Control System: Homework 04 for Unit 4A, 4B: Feedback Analysis
Assigned: Oct 14, 2022 Due: Oct 20, 2022 (23:59)
Please read the following problems and their solutions. Then, choose one of them and edit your solution for the selected problem. Submit your homework file in PDF format to the NTU-Cool website.

1. (U4A: Sensitivity)

4. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{s(s+a)}.$$

(a) Compute the sensitivity of the closed-loop transfer function to changes in the parameter A.

(b) Compute the sensitivity of the closed-loop transfer function to changes in the parameter a.

(c) If the unity gain in the feedback changes to a value of $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function with respect to β.

Solution:

(a)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\dfrac{A}{s(s+a)}}{1+\dfrac{A}{s(s+a)}} = \frac{A}{s^2+as+A},$$

$$\frac{dT}{dA} = \frac{(s^2+as+A)-A}{(s^2+as+A)^2},$$

$$S_A^T = \frac{A}{T}\frac{dT}{dA} = \frac{A(s^2+as+A)}{A}\frac{s^2+as}{(s^2+as+A)^2} = \frac{s(s+a)}{s(s+a)+A}.$$

(b)

$$\frac{dT}{da} = \frac{-sA}{(s^2+as+A)^2}.$$

$$\frac{a}{T}\frac{dT}{da} = \frac{a(s^2+as+A)}{A}\frac{-sA}{(s^2+as+A)^2}.$$

$$\mathcal{S}_a^T = \frac{-as}{s(s+a)+A}.$$

(c) In this case,

$$T(s) = \frac{G(s)}{1+\beta G(s)},$$

$$\frac{dT}{d\beta} = \frac{-G(s)^2}{(1+\beta G(s))^2},$$

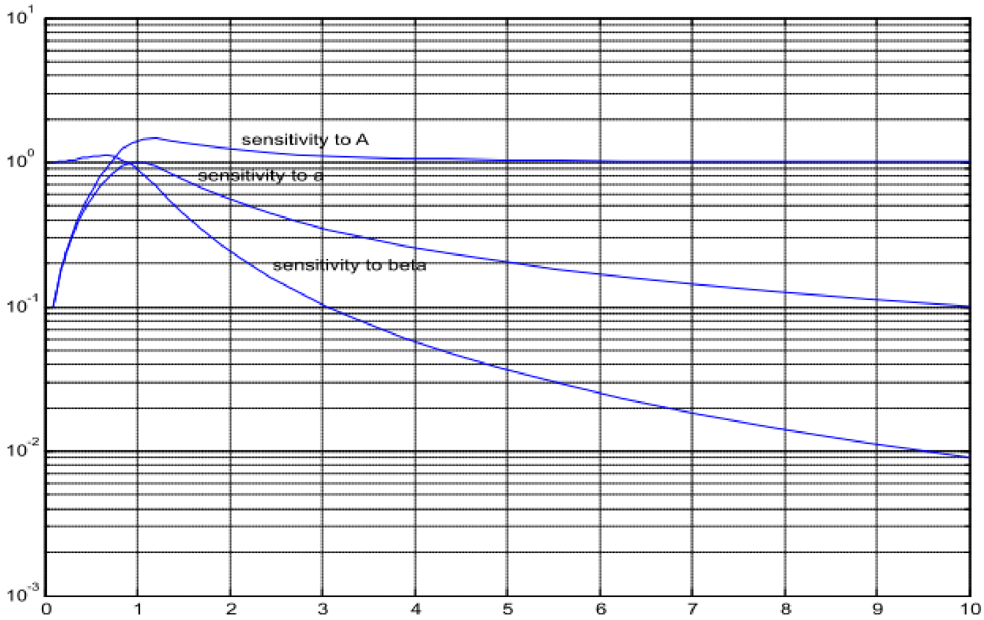
$$\frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1+\beta G)}{G}\frac{-G^2}{(1+\beta G)^2} = \frac{-\beta G}{1+\beta G},$$

$$\mathcal{S}_\beta^T = \frac{\dfrac{-\beta A}{s(s+A)}}{1+\dfrac{\beta A}{s(s+a)}} = \frac{-\beta A}{s(s+a)+\beta A}.$$

- If $a = A = 1$,the transfer function is most sensitive to variations in a and A near $\omega = 1$ rad/sec .
- The steady-state response is not affected by variations in A and a ($\mathcal{S}_A^T(0)$ and $\mathcal{S}_a^T(0)$ are both zero).
- The steady-state response *is* heavily dependent on β since $|\mathcal{S}_\beta^T(0)|$ $= 1.0$

See attached plots of sensitivities versus radian frequency for $a = A = 1.0$.

Sensitivity function frequency response follows.



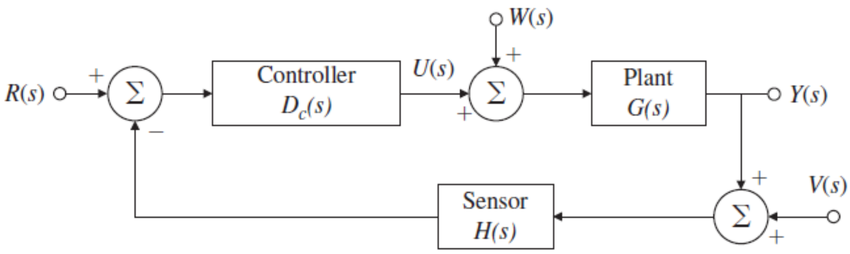
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
0
1
2
3
4
5
6
7
8
9
10
sensitivity to A
sensitivity to a
sensitivity to beta

2. (U4B: Steady-State Error and System Type)

8. A standard feedback control block diagram is shown in Fig. 4.5 with

$$G(s) = \frac{1.5}{s};\ \ D_c(s) = \frac{(s+9)}{(s+3)}; H(s) = \frac{70}{(s+70)}; V(s) = 0.$$

(a) Let $W = 0$ and compute the transfer function from R to Y.
(b) Let $R = 0$ and compute the transfer function from W to Y.
(c) What is the tracking error if R a unit-step input and W = 0?
(d) What is the tracking error if R is a unit-ramp input and W = 0?
(e) What is the system type with respect to the reference inputs and the corresponding error coefficient?



Solution:

(a)

$$\frac{Y(s)}{R(s)} = T(s) = F(s)\frac{D_c(s)G(s)}{1 + D_c(s)G(s)H(s)} = \frac{1.5(s+9)(s+70)}{s^3 + 73s^2 + 315s + 945}$$

(b)

$$\frac{Y(s)}{W(s)} = T(s) = \frac{G(s)}{1 + D_c(s)G(s)H(s)} = \frac{1.5(s+3)(s+70)}{s^3 + 73s^2 + 315s + 945}$$

(c)

$$\frac{E(s)}{R(s)} = 1 - T(s) = S(s) = \frac{s^3 + 71.5s^2 + 196.5s}{s^3 + 73s^2 + 315s + 945}.$$

With $R(s) = \frac{1}{s}$,

$$e_{step}(\infty) = \lim_{s\to 0} sE(s) = 0.$$

(d)

With $R(s) = \frac{1}{s^2}$,

$$e_{ramp}(\infty) = \lim_{s\to 0} sE(s) = 0.2.$$

(e) System is Type 1 and K_v is

$$K_v = \frac{1}{|e_{ramp(\infty)}|} = 5\ sec^{-1}.$$

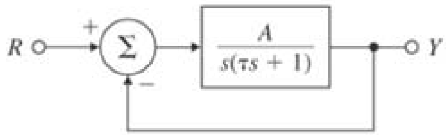
3. (U4B: System Type and Tracking)

14. Consider the system shown in Fig. 4.36(a).

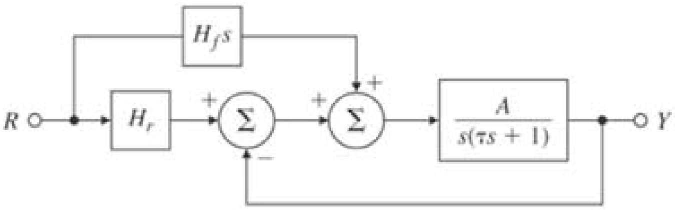
    (a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_o t1(t)$.

    (b) For the modified system with a feed forward path shown in Fig.4.36(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.

    (c) Is the resulting Type 2 property of this system robust with respect to changes in H_f i.e., will the system remain Type 2 if H_f changes slightly?



(a)



(b)

Solution:

(a) System is Type 1 since it is unity feedback and has a pole at $s = 0$in the forward path. Also,

$$E(s) = [1 - \mathcal{T}(s)]R(s),$$

$$= \left[\frac{1}{1+G(s)}\right] R(s),$$

$$= \frac{s(\tau s+1)}{s(\tau s+1)+A}\frac{r_o}{s^2}.$$

The steady-state tracking error using the FVT (assuming stability) is,

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{r_o}{A}.$$

(b)

$$Y(s) = \frac{A}{s(\tau s+1)}U(s),$$
$$U(s) = H_f sR(s) + H_r R(s) - Y(s),$$
$$Y(s) = \frac{A(H_f s + H_r)}{s(\tau s+1)+A}R(s).$$

The tracking error is,

$$E(s) = R(s) - Y(s),$$

$$= \frac{s(\tau s+1) + A - A(H_f s + H_r)}{s(\tau s+1)+K}R(s),$$
$$= \frac{\tau s^2 + (1 - AH_f)s + A(1-H_r)}{s(\tau s+1)+A}.$$

To get zero steady-state error with respect to a ramp, the numerator in the above equation must have a factor s^2. For this to happen, let

$$H_r = 1,$$
$$AH_f = 1.$$

Then
$$E(s) = \frac{\tau s^2}{s(\tau s+1)+A}R(s)$$

and, with $R(s) = \frac{r_o}{s^2}$, apply the FVT (assuming stability) to obtain

$$e_{ss} = 0.$$

Thus the system will be Type 2 with $K_a =$ ~~$\frac{\tau}{A}$~~. $K_a = \frac{A}{\tau}$

(c) No, the system is not robust Type 2 because the property is lost if either H_r or H_f changes slightly.

參考觀摩的作業 1. (Sensitivity)
作者： b08901154，王煒騰 **理由：** 討論不同 sensitivity 函數在非 unity feedback 的情況以及其極值，最後 steady-state response 部分不太清楚
作者： b08901176，陳育楷 **理由：** 討論不同二階系統的 sensitivity 函數，並用 matlab 作圖展示
作者： b10901125，賴禹宏 **理由：** 討論 n 個 zero 和 m 個 pole 的 G(s)對系統增益 k，zero，pole 位置的 sensitivity 函數

1. (U4A: Sensitivity)

4. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{s(s+a)}.$$

(a) Compute the sensitivity of the closed-loop transfer function to changes in the parameter A.

(b) Compute the sensitivity of the closed-loop transfer function to changes in the parameter a.

(c) If the unity gain in the feedback changes to a value of $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function with respect to β.

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$$G(s) = \frac{A}{s(s+a)} \qquad \text{feedback } H(s) = \beta$$

$$T(s) = \frac{G(s)}{1+G(s)\beta} = \frac{A}{s(s+a)+A\beta} = \frac{A}{s^2+as+A\beta}$$

(a) $$S_A^T = \frac{\frac{dT}{T}}{\frac{dA}{A}} = \frac{A}{T}\frac{dT}{dA} = \frac{A}{T}\left(\frac{(s^2+as+A\beta)-A\beta}{(s^2+as+A\beta)^2}\right)$$

$$= \frac{s^2+as}{s^2+as+A\beta} \overset{\beta=1}{\Longrightarrow} \boxed{S_A^T = \frac{s^2+as}{s^2+as+A}}$$

(b) $$S_a^T = \frac{a}{T}\frac{dT}{da} = \frac{a}{A}(s^2+as+A\beta)\left(\frac{0-As}{(s^2+as+A\beta)^2}\right)$$

$$= \frac{-as}{s^2+as+A\beta} \overset{\beta=1}{\Longrightarrow} \boxed{S_a^T = \frac{-as}{s^2+as+A}}$$

(c) $$S_\beta^T = \frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta}{A}(s^2+as+A\beta)\left(\frac{-A^2}{(s^2+as+A\beta)^2}\right)$$

$$= \boxed{\frac{-\beta A}{s^2+as+A\beta}}$$

(d) discussion

I: $$|S^T_A(w)|^2 = \left|\frac{-w^2+jaw}{(A-w^2)+jaw}\right|^2 = \frac{a^2w^2+w^4}{(A-w^2)^2+a^2w^2}$$

$$\frac{d|S^T_A(w)|^2}{dw} = \frac{(2a^2w+4w^3)[a^2w^2+(A-w^2)^2]-[a^2w^2+w^4][-4w(A-w^2)+2a^2w]}{[(A-w^2)^2+a^2w^2]^2} = 0$$

$\Rightarrow (4w^3+2a^2w)(w^4+(a^2-2A)w^2+A^2) = [w^4+a^2w^2][4w^3+(-4A+2a^2)w]$

$\Rightarrow w^7(4)+w^5(4a^2-8A+2a^2)+w^3(4A^2+2a^4-4a^2A)+w(2a^2)$
$= w^7(4)+w^5(-4A+2a^2+4a^2)+w^3(-4a^2A+2a^4)$

$\Rightarrow w^5(-4A)+w^3(4A^2)+w(2a^2)=0$

$\Rightarrow -2w(2Aw^4-2A^2w^3-a^2)=0$

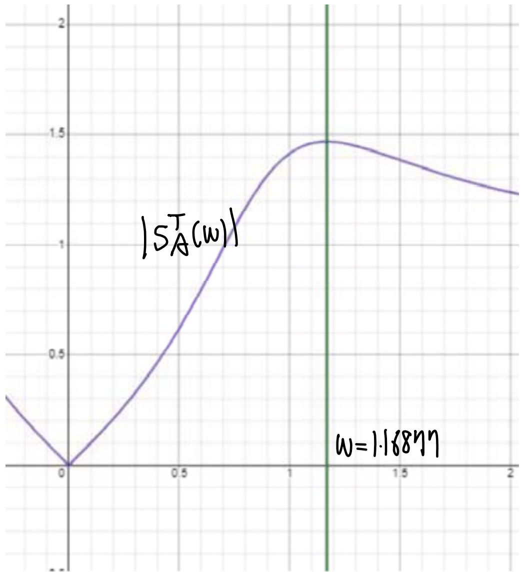
$\Rightarrow w=0,\quad w^2 = \frac{2A^2 \pm \sqrt{4A^4+8Aa^2}}{4A} = \frac{A}{2} \pm \frac{1}{2A}\sqrt{A^4+2Aa^2}$

so when $\boxed{w=\sqrt{\frac{A}{2}+\frac{1}{2}\sqrt{A^2+\frac{2a^2}{A}}},}$

$|S^T_A(w)|$ will have maximum value

If $A=a=1$, $w=1.16877$,

$|S^T_A(w)|$ reaches it's maximum value in the graph.



II: $$|S^T_a(w)|^2 = \left|\frac{-jaw}{(A-w^2)+jaw}\right|^2 = \frac{a^2w^2}{(A-w^2)^2+a^2w^2}$$

$$\frac{d|S^T_a(w)|^2}{dw} = \frac{(2a^2w)[a^2w^2+(A-w^2)^2]-[a^2w^2][-4w(A-w^2)+2a^2w]}{[(A-w^2)^2+a^2w^2]^2} = 0$$

$\Rightarrow (2a^2w)(w^4+(a^2-2A)w^2+A^2) = [a^2w^2][4w^3+(-4A+2a^2)w]$

$\Rightarrow w^5(2a^2)+w^3(2a^4-4a^2A)+w(2a^2A^2) = w^5(4a^2)+w^3(-4a^2A+2a^4)$

$\Rightarrow w^5(2a^2)+w(-2a^2A^2)=0$

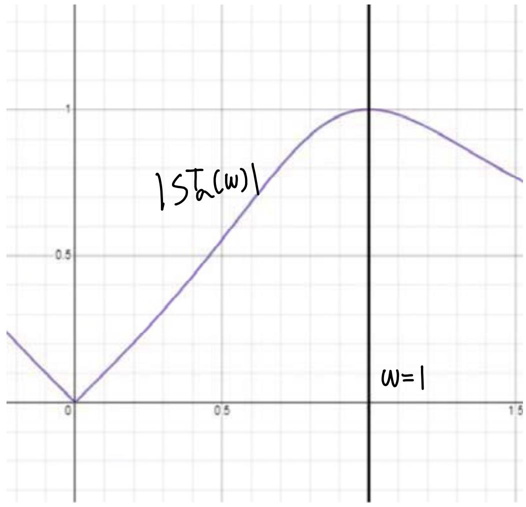
$\Rightarrow w=0,\quad w=\sqrt{A}$

So when $\boxed{\omega = \sqrt{A}}$

$|S_a^T(\omega)|$ will have maximam value

If $A = a = 1$, $\omega = 1$,

$|S_a^T(\omega)|$ reaches it's maximam value in the graph.



III:

$$|S_a^T(\omega)|^2 = \left|\frac{1}{(A\beta - \omega^2) + ja\omega}\right|^2 = \frac{1}{(A\beta - \omega^2)^2 + a^2\omega^2}$$

$$\frac{d|S_a^T(\omega)|^2}{d\omega} = \frac{-[-4\omega(A\beta - \omega^2) + 2a^2\omega]}{[(A\beta - \omega^2)^2 + a^2\omega^2]^2} = 0$$

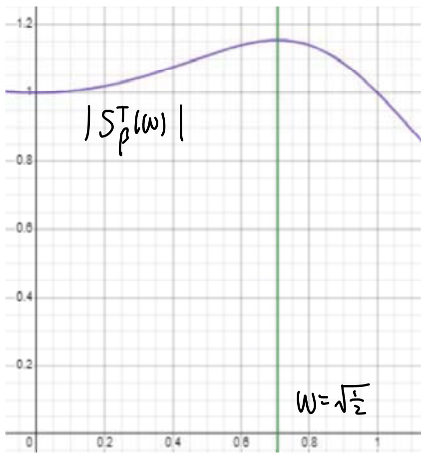
$$\Rightarrow 2a^2\omega = 4\omega(A\beta - \omega^2) \Rightarrow \omega^2 = \frac{1}{4}(4A\beta - 2a^2), \omega = 0$$

So when $\boxed{\omega = \sqrt{A\beta - \frac{a^2}{2}}}$

$|S_\beta^T(\omega)|$ will have maximam value

If $A = a = \beta = 1$, $\omega = \sqrt{\frac{1}{2}}$

$|S_\beta^T(\omega)|$ reaches it's maximam value in the graph.



IV: steady-state response

$$\lim_{s \to 0} s \cdot T(s) = Y_{ss} \Rightarrow S_A^{Y_{ss}} = \frac{A}{Y_{ss}} \frac{dY_{ss}}{dA} = \lim_{s \to 0} \frac{A}{s \cdot T(s)} \frac{d(s \cdot T(s))}{dA}$$

$$= \lim_{s \to 0} \frac{A}{s \cdot T(s)} \cdot s \cdot \frac{dT(s)}{dA} = \lim_{s \to 0} S_A^T(s) = S_A^T(0) = 0$$

also, $S_a^{Y_{ss}} = S_a^T(0) = 0$, $S_\beta^{Y_{ss}} = S_\beta^T(0) = -1$

So the steady-state response will affected by $\Delta\beta$ but not Δa, ΔA

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HW4

1. (U4A: Sensitivity)

1-1. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{(s+a)(s+b)}$$

(1) Compute the sensitivity of the closed-loop transfer function to change in the parameter A.

(2) Compute the sensitivity of the closed-loop transfer function to change in the parameter a.

(3) If the unity gain in the feedback changes to a value $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function to change in the parameter β.

1-2. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)}$$

(4) Compute the sensitivity of the closed-loop transfer function to change in the parameter A.

(5) Compute the sensitivity of the closed-loop transfer function to change in the parameter w_1.

(6) If the unity gain in the feedback changes to a value $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function to change in the parameter β.

1-3 .

(7) Plot the Bode plot of S_A^T, S_a^T, and S_β^T of these two case if $a = 1$, $b = 100$, $A = 60\text{dB}$, $\beta = 20\text{dB}$.

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Solution:

(1)

$$\text{Let } H(s) = (s+a)(s+b)\text{ , which is indepndent of A.}$$

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{A}{H(s)+A}$$

$$\frac{dT}{dA} = \frac{(H(s)+A)-A}{(H(s)+A)^2}$$

$$S_A^T = \frac{A}{T}\frac{dT}{dA} = (H(s)+A)\frac{H(s)}{(H(s)+A)^2} = \boxed{\frac{(s+a)(s+b)}{(s+a)(s+b)+A}} = \boxed{\frac{1}{1+G(s)}}$$

(2)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{A}{s^2+(a+b)s+ab+A}$$

$$\frac{dT}{da} = \frac{-A(s+b)}{[s^2+(a+b)s+ab+A]^2} = \frac{-A(s+b)}{[(s+a)(s+b)+A]^2}$$

$$S_a^T = \frac{a}{T}\frac{dT}{da} = a\frac{(s+a)(s+b)+A}{A}\frac{-A(s+b)}{[(s+a)(s+b)+A]^2} = \boxed{\frac{-a(s+b)}{(s+a)(s+b)+A}} = \boxed{\frac{\frac{-a}{s+a}}{1+G(s)}}$$

(3)

$$T(s) = \frac{G(s)}{1+\beta G(s)}$$

$$\frac{dT}{d\beta} = \frac{-G^2}{(1+\beta G)^2}$$

$$S_\beta^T = \frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1+\beta G)}{G}\frac{-G^2}{(1+\beta G)^2} = \boxed{\frac{-\beta G}{1+\beta G}} = \boxed{\frac{-\beta A}{(s+a)(s+b)+\beta A}}$$

(4)

$$\text{Let } H(s)=\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)\text{, which is indepndent of A.}$$

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{A}{H(s)+A}\,,\qquad \frac{dT}{dA}=\frac{(H(s)+A)-A}{(H(s)+A)^2}$$

$$S_A^T=\frac{A}{T}\frac{dT}{dA}=(H(s)+A)\frac{H(s)}{(H(s)+A)^2}=\boxed{\frac{\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)}{\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)+A}}=\boxed{\frac{1}{1+G(s)}}$$

(5)

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{abA}{s^2+(a+b)s+ab+abA}$$

$$\frac{dT}{da}=\frac{(bA)(s^2+(a+b)s+ab+abA)-(abA)(s+b+bA)}{[s^2+(a+b)s+ab+abA]^2}$$

$$=\frac{(bA)s(s+b)}{[s^2+(a+b)s+ab+abA]^2}$$

$$S_a^T=\frac{a}{T}\frac{dT}{da}=\frac{s^2+(a+b)s+ab+abA}{bA}\frac{(bA)s(s+b)}{[s^2+(a+b)s+ab+abA]^2}$$

$$=\frac{s(s+b)}{s^2+(a+b)s+ab+abA}=\boxed{\frac{\frac{s}{a}\left(1+\frac{s}{b}\right)}{\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)+A}}=\boxed{\frac{\frac{s}{s+a}}{1+G(s)}}$$

(6)

$$T(s)=\frac{G(s)}{1+\beta G(s)}$$

$$\frac{dT}{d\beta}=\frac{-G^2}{(1+\beta G)^2}$$

$$S_\beta^T=\frac{\beta}{T}\frac{dT}{d\beta}=\frac{\beta(1+\beta G)}{G}\frac{-G^2}{(1+\beta G)^2}=\boxed{\frac{-\beta G}{1+\beta G}}=\boxed{\frac{-\beta A}{\left(1+\frac{s}{a}\right)\left(1+\frac{s}{b}\right)+\beta A}}$$

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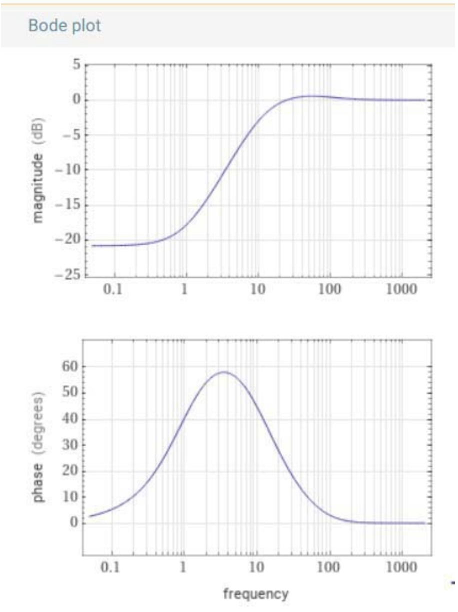
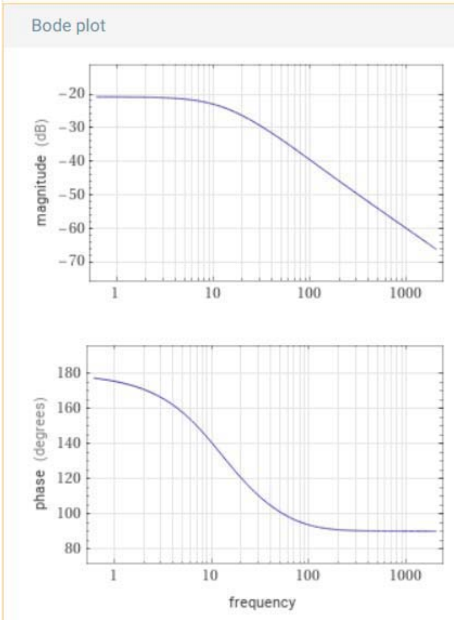
(7) a =1, b = 100, A = 60dB, β = 20dB.

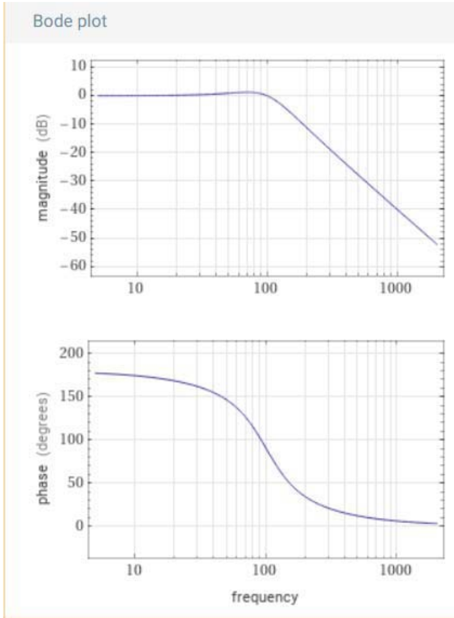
1-1 S_A^T



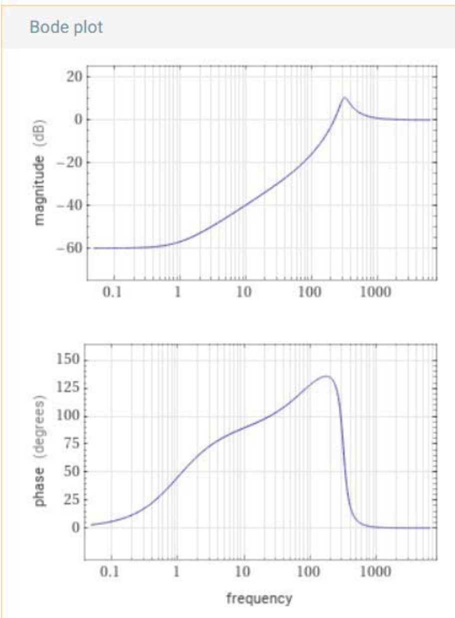
1-1 S_a^T



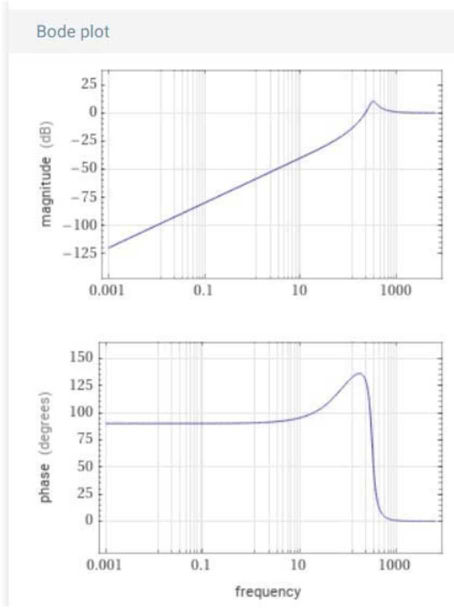
1-1 S_β^T



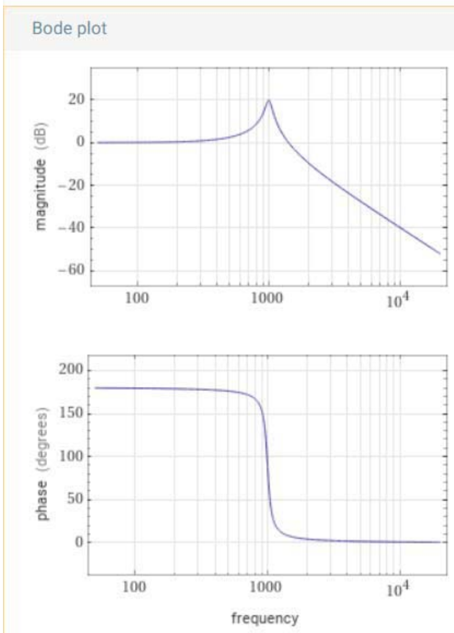
1-2 S_A^T



1-2 S_a^T



1-2 S_β^T



Observation:

- For two poles control system

$$S_A^T = \frac{1}{1+G}, \qquad S_\beta^T = \frac{-\beta G}{1+\beta G}$$

- For these two case of expression of two poles control system,

$$S_A^T(0) \approx 0, S_a^T(0) \approx 0, \text{and } |S_\beta^T(0)| \approx 1,$$

Thus, the steady-state response of these two case is not affected by variations in parameter A and parameter a, but heavily dependent on parameter β.

HW 04	Control Systems, Fall 2022, NTU-EE
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Problem 1

We calculate the sensitivities of different parameters for a general open-loop transfer function (G(s)) in closed-loop case (T(s) = G(s)/(1+G(s)))

$$G(s) = \frac{k(s+z_1)\cdots(s+z_n)}{(s+p_1)\cdots(s+p_n)} = \frac{k\prod_i(s+z_i)}{\prod_j(s+p_j)}, \quad T(s) = \frac{k\prod_i(s+z_i)}{\prod_j(s+p_j) + k\prod_i(s+z_i)}$$

- $$\frac{dT}{dK} = \frac{\prod_i(s+z_i)}{\prod_j(s+p_j)+k\prod_i(s+z_i)} - \frac{(k\prod_i(s+z_i))(\prod_i(s+z_i))}{(\prod_j(s+p_j)+k\prod_i(s+z_i))^2}$$
$$= \frac{\prod_j(s+p_j)\prod_i(s+z_i)}{(\prod_j(s+p_j)+k\prod_i(s+z_i))^2}$$
$$S_K^T = \frac{\prod_j(s+p_j)}{\prod_j(s+p_j)+k\prod_i(s+z_i)} = 1 - T(s) = \frac{1}{1+G} \quad \#$$

- $$\frac{dT}{dz_\alpha} = \frac{k}{s+z_\alpha}\frac{dT}{dK}, \quad S_{z_\alpha}^T = \frac{z_\alpha}{s+z_\alpha} S_K^T = \frac{z_\alpha}{s+z_\alpha}\frac{\prod_j(s+p_j)}{\prod_j(s+p_j)+k\prod_i(s+z_i)}$$

- $$\frac{dT}{dp_\beta} = \frac{-k\prod_i(s+z_i)}{(\prod_j(s+p_j)+k\prod_i(s+z_i))^2}\cdot\frac{\prod_j(s+p_j)}{s+p_\beta}$$
$$S_{p_\beta}^T = \frac{-p_\beta}{s+p_\beta}S_K^T = \frac{-p_\beta}{s+p_\beta}\frac{\prod_j(s+p_j)}{\prod_j(s+p_j)+k\prod_i(s+z_i)}$$

※ Steady state response have the same sensitivity for every parameters, ($S_K^T(0) = \frac{\prod_j p_j}{\prod_j p_j + k\prod_i z_i}$)

※ Sensitivity become infinity when $1+G(s)=0$, and for individual $S^T_{z_\alpha \text{ or } p_\beta}$ reach infinity at $s=-z_\alpha$ or $-p_\beta$

※ It may be helpful to study the minimization of $S^T_{z_\alpha}$ and $S^T_{p_\beta}$ in vector space.

參考觀摩的作業

2. (Steady-State Error and System Type)

作者：b08901085，施彥宇

理由：參考論文並討論 controller 為 nonlinear memoryless 的情況，建議將論文想表達的內容用自己的話講出來，而不是將論文內容部份搬到作業內，比如控制器 Lipschitz 條件的意義以及討論 G(jw) root locus 的意義。式(3), (8), (9)有缺漏符號或打錯字

HW04 – Unit 4, Feedback Analysis

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- **Question** : Consider the system shown in Fig.1(a)
  a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_0 t1(t)$.
  b) For the modified system with a feedback forward path shown in Fig .1(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.
  c) In the resulting Type 2 property of this system robust with respect to changes in H_f i.e., will the system remain Type 2 if H_f changes slightly?

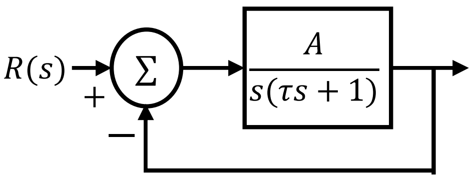


Figure 1(a).

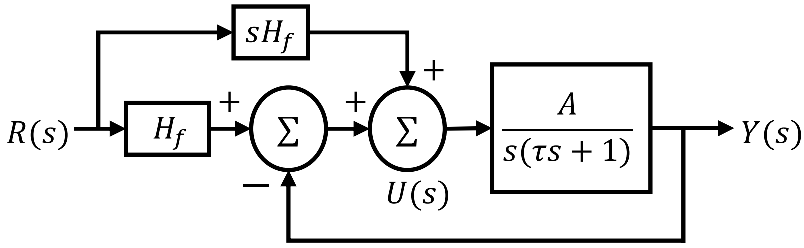


Figure 1(b).

- **Solution :**
  a) We can observe that the system has one pole at $s = 0$ in forward path and a unity feedback, as a result, this system is Type 1. And also,

$$E(s) = (1 - T\{s\})R(s) = \frac{1}{1+G(s)}R(s), \text{ where}$$

$$G(s) = \frac{A}{s(\tau s+1)}, \text{ and } R(s) = \frac{r_0}{s^2}. \quad (1).$$

Thus, the steady-state tracking error is :

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{s r_0}{s^2}\frac{s(\tau s+1)}{s(\tau s+1)+A} = \frac{r_0}{A} \quad (2).$$

  b) The relationship of $Y(s), U(s)$ and $R(s)$ is that :

$$Y(s) = \frac{A}{s(\tau s+1)}R(s) \quad (3).$$

$$U(s) = sH_f R(s) + H_r R(s) - Y(s) \quad (4).$$

Substituting equation (4). into (3). , we get the transfer function :

$$Y(s) = \frac{A(H_f s + H_r)}{s(\tau s+1)+A}R(s) \quad (5).$$

Therefore, the tracking error is :

$$E(s) = R(s) - Y(s) = \frac{\tau s^2 + (1 - AH_f)s + A(1 - H_r)}{s(\tau s + 1) + A}R(s) \quad (6).$$

- **Solution :**

To get the zero steady-state error with respect to a ramp, the numerator in equation (6). must have a factor of s^2. In order to satisfy this condition, the coefficient H_f and AH_f are :

$$H_r = 1\,; \qquad AH_f = 1 \qquad (7).$$

Then the tracking error becomes :

$$E(s) = \frac{\tau s^2}{s(\tau s+1)+} R(s)\text{ , while } R(s) = \frac{r_0}{s^2} \qquad (8).$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{sr_0}{s^2}\frac{\tau s^2}{s(\tau s+1)+} = 0 \qquad (9).$$

For Type 2 error, new functions $E'(s)$ and $R'(s)$ will become :

$$E'(s) = \frac{\tau s^2}{s(\tau s+1)+} R'(s) = \frac{r_0}{s^3}\frac{\tau s^2}{s(\tau s+1)+}\text{ , while } R'(s) = \frac{r_0}{s^3}\text{ is a parabola}$$

input. Hence, the steady-state error $e_{ss} = K_a^{-1}$, as a result, K_a is :

$$K_a = \left(\lim_{s\to 0} sE'(s)\right)^{-1} = \left(\lim_{s\to 0}\frac{s}{s^3}\frac{\tau s^2}{s(\tau s+1)+\mathrm{A}}\right)^{-1} = \left(\frac{\tau}{A}\right)^{-1} = \frac{A}{\tau} \quad (10).$$

c) This system is not robust because the property is lost if either H_f or H_r changes slightly.

- **What I can do more :**

All the formula in textbook are calculating error in linear system. However, most of the systems in the world are nonlinear. As a result, I think I can take a look at some methods analyzing nonlinear systems. Starting from normal linear system, Figure 2(a). demonstrates a block diagram of linear control system. And we take N into account, which is a memoryless nonlinear element. This system is shown in Figure 2(b).

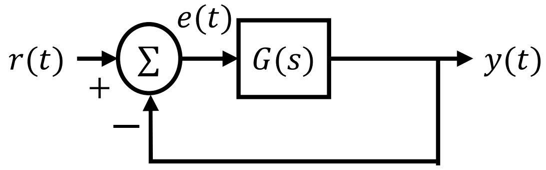


Figure 2(a).

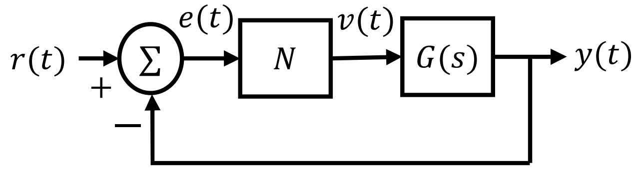


Figure 2(b).

Referring to Figure 2, the relationship between $y, r, e,$ and v is :

$$y(t) = L\{v(t)\} + z(t),\ \ t \ge 0 \qquad (11).$$

$z(t)$means the initial condition. And the operator L is the convolution sum which is defined by :

$$L\{x(t)\} = \int_0^t g(t-\tau)x(\tau)d\tau\,,\ \ t \ge 0 \qquad (12).$$

And $G(s), Z(s)$ is the Laplace transform of $g(t)$, and $z(t)$, respectively. We can suppose that the nonlinear portion of this system, $N\{x(t)\} = \eta[x(t)],\ t \geq 0$, where η means a mapping from $\mathcal{R}$ to $\mathcal{R}$ such that $\eta(0) = 0$ and there are two real constants α and β such that :

$$\alpha \leq \frac{\eta(a)-\eta(b)}{a-b} \leq \beta,\ \text{for } \alpha \neq \beta \qquad (13).$$

From equation (11). (12). and Figure 2, the system relationship is :

$$r(t) - z(t) = e(t) + \int_0^t g(t-\tau)\eta[e(\tau)]d\tau\,,\ \ t \geq 0 \qquad (14).$$

Then, we assume three conditions and suppose that one of these conditions satisfied,

Condition (a). : $0 < \alpha < \beta$, and the locus of $G(j\omega)$ for $-\infty < \omega < \infty$, is bounded away from the circle C_1 of radius is $0.5(\alpha^{-1} - \beta^{-1})$ centered on the complex plain $[-0.5(\alpha^{-1} + \beta^{-1}), 0]$; and encircles C_1 in the counterclockwise direction n_p times, with Nyquist-locus identification. Where n_p is the number of poles of G with positive real parts.

Condition (b). : $0 = \alpha < \beta$, G has no poles in RHP, and $\mathcal{R}e[G(j\omega)] \succ \beta^{-1}$ for all real ω.

Condition (c). : $\alpha < 0 < \beta$, G has no poles in RHP, and the locus of $G(j\omega)$ for $-\infty < \omega < \infty$ is contained with a circle C_2 of radius is $0.5(\beta^{-1} - \alpha^{-1})$ centered at complex plain $[-0.5(\alpha^{-1} + \beta^{-1}), 0]$.

And the rational function $H(s)$ is defined by :

$$H(s) = \frac{G(s)}{1+c_0 G(s)}, c_0 = \frac{\alpha+\beta}{2} \qquad (15).$$

And we define an operator $\bar{\eta}(a) = \eta(a) - c_0 a$, assuming the above three conditions met, then for each $\gamma \in \mathcal{R}$, there exist a unique $\zeta \in \mathcal{R}$ such that

$$\gamma = \zeta + H(0)\bar{\eta}(\zeta) \qquad (16).$$

and one has

$$\zeta = \lim_{k\to\infty} \zeta_k\ \text{, where } \zeta_{k-1} = \gamma - H(0)\,\bar{\eta}(\zeta_k) \qquad (17).$$

We use the iteration in equation (16). and (17). to approach the performance of nonlinear system using linear calculation. And the initial ζ_0 can be chosen arbitrarily. Also, $c = \frac{\beta-\alpha}{2}|H(0)| < 1$, and

$$|\zeta - \zeta_k| \leq \frac{c^k}{1-c}|\zeta_0 - \gamma + H(0)\bar{\eta}(\zeta_0)|\,,\ k \geq 1 \qquad (18).$$

Referring to equation (16). to (18).,let Θ be the mapping taking γ into ζ. Then, the theorem of the relationship between the nature of e and the system type is in the following :

- **Theorem :** Assuming that the above three conditions are met, and $G(s) = \frac{p(s)}{s^\rho q(s)}$, $Z(s) = \frac{m(s)}{s^\rho q(s)}$, then

    1. If input signal $r(t)$ approaches a limit l as $t \to \infty$, or $\lim_{t\to\infty} r(t) = l$, then $S = \lim_{t\to\infty} e(t)$ exists. Moreover, $S \neq 0$ if and only if $l \neq 0$ and $\rho = 0$. At this moment, $S = \Theta\{\frac{l}{1+c_0 G(0)}\}$.
    2. If the input signal can be written as :

$$r(t) = \sum_{j=0}^{v} a_j t^j \ , \ t \geq 0 \qquad (19).$$

where a_j are real, $v \in \mathcal{N}$, and $a_v \neq 0$, then e has three conditions :

① $v > \rho \ \Rightarrow \ e \to \infty$ as t → ∞.

② $v \leq \rho \ \Rightarrow \ e \to \Theta\left\{\frac{a_v v! q(0)}{c_0 p(0)}\right\}$ as $t \to \infty$.

③ $v = \rho \ \Rightarrow \ e \to 0$ as $t \to \infty$.

By using this theorem, we can approximate the behavior of error in nonlinear systems. However, the proof is quite complicate, as a result, I didn't write it out. It can be seen at References.

- References

[1]. Irwin W. Sandberg and Kelly Kramer Johnson, "Steady-State Errors in Nonlinear Control Systems", *IEEE Trans. Automat. Control*, 1922
[2]. Irwin W. Sandberg and Kelly Kramer Johnson, "On steady-state errors in nonlinear control systems", 1922
[3]. F.G. Tricomi, "Integral Equations.", *New York: Interscience, 1963, pp. 42-47*
[4]. I.W. Sandberg, "A frequency-domain condition for the stability of feedback systems containing a single time-varying nonlinear element," *Bell Syst. Tech. J.*, vol43, pp. 1601-1608, July 1964.

參考觀摩的作業

3. (System Type and Tracking)

作者： b10202032，卓 然

理由： 詳細討論 n 階系統如何在 unity feedback 情況下利用前饋項使 system type 為 n（前饋 n-1~1 項係數不須為 0，只需要常數項能消掉且 n 項係數不為 0 即可）

Control Systems HW
B10202032 物理二 卓然

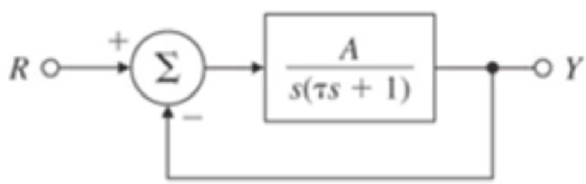
3. (U4B: System Type and Tracking)

14. Consider the system shown in Fig. 4.36(a).

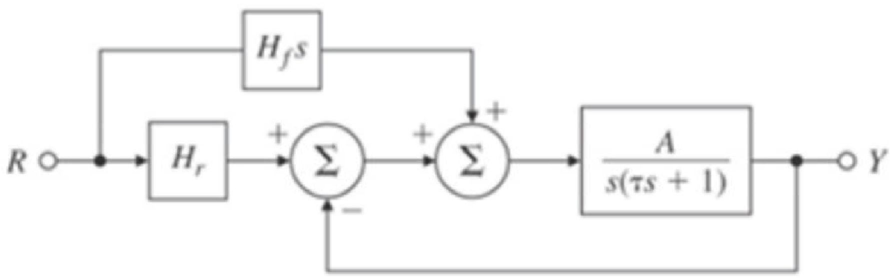
(a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_o t1(t)$.

(b) For the modified system with a feed forward path shown in Fig.4.36(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.

(c) Is the resulting Type 2 property of this system robust with respect to changes in H_f i.e., will the system remain Type 2 if H_f changes slightly?



(a)

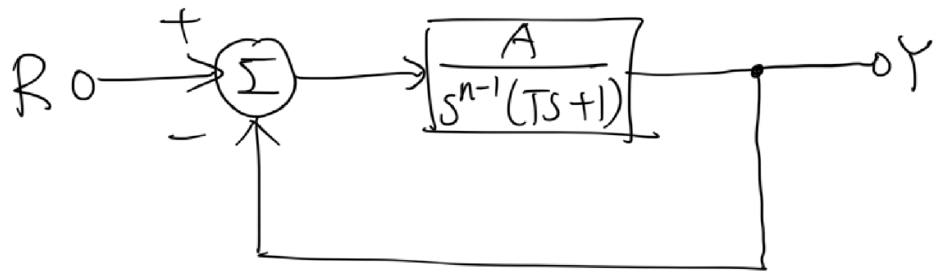


(b)

In the lecture, the discussion about system type was mainly focused on the properties of Type 0, 1, and 2 systems. However, these are not the only system types; a Type n system is one that can reasonably track a polynomial input signal of degree at most n. Based on this definition, for every $n \in \mathbb{N} \cup \{0\}$, "a Type n system" makes sense, even though it is not practical. This homework discusses Problem 3 by altering some parts of the functions shown in the block diagrams to make them Type n systems.

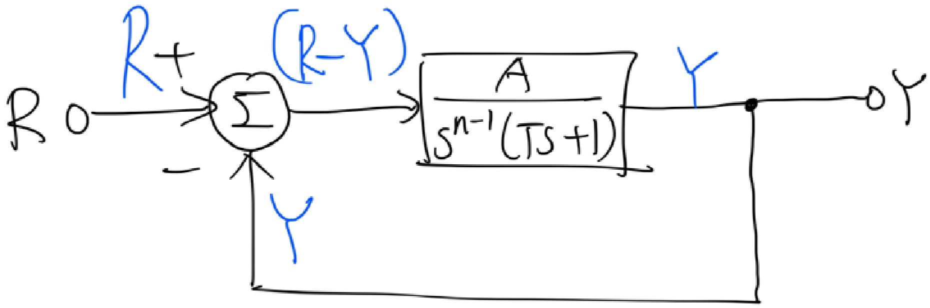
First, I define K_i as the position error constant of a Type i system: $K_i \triangleq \lim_{s \to 0} s^i G D_{ci}(s)$. For example, $K_p \triangleq K_0$, $K_v \triangleq K_1$, $K_a \triangleq K_2$.

(a) Consider the system



with the reference input $r(t) = r_0 \frac{t^{n-1}}{(n-1)!} 1(t)$.

This system is a Type $n-1$ system since there are $n-1$ poles at $s=0$. Now calculate the steady state tracking error e_{ss}:



$$(R-Y)\frac{A}{s^{n-1}(\tau s+1)} = Y \Rightarrow \frac{RA}{s^{n-1}(\tau s+1)} = Y\left(1+\frac{A}{s^{n-1}(\tau s+1)}\right)$$

$$\Rightarrow RA = Y\left(A + s^{n-1}(\tau s+1)\right)$$

$$\Rightarrow E(s) = R - Y = R - \frac{RA}{A+s^{n-1}(\tau s+1)} = R\left(\frac{s^{n-1}(\tau s+1)}{A+s^{n-1}(\tau s+1)}\right)$$

Since $R = \mathcal{L}\{r_0 \frac{t^{n-1}}{(n-1)!} 1(t)\} = \frac{r_0}{s^n}$,

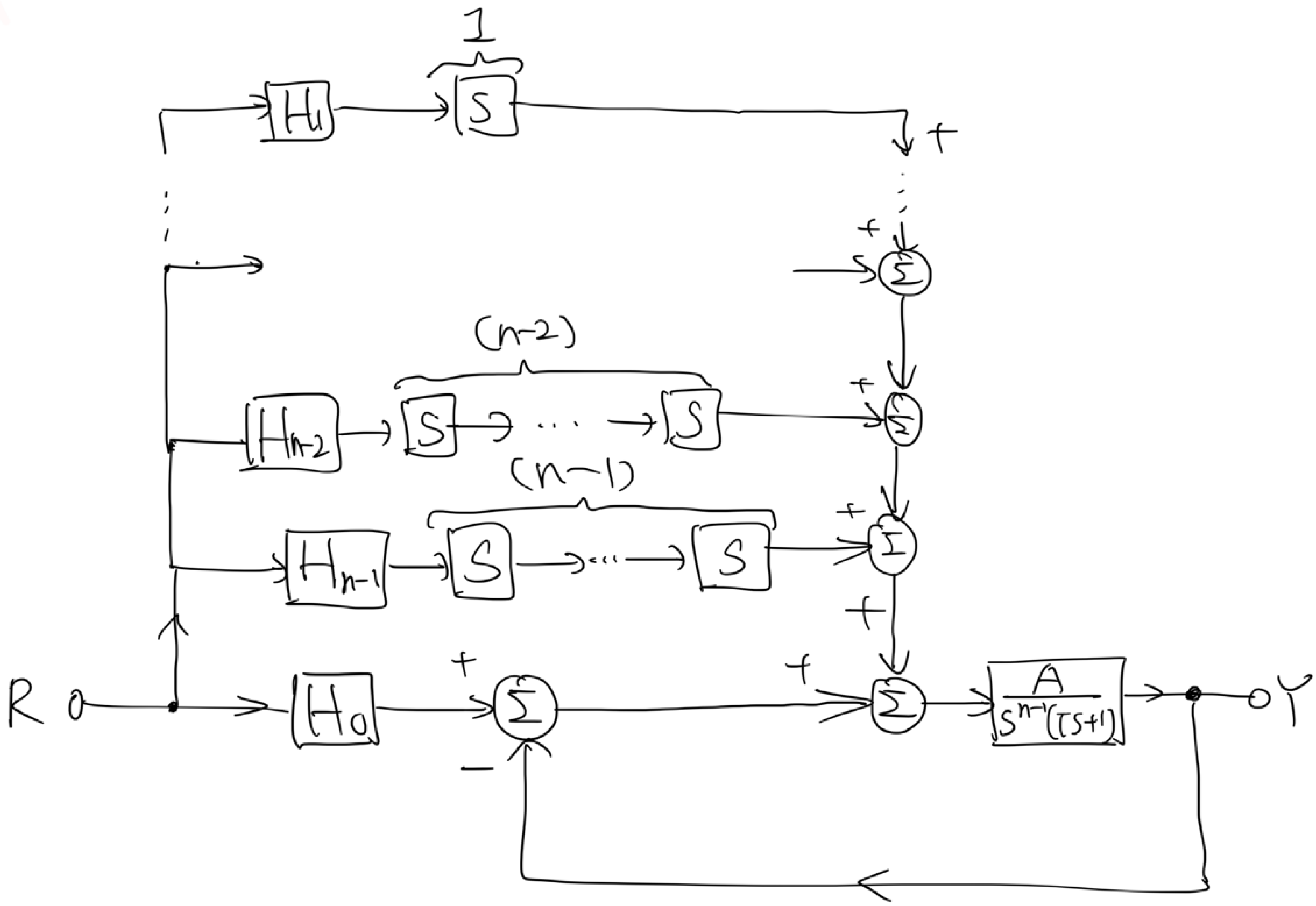
$$E(s) = \left(\frac{r_0}{s^n}\right)\frac{s^{n-1}(\tau s+1)}{A+s^{n-1}(\tau s+1)}$$

$$\therefore e_{ss} = \lim_{t\to\infty} e(t) \underset{\text{FVT}}{=} \lim_{s\to 0} s \cdot \frac{r_0}{s^n}\frac{s^{n-1}(\tau s+1)}{A+s^{n-1}(\tau s+1)}$$

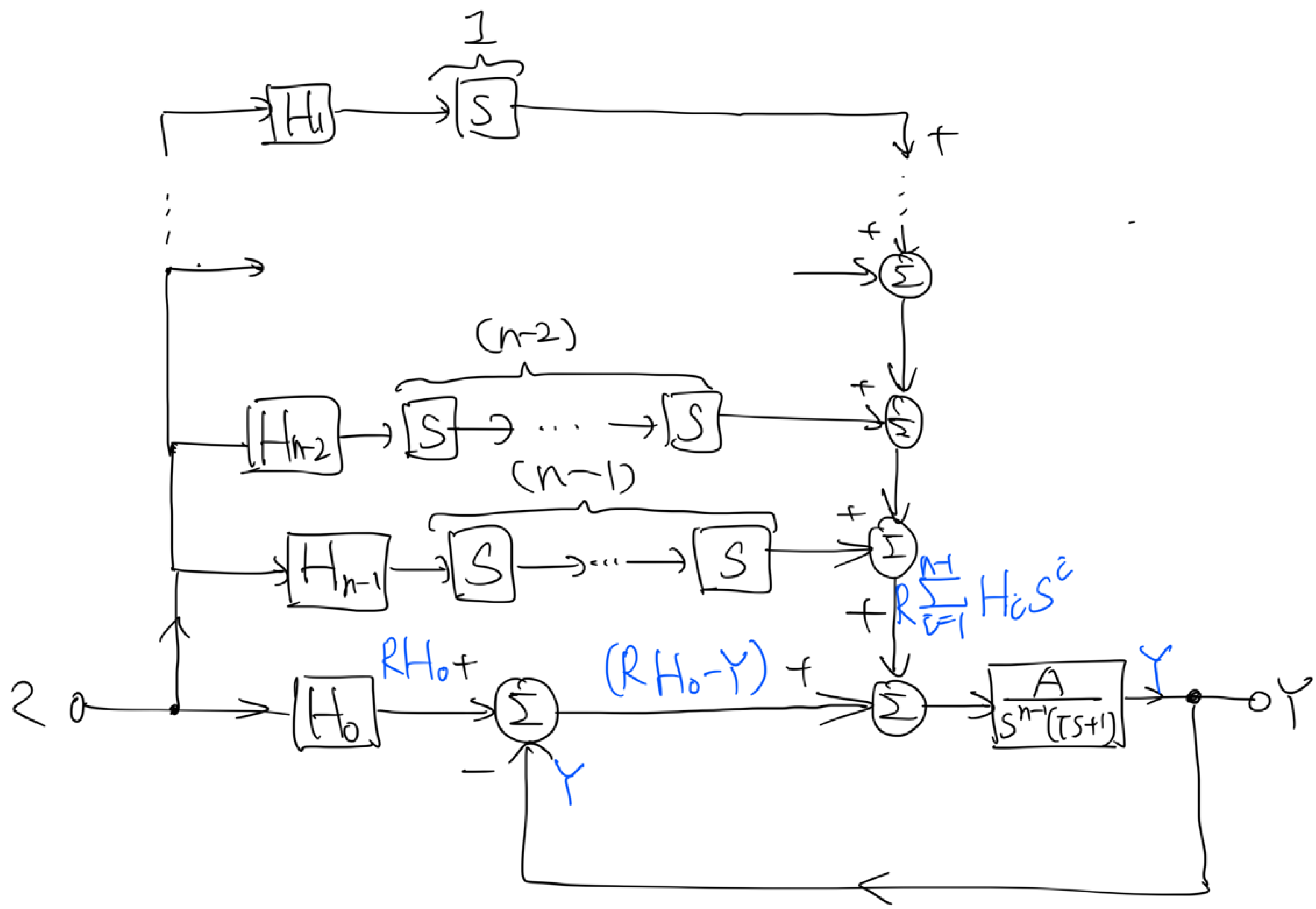
$$= \lim_{s\to 0} r_0 \cdot \frac{\tau s+1}{A+s^{n-1}(\tau s+1)} = r_0 \cdot \frac{1}{A} = \frac{r_0}{A}.$$

This result doesn't depend on the system type, n.

(b) Consider the feedforward system



I want to make it a Type n system.



$$Y=\left(RH_0-Y+R\sum_{i=1}^{n-1}H_is^i\right)\left(\frac{A}{s^{n-1}(\tau s+1)}\right)$$

$$\Rightarrow s^{n-1}(\tau s+1)Y+AY=R\sum_{i=0}^{n-1}H_is^iA$$

$$\Rightarrow Y=\frac{AR\sum_{i=0}^{n-1}H_is^i}{A+s^{n-1}(\tau s+1)};$$

$$E(s)=R-Y=R\left(1-\frac{A\sum_{i=0}^{n-1}H_is^i}{A+s^{n-1}(\tau s+1)}\right)=R\left(\frac{A+s^{n-1}(\tau s+1)-A\sum_{i=0}^{n-1}H_is^i}{A+s^{n-1}(\tau s+1)}\right)$$

$$=\frac{r_0}{s^n}\cdot\frac{A+s^{n-1}(\tau s+1)-A\sum_{i=0}^{n-1}H_is^i}{A+s^{n-1}(\tau s+1)}$$

$$\Rightarrow sE(s)=\frac{r_0}{s^{n-1}}\cdot\frac{A+s^{n-1}(\tau s+1)-A\sum_{i=0}^{n-1}H_is^i}{A+s^{n-1}(\tau s+1)}$$, to make the system a Type n one, we must make $e_{ss}=\lim_{s\to 0}sE(s)=0$.

Therefore, $A+s^{n-1}(\tau s+1)-\sum_{i=0}^{n-1}H_is^i=\tau s^n$

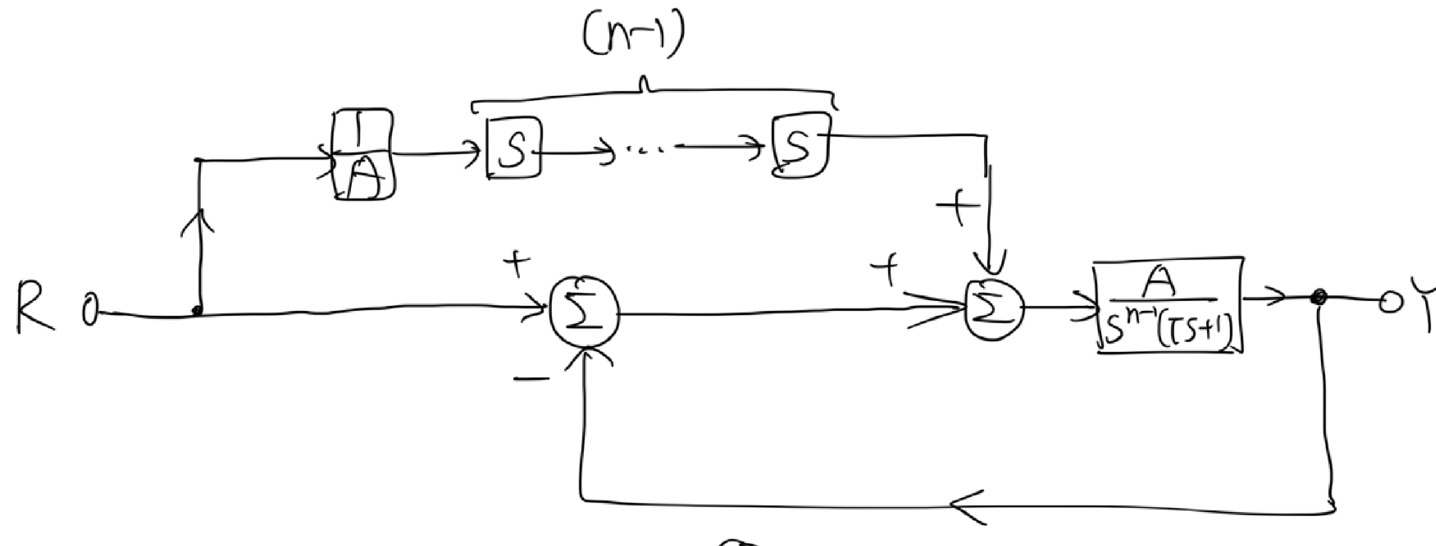
$$\Rightarrow s^{n-1}-A\left(\sum_{i=0}^{n-1}H_is^i\right)+A=0$$

$$\Rightarrow (1-AH_{n-1})s^{n-1}-A\left(\sum_{i=1}^{n-2}H_is^i\right)+(A-AH_0)=0$$

$$\Rightarrow \begin{cases}1-AH_{n-1}=0\\ H_iA=0 \quad (i=1,\dots,n-2)\\ A(1-H_0)=0\end{cases} \xrightarrow{A\neq 0} \begin{cases}H_{n-1}=\frac{1}{A}\\ H_i=0 \quad (i=1,\dots,n-2)\\ H_0=1.\end{cases}$$

($A\neq 0$ ← the nontrivial case)

Therefore, the system can be simplified to:



which is much more simpler.

Now, $K_n = \lim_{s \to 0} s^n GD_{s2}$

$$E(s) = R - Y = R - \frac{AR(H + \frac{1}{A}s^{n-1})}{A + s^{n-1}(\tau s+1)} = R\left(\frac{s^{n-1}(\tau s+1) - s^{n-1}}{A + s^{n-1}(\tau s+1)}\right)$$

$$= \frac{R\tau s^n}{A + s^{n-1}(\tau s+1)} = \frac{R}{1 + GD_{s2}}$$

$$\Rightarrow GD_{s2} = \frac{A + s^{n-1}(\tau s+1)}{\tau s^n}$$

$$\therefore K_n = \lim_{s \to 0} s^n \frac{A + s^{n-1}(\tau s+1)}{\tau s^n} = \lim_{s \to 0} \frac{A + s^{n-1}(\tau s+1)}{\tau} = \frac{A}{\tau}.$$

(c) Absolutely not. In fact, if any one of

$$\begin{cases} H_{n-1} = \frac{1}{A} \\ H_i = 0 \quad (i = 1, \dots, n-2) \\ H_0 = 1. \end{cases}$$

fails, it is not a Type n system.

Check. Taking $n=2$: (a) It's a Type 1 system, $e_{ss} = \frac{r_0}{A}$.

(b) $H_1 = \frac{1}{A}$, $H_0 = 1$, $K_2 = K_a = \frac{A}{\tau}$.

(c) No.

This matches the reference answer provided.

Conclusion 1. It's not that hard to construct a Type n system: we can do this if $H_{n-1} = \frac{1}{A}$, and we don't conneet the H_i, $i = 1, \dots, n-1$ loop, and we don't put anything on the H_0 loop.

2. Some results remain constant (like e_{ss} in (a), K_n in (b)) for different n in this problem.

3. To build a Type n system, we need (n−1) differentiator. As $n \to \infty$, $(n-1) \to \infty$.