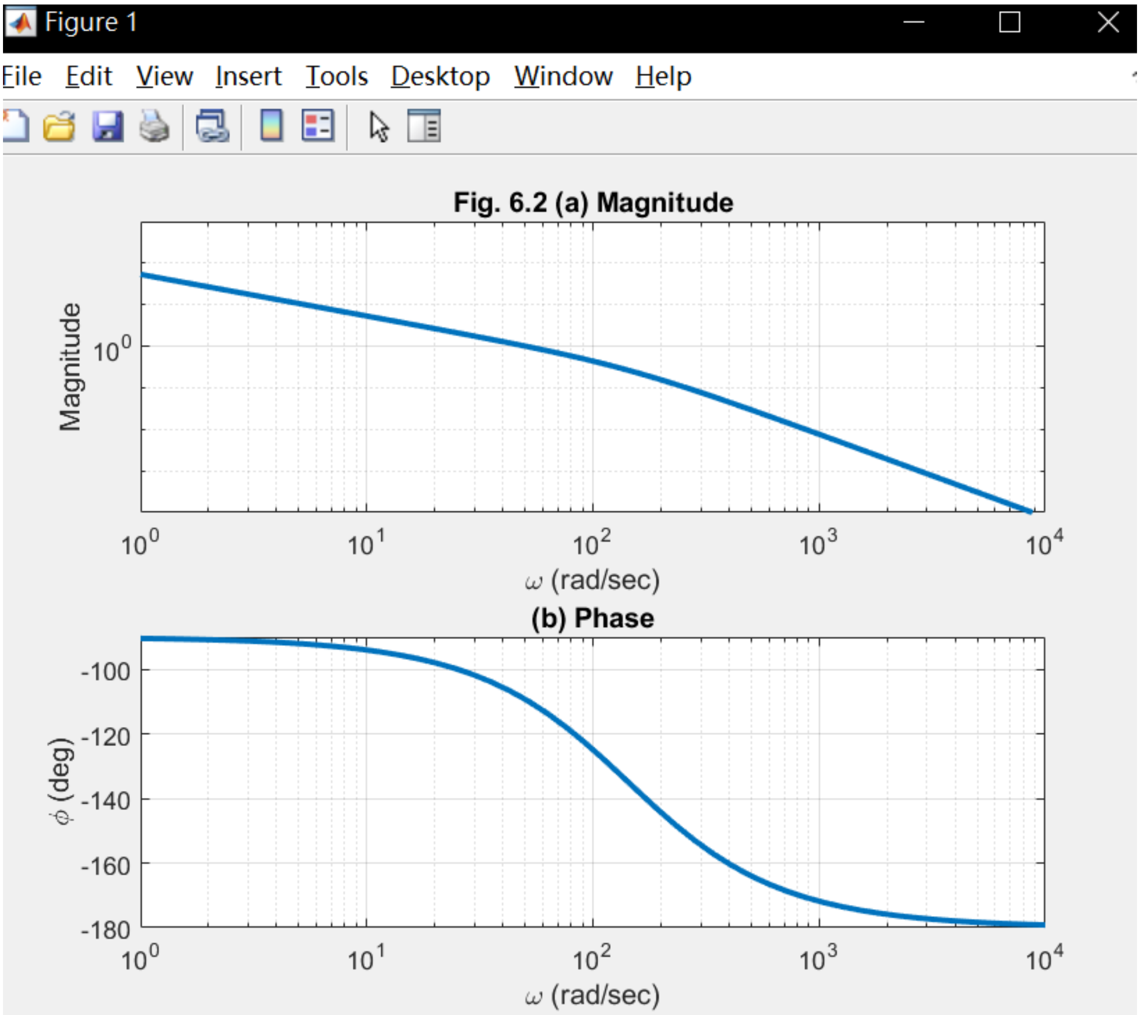
就是最後得到轉移函數

$$\frac{\Theta_m(j\omega)}{V_a(j\omega)} = \frac{0.017836}{2.32226529 * 10^{-6}(j\omega)^2 + (0.00031637 + 0.000164871)j\omega}$$

$$= \frac{7680.43172}{j\omega(j\omega + 143.668016)}$$

Time constant $\tau = \frac{1}{143.668016} = 6.96049147 * 10^{-3}$ (6.96ms 左右，還算合理)

根據結果，在 Matlab 中繪製 Bode Plot。



當$\omega \ll 143.668$ 時，轉移函數近似於$\frac{7680.43172}{j\omega}$

Magnitude 的斜率一開始為 -10dB/decade，一旦到了 Breaking Point，由於 Pole 的作用，斜率會變為-20dB/decade。

Phase 方面，一開始的 Phase 為$\tan^{-1}\frac{7680.43172}{-1} \approx -90^{o}$，在 Breaking Point 附近再降 45 度，而最終變為約-180 度。

轉移函數與繪圖結果相符。

參考觀摩的作業 . (Bode lot and Frequency ropert es)
作者：b0890108， 理由： 與 位的
作者：b09901152， 理由： 並 出 與 b 的波德 圖
作者：b10202032， 理由： 出波德圖

HW08 – Unit 6, Bode Plot

學號：B08901085　　　系級：電機四　　　姓名：施彥宇

- **Question :**

For the open-loop transfer functions of the unity feedback control systems, given below, sketch the Bode magnitude and phase plots. Find their gain margins, gai crossover frequencies, and phase crossover frequencies.

$(a). L(s) = \frac{10}{s(s+10)}$

$(b). L(s) = \frac{50}{s(0.5s+1)^2}$

- **Solution :**

I will sketch the Bode plot by hands and then verify my answer by MATLAB.

$(a). L(s) = \frac{10}{s(s+10)}$

✓ Break frequency : $10\ (rad/s)$.

✓ Asymptotes : low-frequency asymptote $L(j\omega) = \frac{10}{j\omega}$ for $\omega < 0.1$.

$\omega \ll 10$: slope = -1 (or -20db per decade).

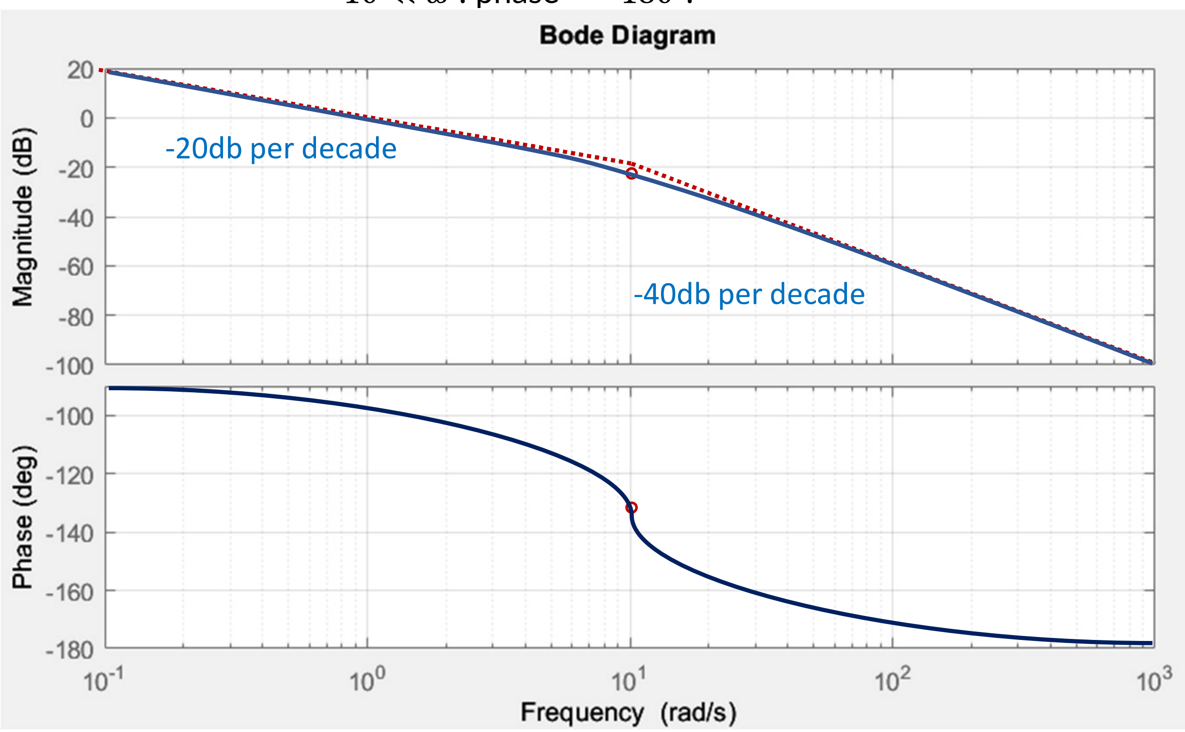
$10 \ll \omega$: slope = -2 (or -40db per decade).

✓ Phase : : low-frequency asymptote $L(j\omega) = \frac{10}{j\omega}$ for $\omega < 0.1$.

$\omega \ll 10$: phase = $-90°$.

$\omega = 10$: phase = $-135°$.

$10 \ll \omega$: phase = $-180°$.



- **Solution :**

Also, we can calculate the gain crossover frequency and phase crossover frequency :

$$\omega_{gc} \Rightarrow |L(j\omega_{gc})| = \left|\frac{10}{-\omega_{gc}^2+10j\omega_{gc}}\right| = \frac{10}{\sqrt{\omega_{gc}^4+100\omega_{gc}^2}} = 1 \quad (1).$$

$$\rightarrow \omega_{gc}^4 + 100\omega_{gc}^2 - 100 = 0 \quad \Rightarrow \quad \omega_{gc}^2 = -50 \pm 10\sqrt{26} \ (2).$$

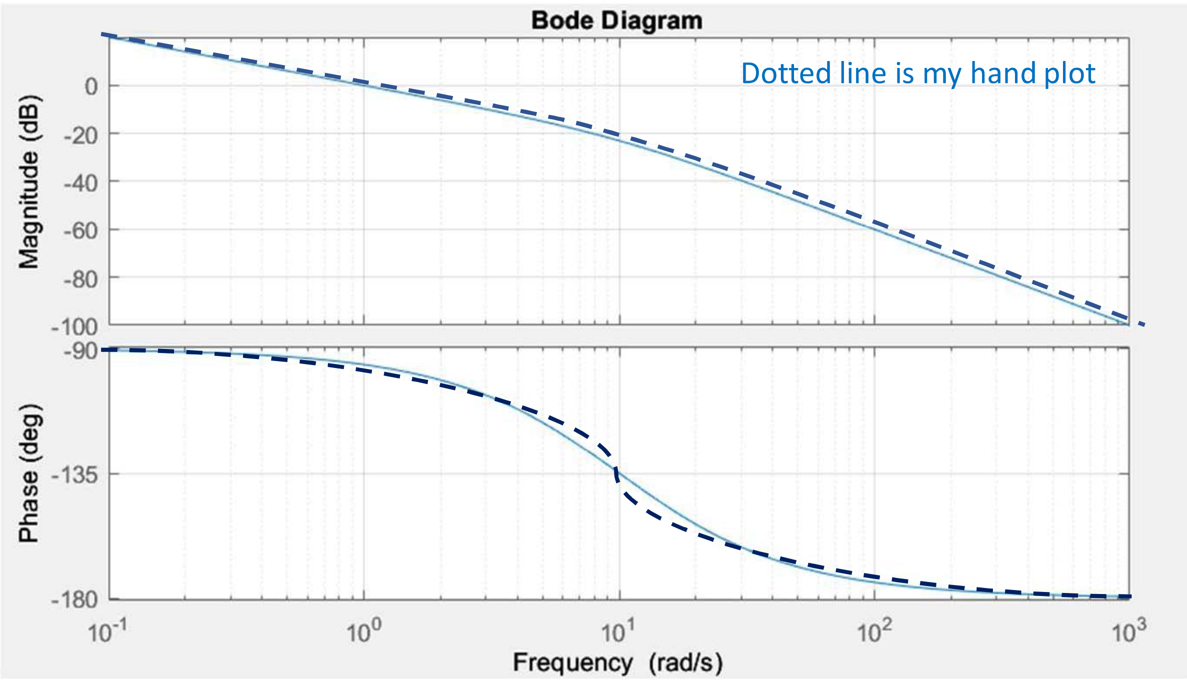
$$\rightarrow \omega_{gc} = \pm\sqrt{-50 \pm 10\sqrt{26}} = \pm 0.9949, \ \pm 10.04j \quad (3).$$

From (3). , choosing the reasonable answer, the gain crossover frequency $\omega_{gc} = 0.9949\ rad/s$. Thus, phase margin can be calculated.

$$PM = -90° - \tan^{-1}\frac{\omega_{gc}}{10} + 180° = -90 - 5.68° + 180° = 84.32° \quad (4).$$

The phase crossover frequency for this system is infinity because the phase of this system will approach to $-180°$ as frequency goes to infinity, but it never reach $-180°$. Therefore, both phase crossover frequency and gain margin is infinity in this situation.

The following is the Bode plot drawn by MATLAB, and I overlapped my hand plot and the result. Also, I use code to calculate the margins. The result is very close to my answer.




```
>> [gm, pm, wpc, wgc]

ans =

       Inf    84.3180        Inf     0.9950
```

- **Solution :**

  $(b).\ L(s) = \frac{50}{s(0.5s+1)^2}$

  ✓ Break frequency : 2 (rad/s).

  ✓ Asymptotes : low-frequency asymptote $L(j\omega) = \frac{50}{j\omega}$ for $\omega < 0.1$.

  $\omega \ll 2$: slope = -1 (or -20db per decade).

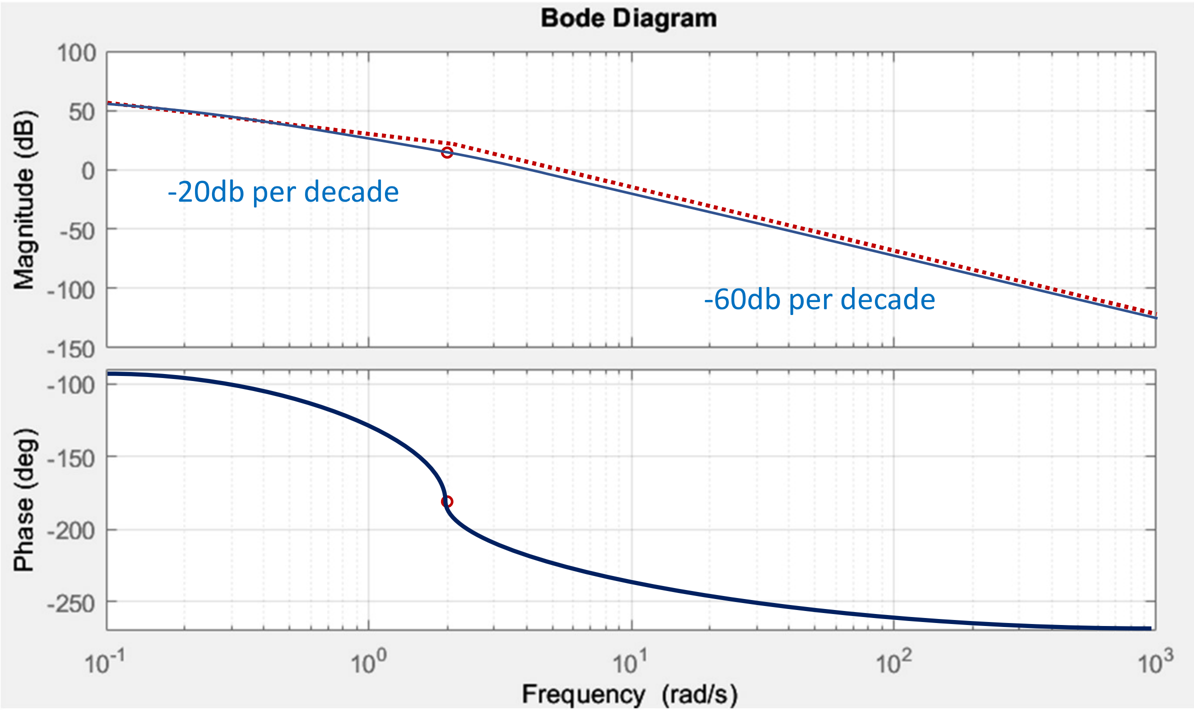
  $2 \ll \omega$: slope = -3 (or -60db per decade).

  ✓ Phase : : low-frequency asymptote $L(j\omega) = \frac{50}{j\omega}$ for $\omega < 0.1$.

  $\omega \ll 2$: phase = $-90°$.

  $\omega = 2$: phase = $-180°$.

  $2 \ll \omega$: phase = $-270°$.



$$\omega_{gc} \Rightarrow |L(j\omega_{gc})| = \left|\frac{50}{-\omega_{gc}^2+(\omega_{gc}-0.25\omega_{gc}^3)j}\right| = \frac{50}{\sqrt{\omega_{gc}^4+(\omega_{gc}-0.25\omega_{gc}^3)^2}} = 1 \qquad (5).$$

$$\rightarrow 0.0625\omega_{gc}^6 + 0.5\omega_{gc}^4 + \omega_{gc}^2 - 2500 = 0 \qquad (6).$$

$$\rightarrow \omega_{gc}^2(\omega_{gc}^2+4)^2 = \frac{2500}{0.0625} \Rightarrow \omega_{gc}(\omega_{gc}^2+4) = \pm 200 \qquad (7).$$

Now the question becomes solving two 3-order equations :

$$x^3 + 4x + 200 = 0 \qquad (8).$$

$$x^3 + 4x - 200 = 0 \qquad (9).$$

Suppose the equation in standard form to be : $x^3 + px + q = 0$ (10).

If $x = (u+v)$, then $x^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$$= (u+v)^3 - 3uv(u+v) - (u^3+v^3) \qquad (11).$$

- **Solution :**

$$= (u+v)^3 - 3uv(u+v) - (u^3+v^3) \tag{11.}$$

Comparing with equation (10). and (11)., we can get :

$$p = -3uv \tag{12.}$$

$$q = -u^3 - v^3 \tag{13.}$$

At this moment, the question becomes solving u, v with p, q. Therefore, one can set a new variation z, and solving $(z-u^3)(z-v^3) = 0$:

$$(z-u^3)(z-v^3) = z^2 - (u^3+v^3)z + u^3v^3$$

$$= z^2 - qz - \frac{p^3}{27} = 0 \tag{14.}$$

This is a standard form second-order equation, and the solution of (14). is

$z = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$, which represent :

$$u^3 = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = a \ \Rightarrow \ u = \sqrt[3]{a}\, , \omega\sqrt[3]{a}\, , \omega^2\sqrt[3]{a} \tag{15.}$$

$$v^3 = \frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = b \ \Rightarrow \ v = \sqrt[3]{b}\, , \omega\sqrt[3]{b}\, , \omega^2\sqrt[3]{b} \tag{16.}$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$, and $\omega^2 = \bar{\omega}$, $\omega^3 = 1$.

It seems that $x = u + v$ have nine possibilities, but $uv = -\frac{p}{3}$ is real. As a result, x only have three solutions :

$$x = \sqrt[3]{a} + \sqrt[3]{b}\, ,\ \omega\sqrt[3]{a} + \omega^2\sqrt[3]{b}\, ,\ \omega^2\sqrt[3]{a} + \omega\sqrt[3]{b} \tag{17.}$$

To solving equation (8). and (9), substitute (p, q) with $(4{,}200)$, $(4, -200)$

$(\boldsymbol{p}, \boldsymbol{q})$	$(\boldsymbol{a}, \boldsymbol{b})$	$\boldsymbol{x} = (\boldsymbol{u} + \boldsymbol{v})$
$(4, 200)$	$(200.012, -0.012)$	$\sqrt[3]{a} + \sqrt[3]{b} = 5.619$ $\omega\sqrt[3]{a} + \omega^2\sqrt[3]{b} = -2.809 + 4.866j$ $\omega^2\sqrt[3]{a} + \omega\sqrt[3]{b} = -3.153 + 5.065j$
$(4, -200)$	$(0.012, -200.012)$	$\sqrt[3]{a} + \sqrt[3]{b} = 3.153 + 5.065j$ $\omega\sqrt[3]{a} + \omega^2\sqrt[3]{b} = 2.809 - 4.866j$ $\omega^2\sqrt[3]{a} + \omega\sqrt[3]{b} = -5.619$

Table 1, Solutions for equation (8) and (9).

Therefore, from Table 1 we can get the gain crossover frequency

$$\omega_{gc} = 5.619\ rad/s \tag{18.}$$

Thus, the phase margin is :

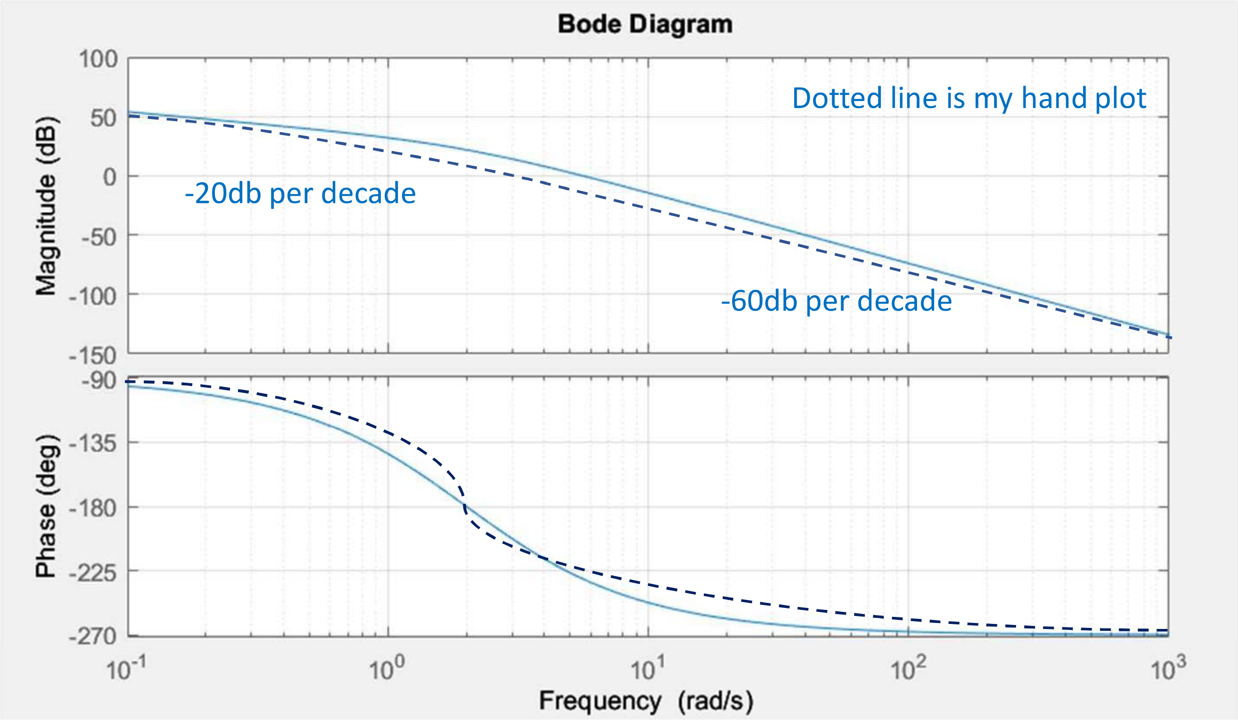
$$PM = \angle L(j\omega_{gc}) + 180° = -90° - 2\tan^{-1}\frac{\omega_{gc}}{0.5} + 180° = -79.83° \tag{19.}$$

- **Solution :**

And for this system, it is obvious that the phase crossover frequency $\omega_{pc} = 2$. As a result, the gain margin is :

$$GM = |L(j\omega_{pc})| = \left|\frac{50}{j\omega_{pc}(0.5j\omega_{pc}+1)^2}\right| = 12.5 = 21.94\ db \qquad (20).$$

The following is the Bode plot drawn by MATLAB.



And the margin calculated by MATLAB is :

```
>> [gm, pm, wpc, wgc]

ans =

  -21.9000  -50.8226    2.0000    5.6202
```

We can see that except phase margin, other results are quite fit. I think this is because $0.1\omega_n = 0.2 < \omega_{gc} = 5.62 < 10\omega_n = 20$. So, the formula of calculating phase margin must be modified as :

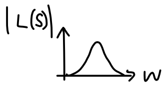
$$PM = -90° - 90°\left(\log_{10}\frac{\omega_{gc}}{\omega_n} + 1\right) + 180° = -40.38° \qquad (21).$$

which is much more accurate.

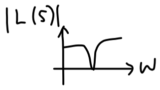
2. (Bode Plot and Frequency Properties) 電机三 B09901152 施文崴

Design a second-order open-loop transfer function of the following filter, and sketch the Bode magnitude and phase plots.

(a) Bandpass filter

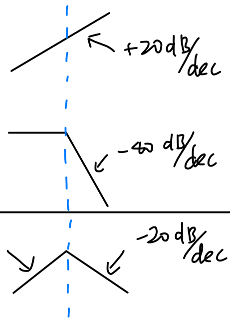


(b) Notch filter

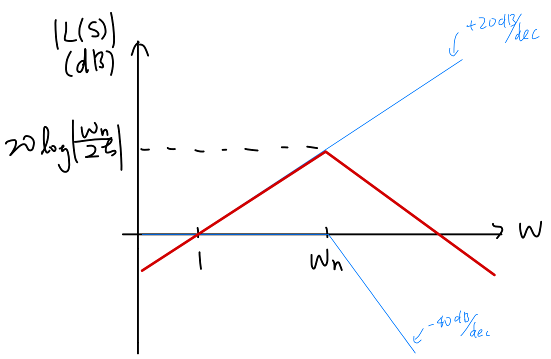


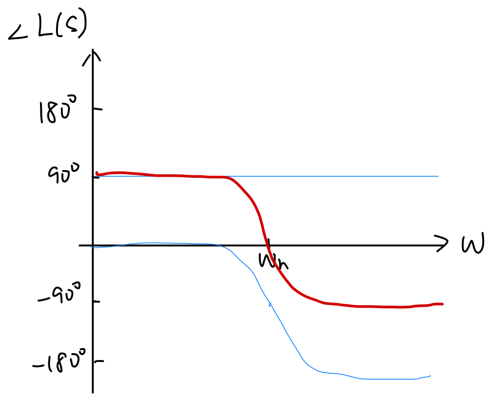
(a) We can use a first-order

+) a second-order

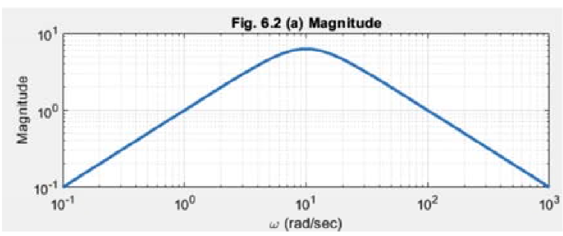


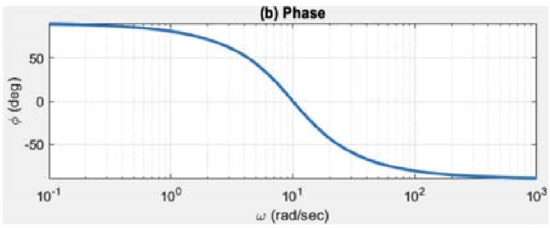
let $L(s) = \dfrac{S}{\left(\frac{S}{W_n}\right)^2 + 2\zeta\left(\frac{S}{W_n}\right) + 1}$





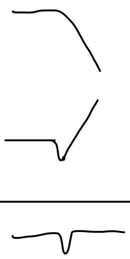
Matlab: ($W_n=10$, $\zeta=0.8$)



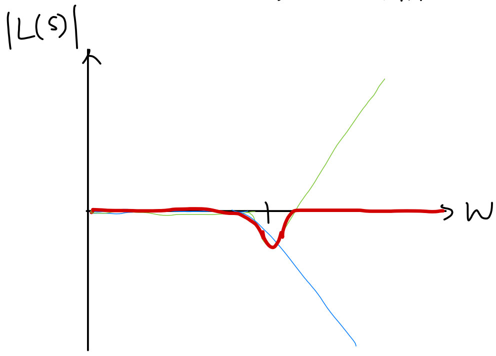


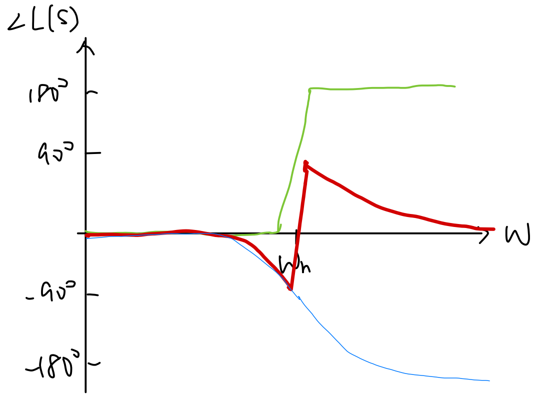
(b) we can use a second-order

\+) a second-order with $\zeta > 0.7$

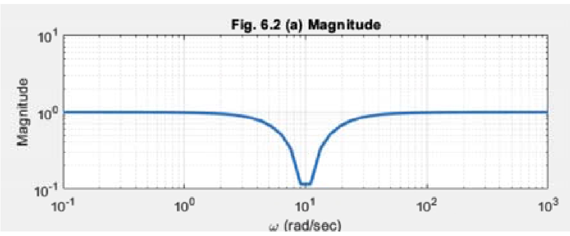


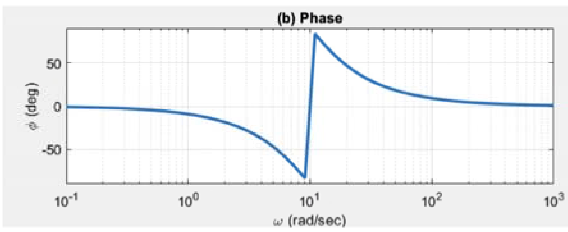
let $L(s) = \dfrac{\left(\frac{s}{\omega_n}\right)^2 + 1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$





Matlab: ($\omega_n = 10$, $\zeta = 0.8$)





\#

Control System HW8
B10202032 物理二 卓然

3. For the open-loop transfer functions of the unity feedback control systems. given below, sketch the Bode magnitude and phase plots. Find their gain margins, gain crossover frequencies, and phase crossover frequencies.

(a) $L(s) = \frac{10}{s[s+10]}$ ← Suppose that [s+10] means exactly the same as (s+10).

(d) $L(s) = \frac{50}{s(0.5s+1)^2}$

I. Sketch The Bode Plots

(a) $L(s) = \frac{10}{s(s+10)} \Rightarrow L(j\omega) = \frac{10}{j\omega(j\omega+10)} = \frac{10}{-\omega^2+10j\omega}$

(i) Magnitude:

$|L(j\omega)| = 10\left|\frac{1}{10j\omega-\omega^2}\right| = \frac{10}{\sqrt{(10\omega)^2+(\omega^2)^2}} = \frac{10}{\sqrt{\omega^4+100\omega^2}}$

$\Rightarrow dB = 20\log_{10}|L(j\omega)| = 20\log_{10}\left(\frac{10}{\sqrt{\omega^4+100\omega^2}}\right) = 20\left(\log_{10}10 - \frac{1}{2}\log_{10}[\omega^2(100+\omega^2)]\right)$

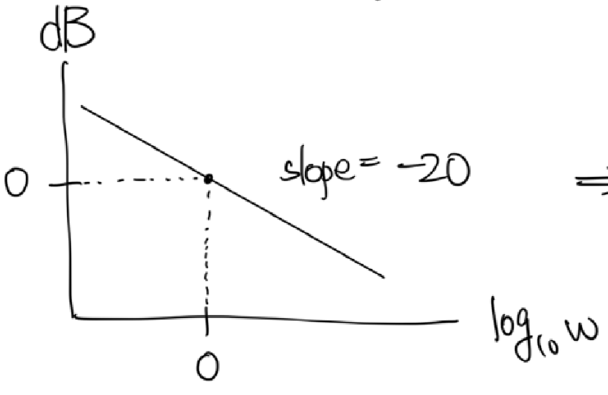
$= 20\left[1-\frac{1}{2}\left(\log_{10}\omega^2+\log_{10}(100+\omega^2)\right)\right] = 20-10\left(2\log_{10}\omega+\log_{10}(100+\omega^2)\right)$

$= 20-20\log_{10}\omega-10\log_{10}(100+\omega^2)$

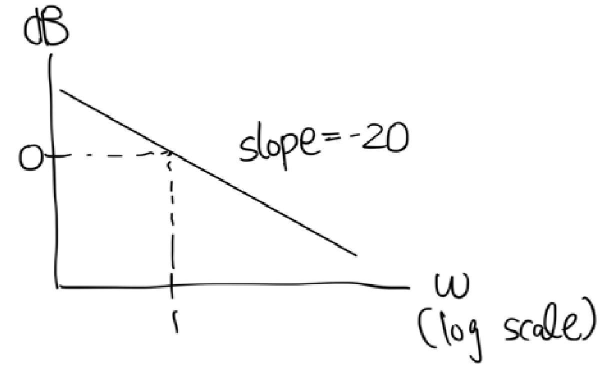
① $\omega \ll 10 \Rightarrow \log_{10}(100+\omega^2) \approx \log_{10}100 = 2$

$\Rightarrow dB \approx 20-20\log_{10}\omega - 10\cdot 2 = -20\log_{10}\omega$

∴



⇒



when $\omega \ll 10$.

② $\omega = 10 \Rightarrow dB = 20-20\log_{10}10-10\log_{10}(100+10^2) = 20-20-10\log_{10}200$

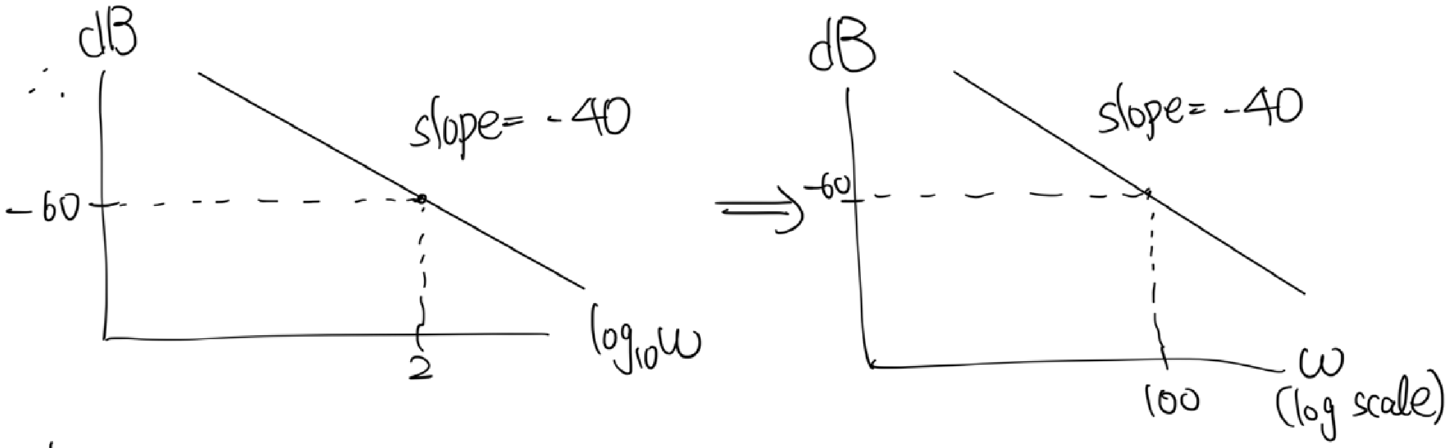
$= -10(\log_{10}100+\log_{10}2) = -10(2+0.3) = -23.$

③ $\omega \gg 10 \Rightarrow \frac{10}{\sqrt{\omega^4+100\omega^2}} \approx \frac{10}{\sqrt{\omega^4}} = \frac{10}{\omega^2}$

$\Rightarrow dB \approx 20\log_{10}\left(\frac{10}{\omega^2}\right) = 20(\log_{10}10-2\log_{10}\omega) = 20(1-2\log_{10}\omega)$

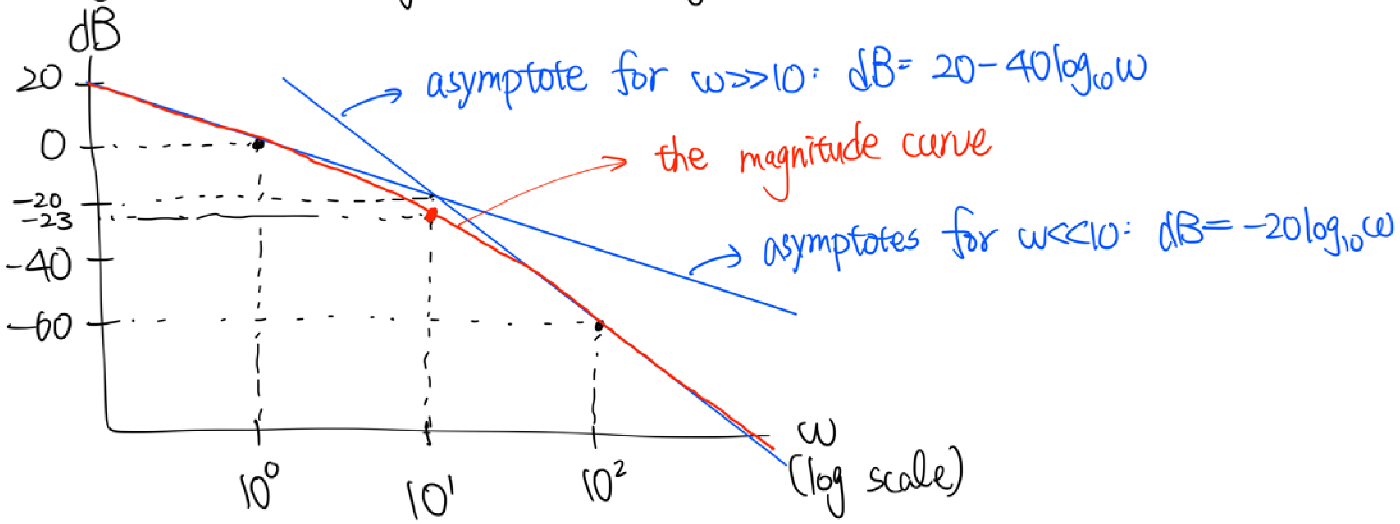
$= 20-40\log_{10}\omega$

①



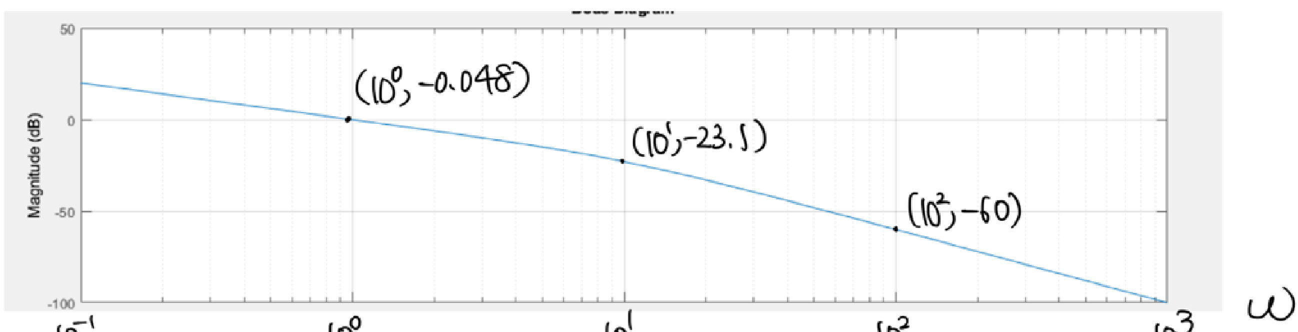
when $\omega >> 10$.

Putting ①②③ together, we get:



The red curve is approximately the magnitude curve.

This result is similar to exact result obtained with Matlab:



This result is different from the reference solution, and I believe the solution is wrong.

(ii) Phase:

$$\measuredangle L(j\omega) = -\measuredangle(-\omega^2+10j\omega) = -\text{“}\tan^{-1}\left(\frac{10\omega}{-\omega^2}\right)\text{”} = \text{“}\tan^{-1}\left(\frac{10}{\omega}\right)\text{”}$$

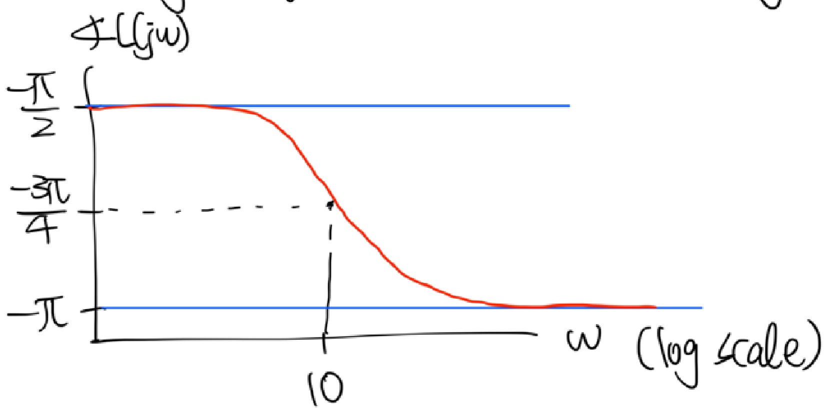
(I add “ ” since now we consider the range of the $\tan^{-1}$ function to be $[-\pi, 0]$, not the default $(-\frac{\pi}{2}, \frac{\pi}{2})$)

① $\omega \ll 10$ $(\omega \to 0^+) \Rightarrow \frac{10}{\omega} \to \infty \Rightarrow \measuredangle L(j\omega) \to -\frac{\pi}{2}^-$

② $\omega = 10 \Rightarrow \frac{10}{\omega} = 1$, $\measuredangle L(j\omega) = \frac{-3\pi}{4}$;

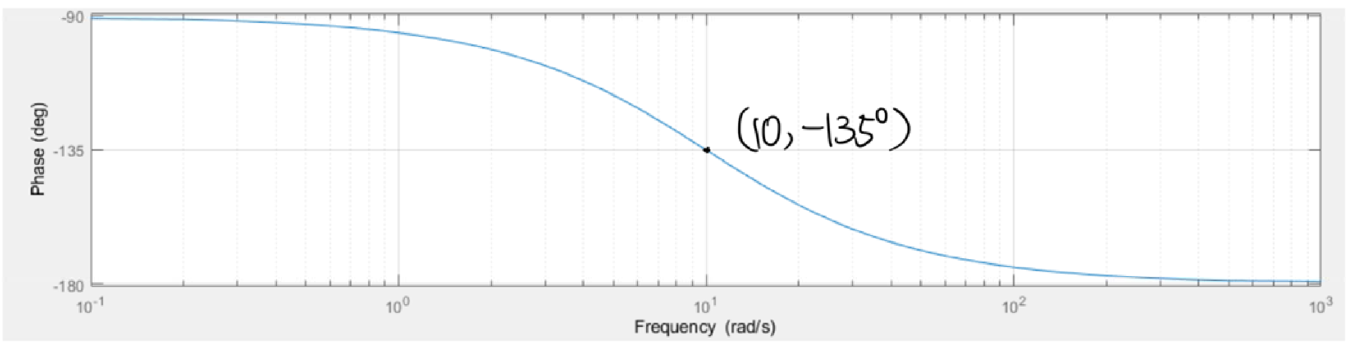
③ $\omega \gg 10$ $(\omega \to \infty) \Rightarrow \frac{10}{\omega} \to 0^+ \Rightarrow \measuredangle L(j\omega) \to -\pi^+$

Putting together ①②③, we get:



The red curve is approximately the phase curve.

This result is similar to exact result obtained with Matlab:



(b) $L(s) = \frac{50}{s(0.5s+1)^2} \Rightarrow L(j\omega) = \frac{50}{j\omega(0.5j\omega+1)^2}$

(i) Magnitude:

$$|L(j\omega)| = 50\left|\frac{1}{j\omega(0.5j\omega+1)^2}\right| = 50\frac{1}{|j\omega|}\left(\frac{1}{|0.5j\omega+1|}\right)^2 = \frac{50}{\omega}\left(\frac{1}{\sqrt{0.25\omega^2+1}}\right)^2$$

$$= \frac{50}{\omega(0.25\omega^2+1)}$$

③

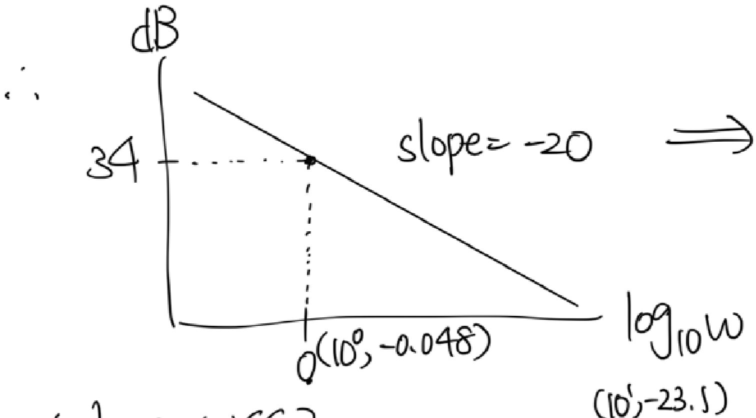
$$\Rightarrow dB = 20\log_{10}|L(j\omega)| = 20\cdot\log_{10}\frac{50}{\omega(0.25\omega^2+1)}$$

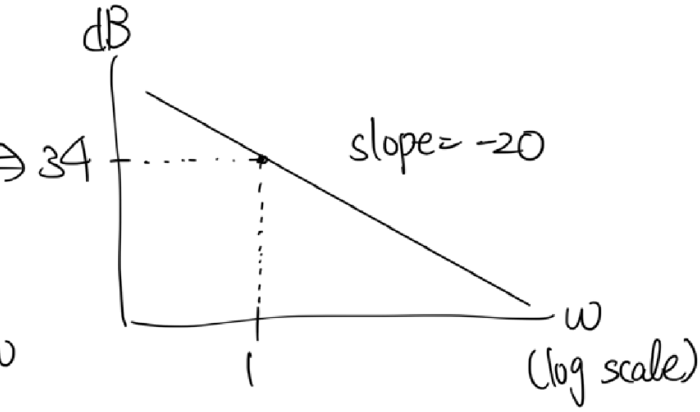
$$= 20\left[\underbrace{\log_{10}50}_{1.7} - \log_{10}\omega - \log_{10}(0.25\omega^2+1)\right]$$

$$= 34 - 20\log_{10}\omega - 20\log_{10}(0.25\omega^2+1)$$

① $\omega \ll 2$: $\log_{10}(0.25\omega^2+1) \approx \log_{10}1 = 0$

$$\Rightarrow dB \approx 34 - 20\log_{10}\omega$$





when $\omega \ll 2$.

$(10^2, -60)$

② $\omega = 2$: $dB = 34 - 20\underbrace{\log_{10}2}_{0.3} - 20\log_{10}\underbrace{(0.25\cdot 2^2+1)}_{2} = 34 - 20\cdot 0.3 - 20\cdot 0.3$

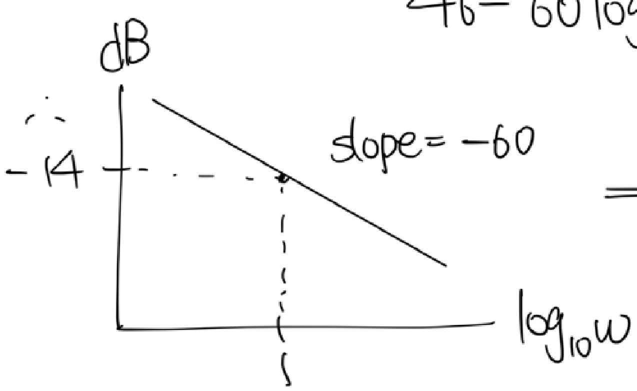
$$= 34 - 12 = 22$$

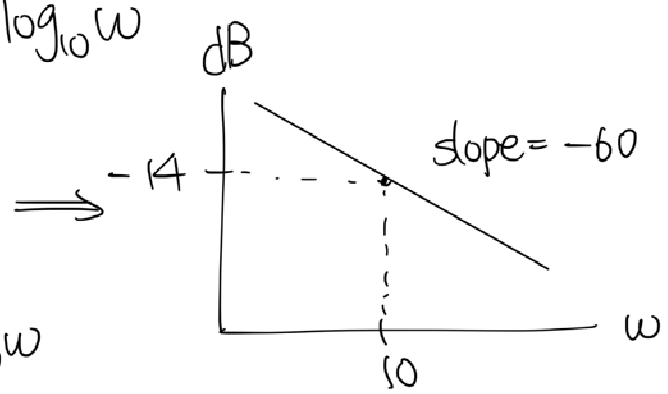
③ $\omega \gg 2$: $dB = 20\log_{10}\frac{50}{\omega(0.25\omega^2+1)} = 20\log_{10}\frac{50}{\omega + 0.25\omega^3}$

$$\approx 20\log_{10}\frac{50}{0.25\omega^3} = 20\log_{10}200\omega^{-3}$$

$$= 20(\underbrace{\log_{10}200}_{2.3} - 3\log_{10}\omega)$$

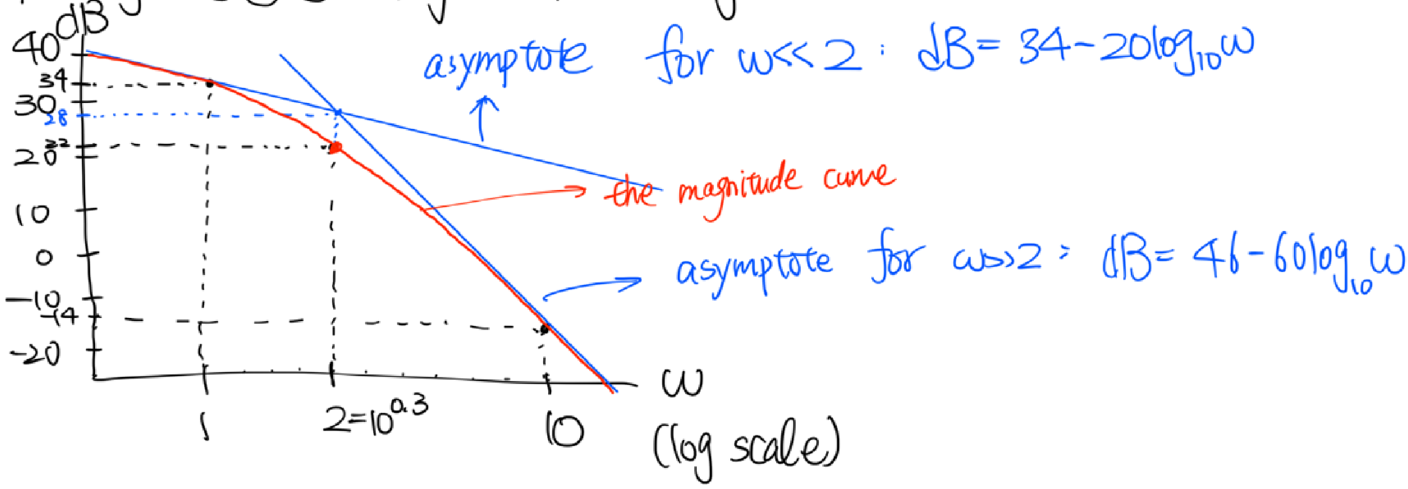
$$= 46 - 60\log_{10}\omega$$





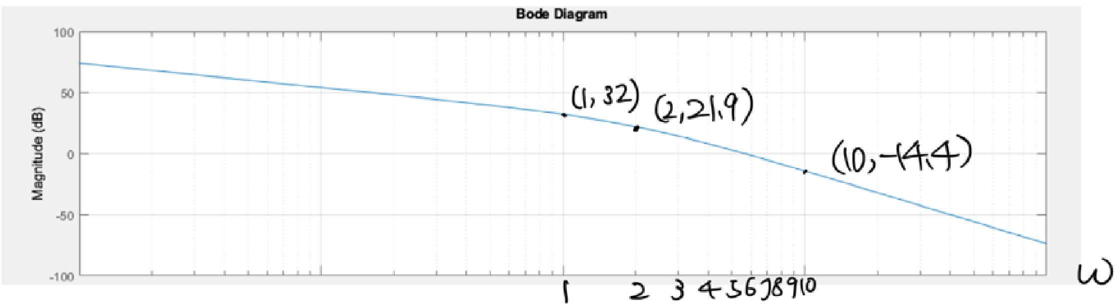
when $\omega \gg 2$

Putting ①②③ together, we get:



The red curve is approximately the magnitude curve.

This result is similar to exact result obtained with Matlab:



This result is different from the reference solution, and I believe the solution is wrong.

(ii) Phase

$$\angle L(j\omega) = -\angle\left(j\omega(0.5j\omega+1)^2\right) = -\angle\left(j\omega(j\omega+1-0.25\omega^2)\right)$$

$$= -\angle\left(-\omega^2 + j(\omega - 0.25\omega^3)\right) = -\text{“}\tan^{-1}\left(\frac{\omega-0.25\omega^3}{-\omega^2}\right)\text{”}$$

$$= \text{“}\tan^{-1}\left(\frac{1}{\omega} - 0.25\omega\right)\text{”}$$

(Here the range is $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$)

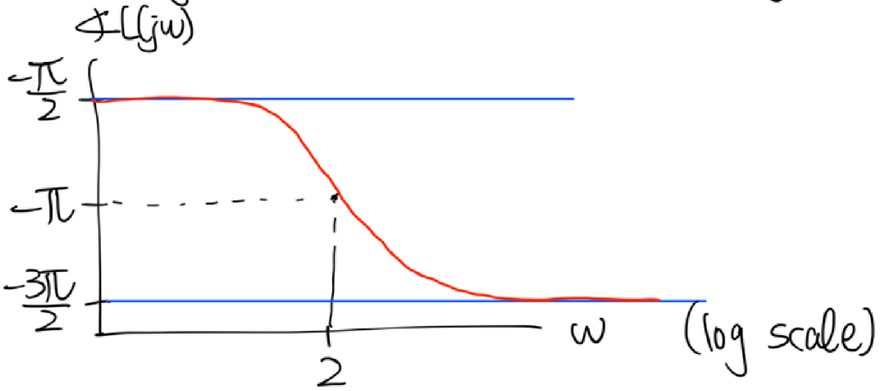
① $\omega \ll 2 \Rightarrow \begin{cases} -0.25\omega \to 0 \\ \frac{1}{\omega} \to \infty \end{cases} \Rightarrow \angle L(j\omega) \to \text{“}\lim_{a\to\infty} \tan^{-1} a\text{”} = -\frac{\pi}{2}^{-}$

② $\omega = 2 \Rightarrow \angle L(j\omega) = \text{“}\tan^{-1}\left(\frac{1}{2} - 0.25\cdot 2\right)\text{”} = \text{“}\tan^{-1} 0\text{”} = -\pi$

③ $\omega \gg 2 \Rightarrow \begin{cases} -0.25\omega \to -\infty \\ \frac{1}{\omega} \to 0 \end{cases} \Rightarrow \angle L(j\omega) \to \text{“}\lim_{a\to-\infty} \tan^{-1} a\text{”} = -\frac{3\pi}{2}^{+}$

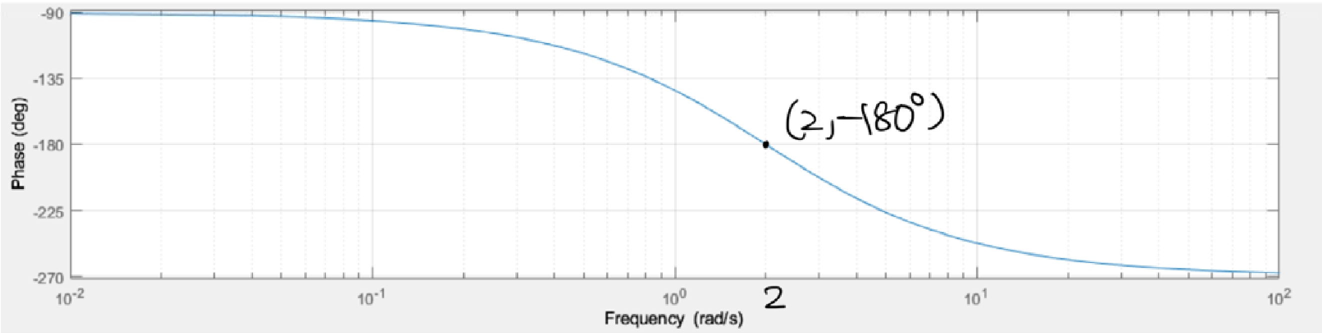
⑤

Putting together ①②③, we get:



The red curve is approximately the phase curve.

This result is similar to exact result obtained with Matlab:



※ Something worthy of note:

(1) The reason I write "$\tan^{-1}$(something)" is that usually, the default of the range of the $\tan^{-1}$ function is $(-\frac{\pi}{2}, \frac{\pi}{2})$; however, here we are discussing about the phase angle of an imagery number, so the angle we want might not lie inside the default range $(-\frac{\pi}{2}, \frac{\pi}{2})$. To remind myself that the range of the $\tan^{-1}$ function now is not the default, I add " " on the function. (A better way to denote them might be $\tan^{-1}(\frac{10}{\omega}) - \pi$ and $\tan^{-1}(\frac{1}{\omega} - 0.25\omega) - \pi$, but I still use " " to make it

look more simple.

(2) I discovered that the intersection points of the two asymptotes for the magnitude part at very small ω and very large ω in both cases ((a) and (b)) happen at the frequency ω_b at which the "$\log_{10}(a+b\omega_b^2)$" term satisfies $a=b\omega_b^2$.

This frequency is exactly the breaking frequency discussed in course.

In (a), the asymptotes are $\begin{cases} dB = 20-40\log_{10}\omega \\ dB = -20\log_{10}\omega \end{cases}$,

and the intersection of the two lines is at

$\omega_b=10$, $dB=-20$. (Close to the exact magnitude $dB=-23$)

We have $\log_{10}(100+\omega_b^2) = \log_{10}(100+100)$, with $100=\omega_b^2$

In (b), the asymptotes are $\begin{cases} dB = 34-20\log_{10}\omega \\ dB = 46-60\log_{10}\omega \end{cases}$,

and the intersection of the two lines is at

$\omega_b=2$, $dB=28$. (Close to the exact magnitude $dB=22$)

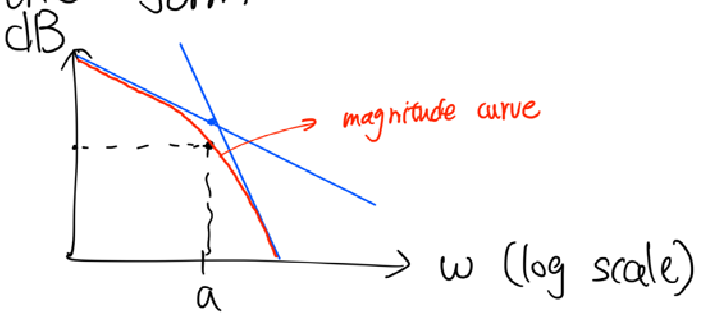
We have $\log_{10}(0.25\omega_b^2+1) = \log(1+1)$, with $0.25\omega_b^2=1$

(3) The transfer function in (a) is of the form $L(s) = \frac{b}{s(s+a)^2}$, and the transfer function in (b) is of the form $L(s) = \frac{b}{s(s+a)^2}$. Despite the difference in the # of poles, the process of determining the Bode plots (especially the magnitude part) are very similar. In fact, the process of determining the plots for a transfer function of the form $L_n(s) = \frac{b}{s(s+a)^n}$ is similar to that of $L_1(s) = \frac{b}{s(s+a)}$.

$L_1(s)$: $|L(j\omega)| = b\frac{1}{|j\omega|}\frac{1}{|j\omega+a|} = \frac{b}{\omega}\frac{1}{\sqrt{a^2+\omega^2}}$

$$\Rightarrow dB = 20\log_{10}\left(\frac{b}{\omega(a^2+\omega^2)^{1/2}}\right) = 20\left(\log_{10}b - \log_{10}\omega - \frac{1}{2}\log_{10}(a^2+\omega^2)\right)$$

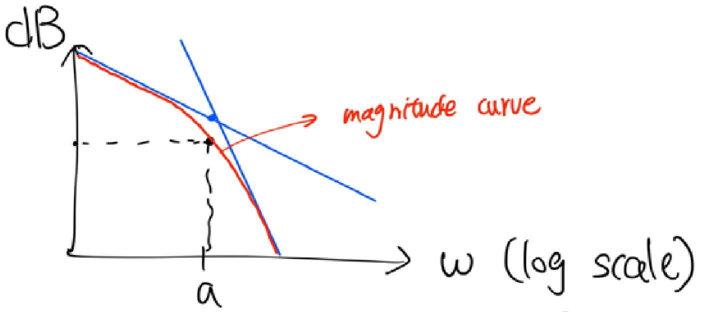
and consider the regions: ① $\omega \ll a$ ② $\omega = a$ ③ $\omega \gg a$ gives the form



$L_n(s)$: $|L(j\omega)| = b\frac{1}{|j\omega|}\left(\frac{1}{|j\omega+a|}\right)^n = \frac{b}{\omega}\frac{1}{(a^2+\omega^2)^{n/2}}$

$$\Rightarrow dB = 20\log\left(\frac{b}{\omega(a^2+\omega^2)^{n/2}}\right) = 20\left(\log_{10}b - \log_{10}\omega - \frac{n}{2}\log(a^2+\omega^2)\right)$$

and consider the regions: ① $\omega \ll a$ ② $\omega = a$ ③ $\omega \gg a$ gives the form



They are similar, regardless of the # of poles.
The phase part also seems interesting, but perhaps I will discuss it as part of the 互動學習週報.

II. Determine The Margin and Crossover Frequencies

Definitions

1. According to the textbook, p.394,

The **gain margin (GM)** is the factor by which the gain can be increased (or decreased in certain cases) before instability results. For the typical case, it can be read directly from the Bode plot (see Fig. 6.15) by measuring the vertical distance between the $|KG(j\omega)|$ curve and the magnitude $= 1$ line at the frequency where $\angle G(j\omega) = -180°$.

2. According to the textbook, p.395

The term **crossover frequency**, ω_c, is often used to refer to the frequency at which the magnitude is unity, or 0 db. ;

According to the website
https://www.sciencedirect.com/topics/engineering/phase-crossover-frequency ,

3. *Gain crossover*

This is the frequency at which the open-loop gain first reaches 1.

These two definitions agree with each other.

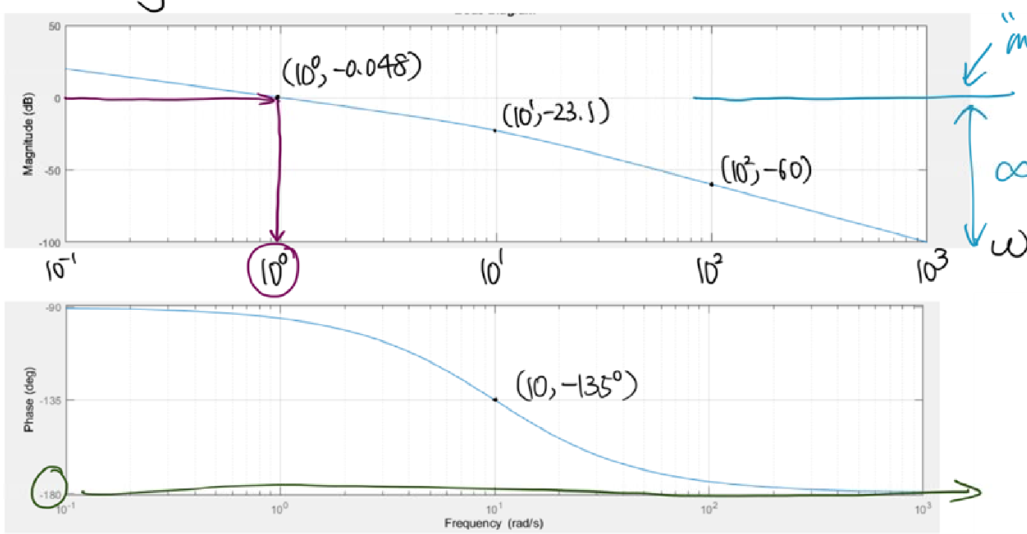
3. According to the website

https://www.sciencedirect.com/topics/engineering/phase-crossover-frequency ,

The phase crossover frequency is the frequency at which the phase angle first reaches –180°

Using these definitions, we can find the gain margins (GM), gain crossover frequencies (denoted ω_g below) and phase crossover frequencies (denoted ω_p below) for the transfer functions in (a), (b).

(a) According to the I part, the Bode plot of (a) is



(i) gain crossover frequency ω_g:

gain reaches $1 \Rightarrow dB = 0$

From the mag. part, we get $\omega_g \approx 10^0 = 1$ (rad/s)

More precisely, we solve for

$$\text{gain} = \frac{10}{\sqrt{\omega_g^4 + 100\omega_g^2}} = 1 \Rightarrow 100 = \omega_g^4 + 100\omega_g^2$$

$\Rightarrow \omega_g^4 + 100\omega_g^2 - 100 = 0$

$\Rightarrow \omega_g^2 = 0.9902,$ ~~-100.99~~ $\Rightarrow \omega_g = 0.9951,$ ~~-0.9951~~

$\Rightarrow$ The precise value is $\omega_g = 0.9951$ rad/s

(ii) phase crossover frequency ω_p:

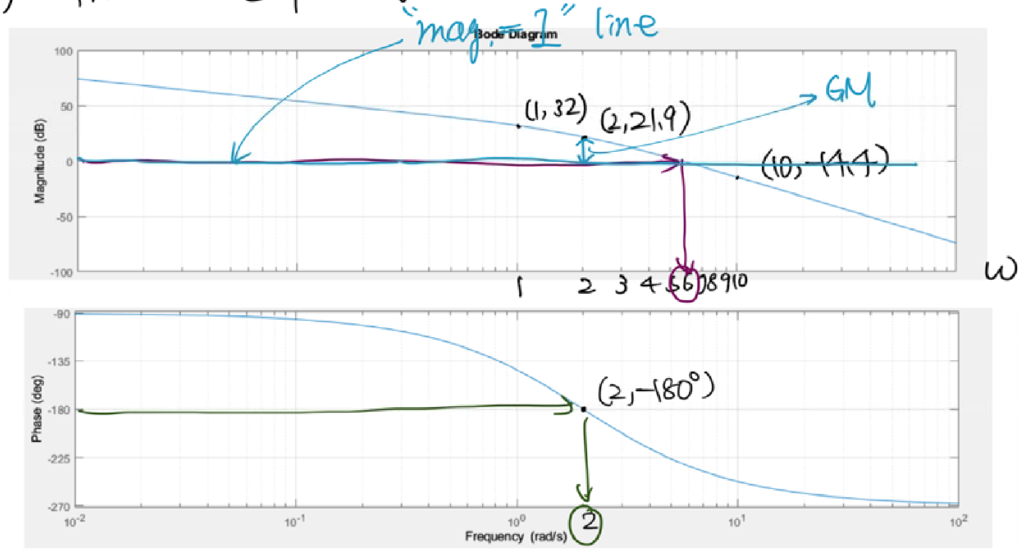
Since $\lim_{\omega\to\infty} \angle L(j\omega) = -180^\circ$, $\omega_p \to \infty$

(iii) gain margin GM:

The frequency considered is $\omega \to \infty$.

$\Rightarrow$ $GM \to \infty$

(b) The Bode plot for (b) is



(i) gain crossover frequency ω_g:

$dB = 0 \Rightarrow \omega_g \approx 6$ rad/s

More precisely, we solve for gain $= \frac{50}{\omega_g(0.25\omega_g^2+1)} = 1$

$\Rightarrow 50 = \omega_g(0.25\omega_g^2+1) \Rightarrow 0.25\omega_g^3 + \omega_g - 50 = 0$

$\Rightarrow \omega_g = 5.6202$, ~~$-2.8101 \pm j5.2621$~~

$\Rightarrow$ The precise value is $\omega_g = 5.6202$ rad/s

(ii) From the phase part, we see that $\omega_p = 2 \text{ rad/s}$

This value is precise.

(iii) The frequency considered is $\omega = 2$.

From the mag. part, we see that $GM = 21.9$

More precisely,

$$|L(j\omega)|\Big|_{\omega=2} = \frac{50}{2(0.25 \cdot 2^2 + 1)} = \frac{50}{2 \cdot 2} = 12.5$$

$$\Rightarrow GM = 20\log_{10}|L(j\omega)|\Big|_{\omega=2} = 20\log_{10}(12.5) = 21.938$$

參考觀摩的作業

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