Control System: Homework 03 for Unit 3E, 3F: Dynamic Response

Assigned: Oct 7, 2022
Due: Oct 13, 2022 (23:59)

Please read the following problems and their solutions. Then, choose one of them and edit your solution for the selected problem. Submit your homework file in PDF format to the NTU-Cool website.

1. (Effect of zeros and additional poles)

41. ▲ Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

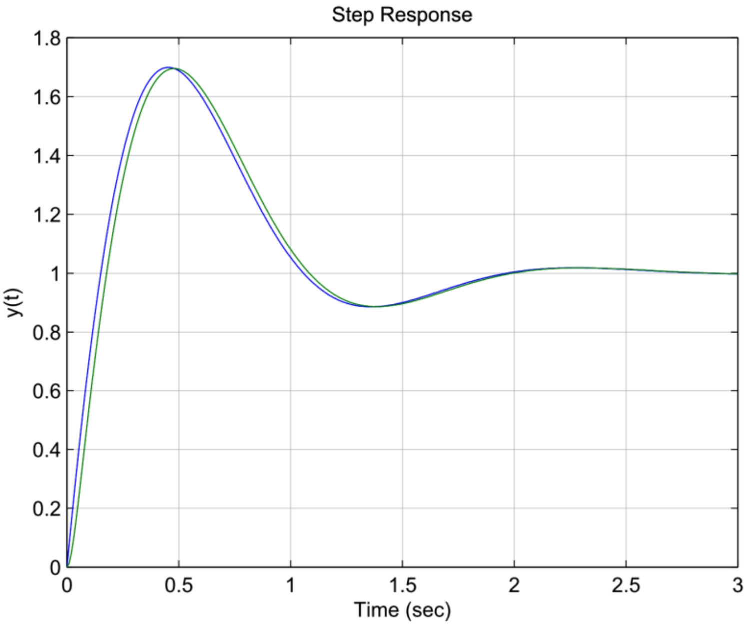
Solution:

From the location of the poles, we notice that the real pole is a factor of 20 away from the complex pair of poles. Therefore, the response of the system is *dominated* by the complex pair of poles.

$$G(s) \approx \frac{(s/2+1)}{[(s/4)^2+s/4+1]}.$$

This is now in the same form as equation (3.72) where $\alpha = 1$, $\zeta = 0.5$ and $\omega_n = 4$. Therefore, Fig. 3.24 suggests an overshoot of over 70%. The step response is the same as shown in Fig. 3.27, for $\alpha = 1$, with more than 70% overshoot and settling time of 3 seconds. The MATLAB plots below confirm this.

```
% Problem 3.41 FPE 8e

num=[1/2, 1];

den1=[1/16, 1/4, 1];

sys1=tf(num,den1);

t=0:.01:3;

y1=step(sys1,t);

den=conv([1/40, 1],den1);

sys=tf(num,den);

y=step(sys,t);

plot(t,y1,t,y);

xlabel('Time (sec)');

ylabel('y(t)');

title('Step Response');

grid on;
```



Problem 3.41: Comparison of step responses: third-order system (green), second-order approximation (blue).

2. (Effect of zeros and additional poles)

46. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \tag{1}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{2}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Solution:

(a) For $G_1(s)$:

$$\begin{aligned} Y_1(s) &= \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)}, \\ H(s) &= k\frac{\prod^j (s-z_j)}{\prod^l (s-p_l)}, \\ R_{p_i} &= \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j (s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j (p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)}. \end{aligned}$$

Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

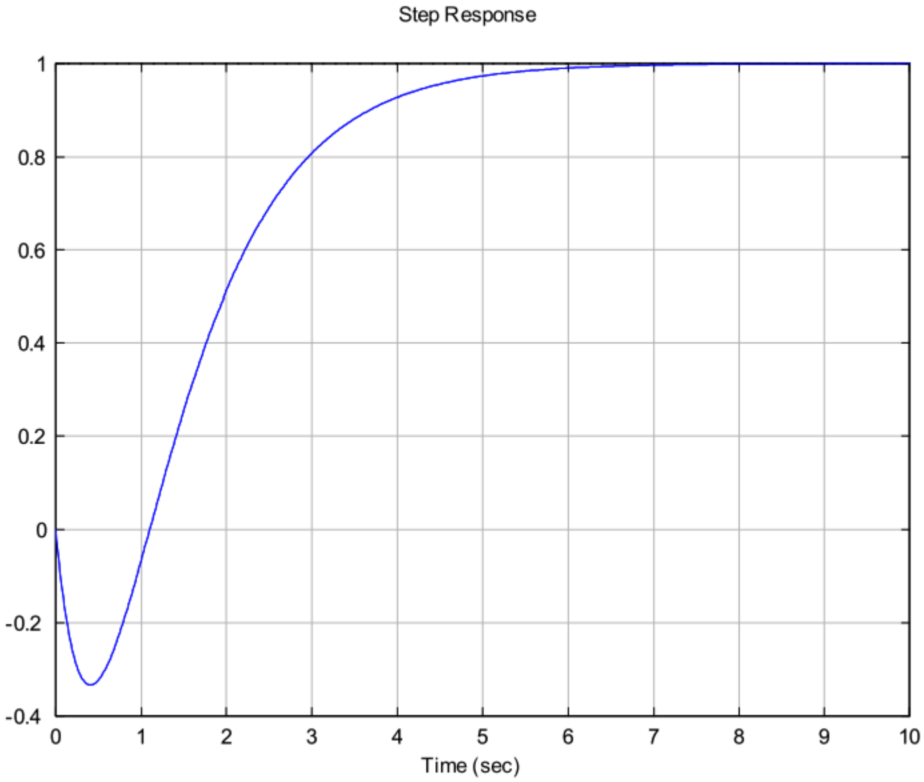
(1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i.

(2) Measure magnitude and phase of these vectors.

(3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

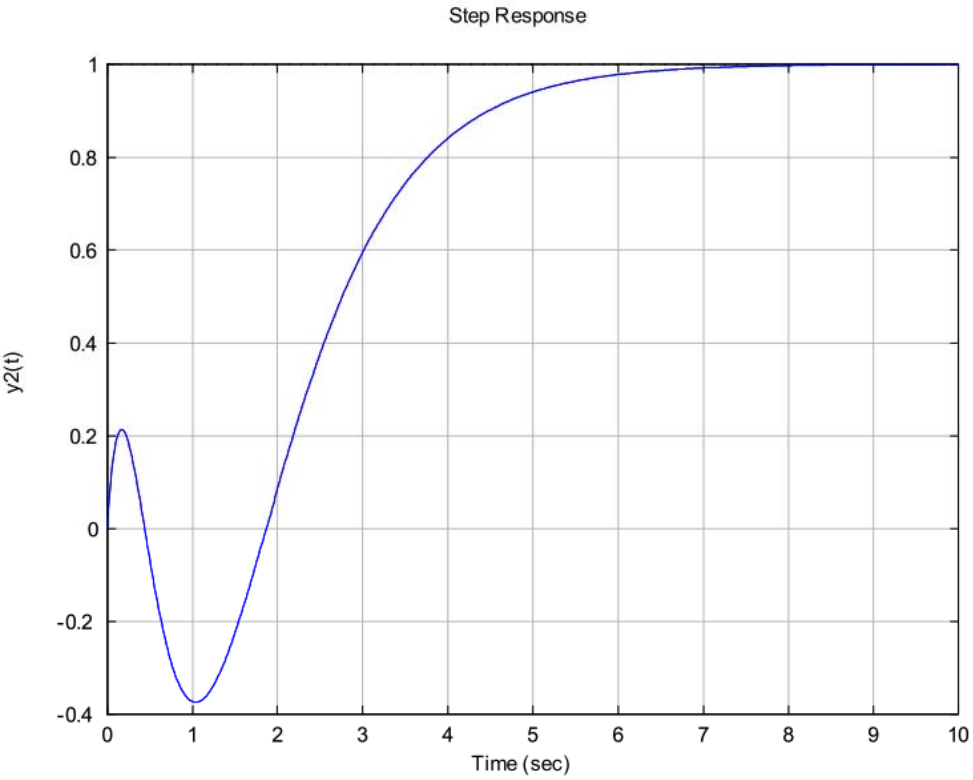
$$
\begin{aligned}
Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2}, \\
y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}.
\end{aligned}
$$



Problem 3.47: Step response for a non-minimum phase system with one *real* RHP zero.

For $G_2(s)$:

$$
\begin{aligned}
Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)}, \\
y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}.
\end{aligned}
$$



Problem 3.47: Step response of a non-minimum phase system with two *real* zeros in the RHP.

(b) The first system presents an "undershoot". The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

(c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain ($G(0) = 1$) Since the system is stable, $y(\infty) = G(0) = 1$, and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$. Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r}(1 - \frac{s}{p_i})}$$

The *numerator* terms can be classified into three types of terms:
(1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.
(2). The second group of terms are of the form $(1 + \alpha_i s)$ with $\alpha_i > 0$.
(3). Finally, the third group of terms are of the form, $(1 + \beta_i s + \alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.
However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.
All the *denominator* terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s\to\infty} s^r G(s)$$

it is seen that the *sign* of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is *odd*, then $y^r(0)$ is *negative* and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$, then we have the desired result.

References

[1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.

[2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.

[3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms, " *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

3. (Stability)

51. A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the *amplitude* of the time response to double (see Fig. 3.65), given some nonzero initial condition.

    (a) For a first-order system, show that the **time to double** is

    $$\tau_2 = \frac{\ln 2}{p},$$

    where p is the pole location in the RHP.

    (b) For a second-order system (with two complex poles in the RHP), show that

    $$\tau_2 = \frac{\ln 2}{-\zeta\omega_n}.$$

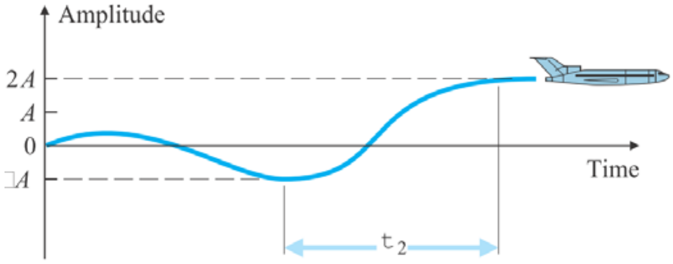


Figure 3.65: Time to double

Solution:

(a) First-order system, $H(s)$ could be:

$$\begin{aligned}
H(s) &= \frac{k}{(s-p)}, \\
h(t) &= \mathcal{L}^{-1}\left[H(s)\right] = ke^{pt}, \\
h(\tau_0) &= ke^{p\tau_0}, \\
h\left(\tau_0+\tau_2\right) &= 2h\left(\tau_0\right) = ke^{p(\tau_0+\tau_2)},
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \qquad & 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2}, \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{p}.
\end{aligned}$$

$$\begin{aligned}|t_0| &= -y_0\frac{e^{\omega_n|\zeta|t}}{\sqrt{1-|\zeta^2|}},\\ |\tau_0| &= -y_0\frac{e^{\omega_n|\zeta|\tau_0}}{\sqrt{1-|\zeta^2|}},\\ |\tau_0+\tau_2| &= -y_0\frac{e^{\omega_n|\zeta|(\tau_0+\tau_2)}}{\sqrt{1-|\zeta^2|}} = 2\,|\tau_0|\end{aligned}$$

$$\begin{aligned}\Longrightarrow &\qquad e^{\omega_n|\zeta|\tau_2} = 2\\ \Longrightarrow &\qquad \tau_2 = \frac{\ln 2}{\omega_n|\zeta|} = \frac{\ln 2}{-\omega_n\zeta} \qquad (\zeta \leq 0)\end{aligned}$$

Note: This problem shows that $\sigma = \omega_n|\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

(b) Second-order system:

$$y(t) = y_0\frac{e^{\omega_n|\zeta|t}}{\sqrt{1-|\zeta^2|}}\sin\left(\omega_n\sqrt{1-|\zeta^2|}t+\cos^{-1}\zeta\right),$$

where

$$\cos^{-1}\zeta = \cos^{-1}|\zeta| + \pi$$

$$\Longrightarrow y(t) = y_0\frac{e^{\omega_n|\zeta|t}}{\sqrt{1-|\zeta^2|}}(-1)\sin\left(\omega_n\sqrt{1-|\zeta^2|}t+\cos^{-1}|\zeta|\right)$$

Note: Instead of working with a negative ζ, everything is changed to $|\zeta|$.

$$\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1 - |\zeta^2|}}, \\
|\tau_0| &= -y_0 \frac{e^{\omega_n |\zeta| \tau_0}}{\sqrt{1 - |\zeta^2|}}, \\
|\tau_0 + \tau_2| &= -y_0 \frac{e^{\omega_n |\zeta| (\tau_0 + \tau_2)}}{\sqrt{1 - |\zeta^2|}} = 2|\tau_0|
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \qquad & e^{\omega_n |\zeta| \tau_2} = 2 \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{\omega_n |\zeta|} = \frac{\ln 2}{-\omega_n \zeta} \qquad (\zeta \leq 0)
\end{aligned}$$

Note: This problem shows that $\sigma = \omega_n|\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

4. (Stability)

55. The transfer function of a typical tape-drive system is given by

$$KG(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + KG(s) = 0$.

Solution:

$$1 + KG(s) = s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + (2+K)s + 4K = 0.$$

The Routh array is,

$$\begin{array}{lclll}
s^5 & : & 1.0 & 5.1 & 2+K \\
s^4 & : & 1.9 & 6.2 & 4K \\
s^3 & : & a_1 & a_2 & \\
s^2 & : & b_1 & 4K & \\
s^1 & : & c_1 & & \\
s^0 & : & 4K & &
\end{array}$$

where

$$\begin{aligned}
a_1 &= \frac{(1.9)(5.1)-(1)(6.2)}{1.9} = 1.837 \qquad a_2 = \frac{(1.9)(2+K)-(1)(4K)}{1.9} = 2 - 1.1K \\
b_1 &= \frac{(a_1)(6.2)-(a_2)(1.9)}{a_1} = 1.138(K+3.63) \\
c_1 &= \frac{(b_1)(a_2)-(4K)(a_1)}{b_1} = \frac{-(1.25K^2+9.61K-8.26)}{1.138(K+363)} = \frac{-(K+8.47)(K-0.78)}{0.91(K+3.63)}
\end{aligned}$$

For stability we must have all the elements in the first column of the routh array to be positive, and that results in the following set of constraints:

$$\begin{array}{clcl}
(1) & b_1 & = & K + 3.63 > 0 \Longrightarrow K > -3.63, \\
(2) & c_1 & > & 0 \Longrightarrow -8.43 < K < 0.78, \\
(3) & d_1 & > & 0 \Longrightarrow K > 0.
\end{array}$$

Intersection of (1), (2), and (3) $\Longrightarrow 0 < K < 0.78$.

參考觀摩的作業

1. (Effect of Zeros and Additional Poles)

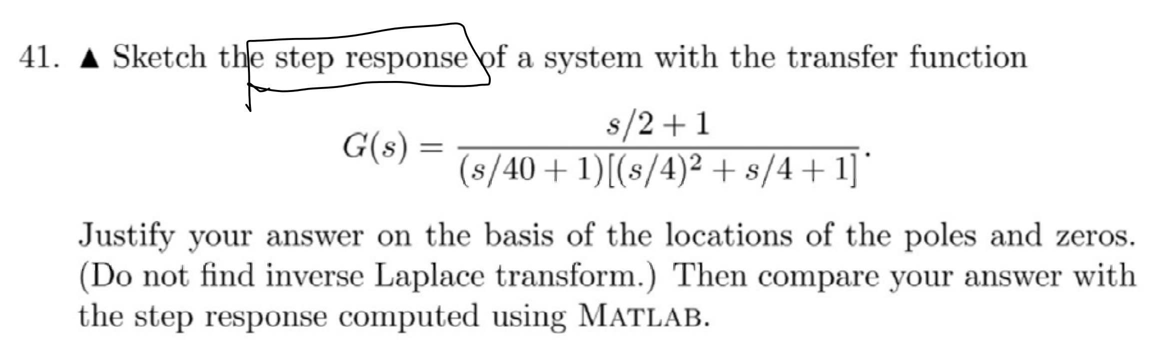
作者：b08611031，易峻葦

理由：討論不同 pole 位置對整個三階系統的影響並畫出 matlab 步階響應討論

作者：b10202032，卓 然

理由：詳細討論不同 pole 對三階系統步階響應的影響以及 overshoot 的關係

B08611031 易峻葦 生機四



41. ▲ Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

$\because$ pole of $(s/40+1) << [(s/4)^2+s/4+1]$

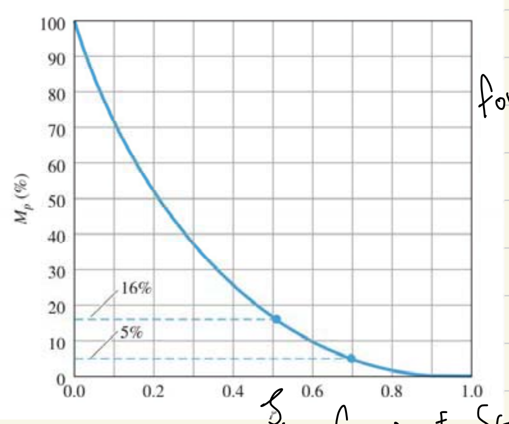
$\therefore G(s) \simeq \frac{s/2+1}{[(s/4)^2+s/4+1]}$

$\omega_n = 4$ $\alpha = 1$

$\zeta = 0.5$

pole $= -2 \pm 2\sqrt{3}j$

zero: -2



ζ for input= S(t)

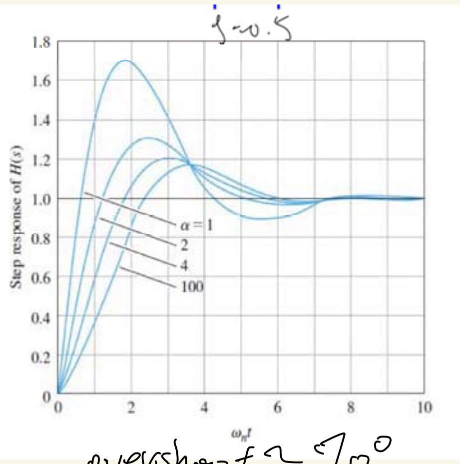
$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

for error= ±1%

$t_s = \frac{4.6}{0.5 \times 4}$

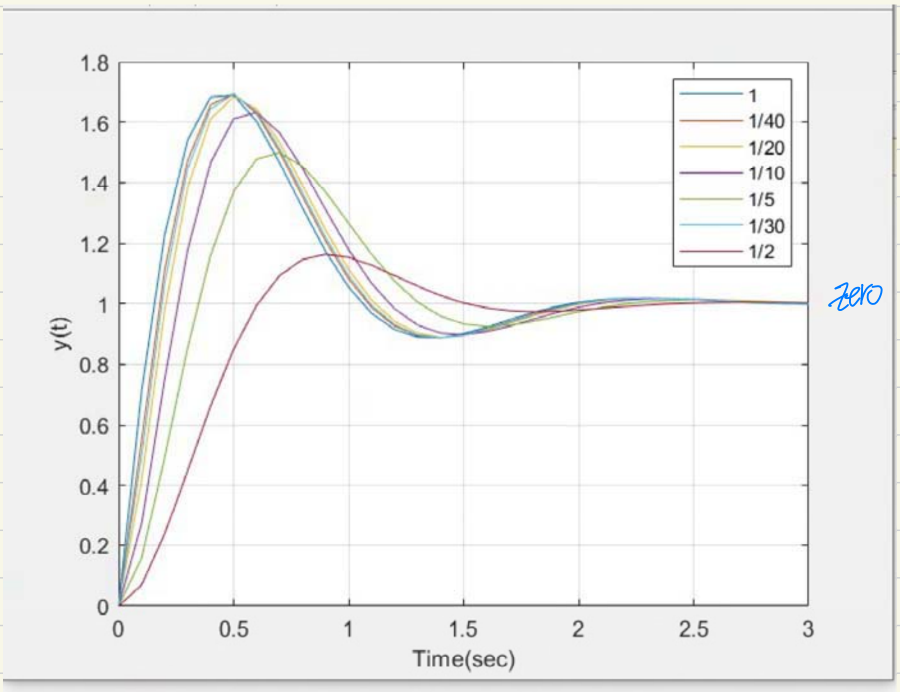
$\simeq 2.3s$

$\zeta = 0.5$



overshoot ≃ 70°

Plot:



zero

除了題意外，還增加了 pole=-30 -20,-10,-5 時的 Unit step response 可以發現，當 pole 越來越接近 -2 時，與 $\frac{s/2+1}{(s+4)^2+(s/4)+1}$ (legend 中為1者) 的 response 相差越來越大，也就是說，該 pole 對整個系統的影響力越來越大，故不能忽略，而當 pole=-2 時，上下分子分母抵消，系統變成典型的簡單二階系統，最大超越量與 Fig 3.24 相符≃16%

Control Systems HW
B10202032 物理二 卓然

1. (Effect of zeros and additional poles)

41. ▲ Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

The method in the reference answer:

Since $\frac{1}{40} \ll \frac{1}{4}, \frac{1}{2}, \cdots$, we can simply neglect the $(\frac{1}{40}s+1)$ term in the denominator, and

$G(s) \approx \frac{\frac{s}{2}+1}{[(\frac{s}{4})^2+\frac{1}{4}s+1]}$. Comparing with $\frac{\frac{s}{\alpha\omega_n\zeta}+1}{(\frac{s}{\omega_n})^2+2\zeta(\frac{s}{\omega_n})+1}$,

we get $\begin{cases} 2=\alpha\omega_n\zeta \\ \omega_n=4 \\ 2\frac{\zeta}{\omega_n}=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} \alpha=1 \\ \omega=4 \\ \zeta=\frac{1}{2} \end{cases}$.

With the figures I'll show below, the time domain functions before and after the second-order approximation are indeed "similar to" each other.

But, as a student in the Department of Physics (?), it is not that comfortable for me to see that one just ignores the effect of the $(\frac{s}{40}+1)$ term [Though we also do things like $\sin\theta = \theta$ when θ is small and $(x+dx)^2 = x^2$

= 2xdx , so perhaps I should be fine with this approximation :)], and I wonder how much we have to "pay" for this approximation (i.e. the error caused by this) Therefore, I do some analyses on this, using Matlab.

I consider the conditions of $G_a(s) \equiv \dfrac{(\frac{s}{2}+1)}{(\frac{s}{a}+1)((\frac{s}{4})^2+\frac{s}{4}+1)}$ for different values of a : $a=1,2,3,\ldots,40$ and compare it with the second-order approximation $\tilde{G}(s) \equiv \dfrac{(\frac{s}{2}+1)}{((\frac{s}{4})^2+\frac{s}{4}+1)}$.

When numerically analyzing the error cause by the approximation, I consider the overshoot ratio M_p. Some results are listed below.

1. From fig. 1 we can see that the step response of the system gets closer to the approximation $\tilde{G}(s)$ for larger value of a. This is reasonable since for larger a, the third pole $s=-a$ is much farther from the center of the s-plane and thus causes less difference for the system.

2. For smaller a's, the difference between $G_a(s)$ and $\tilde{G}(s)$ is much bigger and not negligible, and therefore the approximation fails. For the case $a=1$, $G_1(s) = \dfrac{\frac{s}{2}+1}{(s+1)((\frac{s}{4})^2+\frac{s}{4}+1)}$.

Its time domain response doesn't even have a maximum value (or say, an "overshoot"), and from this we can see that for a small enough, the third order system is not even similar to the second order one. It seems to be an interesting topic to discuss the properties of the 3- order system.

3. Numerically, the overshoot M_p of the "real" system the original problem states, $G_{40}(s) = \frac{\frac{s}{5}+1}{(\frac{s}{40})((\frac{s}{4})^2+\frac{s}{4}+1)}$, and that of the approximated $\tilde{G}(s)$ has an error of $\frac{0.6993-0.6956}{0.6956} = 0.0053 = 0.53\%$, which seems to be acceptable for most real-life cases.

4. Looking at the programming results, to make its approximation $\tilde{G}(s)$'s overshoot be within 1% error,

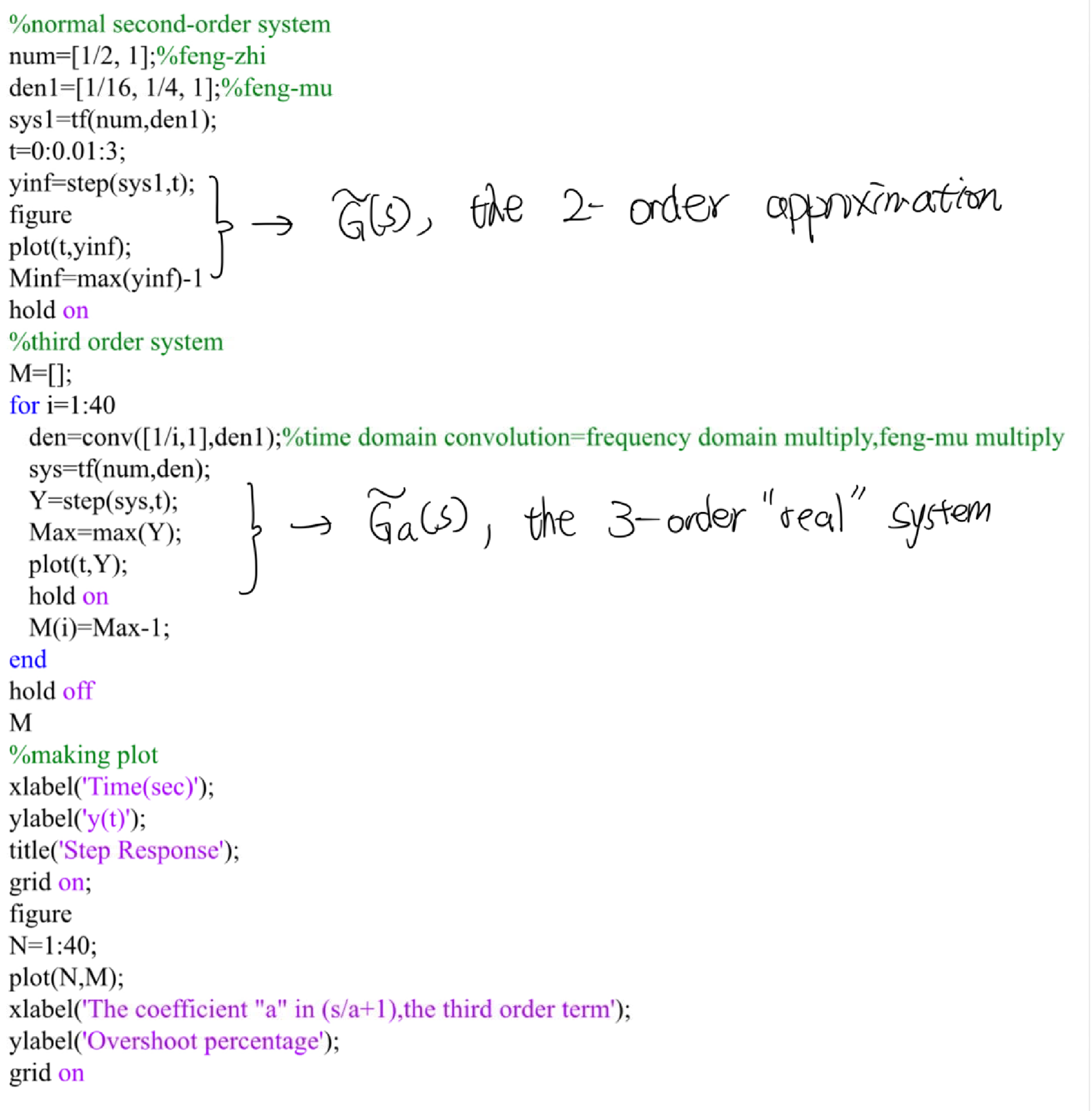
   the largest value of a we can take is $a=30$.

   That is, for $G_{29}(s)$, $G_{28}(s)$, …, $\tilde{G}(s)$ will cause an error $\geq 1\%$ in terms of the overshoot M_p.

5. It's obvious that the "rise time" t_r decreases with larger a, but the rise time of $\tilde{G}(s)$ is the shortest. Unfortunately, I have no enough time

to do analysis on the error in terms of t_r, but this seems to be another interesting topic

The full Matlab Code

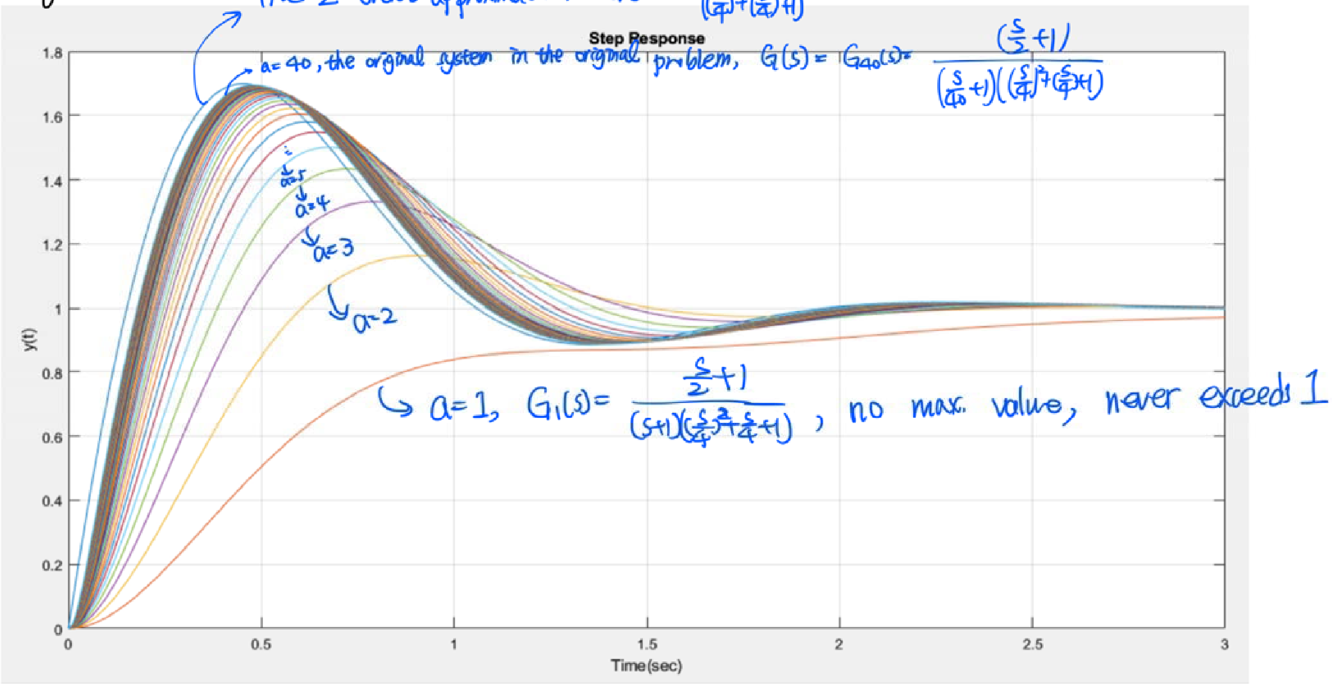


```
%normal second-order system
num=[1/2, 1];%feng-zhi
den1=[1/16, 1/4, 1];%feng-mu
sys1=tf(num,den1);
t=0:0.01:3;
yinf=step(sys1,t);
figure
plot(t,yinf);
Minf=max(yinf)-1
hold on
%third order system
M=[];
for i=1:40
  den=conv([1/i,1],den1);%time domain convolution=frequency domain multiply,feng-mu multiply
  sys=tf(num,den);
  Y=step(sys,t);
  Max=max(Y);
  plot(t,Y);
  hold on
  M(i)=Max-1;
end
hold off
M
%making plot
xlabel('Time(sec)');
ylabel('y(t)');
title('Step Response');
grid on;
figure
N=1:40;
plot(N,M);
xlabel('The coefficient "a" in (s/a+1),the third order term');
ylabel('Overshoot percentage');
grid on
```

→ $\tilde{G}(s)$, the 2-order approximation

→ $\tilde{G}_a(s)$, the 3-order "real" system

Figures



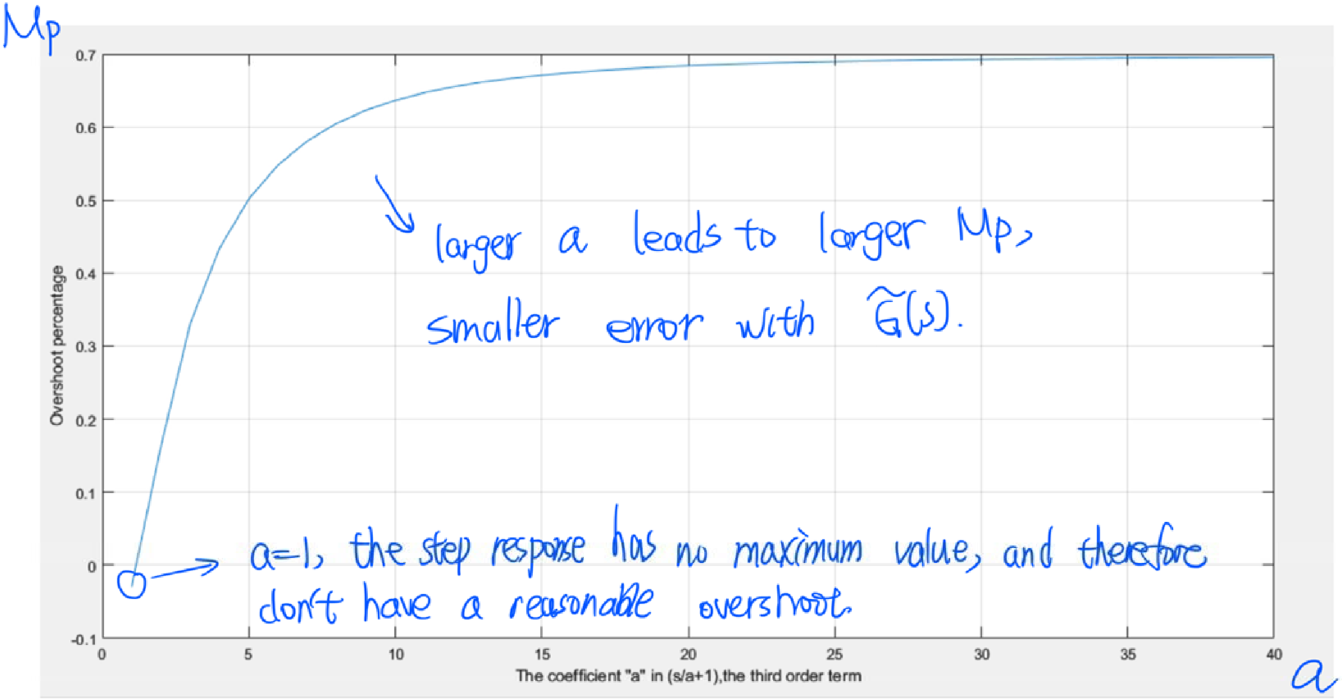
The 2-order approximation, $\tilde{G}(s) = \frac{(\frac{s}{2}+1)}{((\frac{s}{4})^2+(\frac{s}{4})+1)}$

a=40, the original system in the original problem, $G(s) = G_{40}(s) = \frac{(\frac{s}{2}+1)}{(\frac{s}{40}+1)((\frac{s}{4})^2+(\frac{s}{4})+1)}$

a=1, $G_1(s) = \frac{\frac{s}{2}+1}{(s+1)((\frac{s}{4})^2+\frac{s}{4}+1)}$, no max. value, never exceeds 1

- Step responses v.s. time.

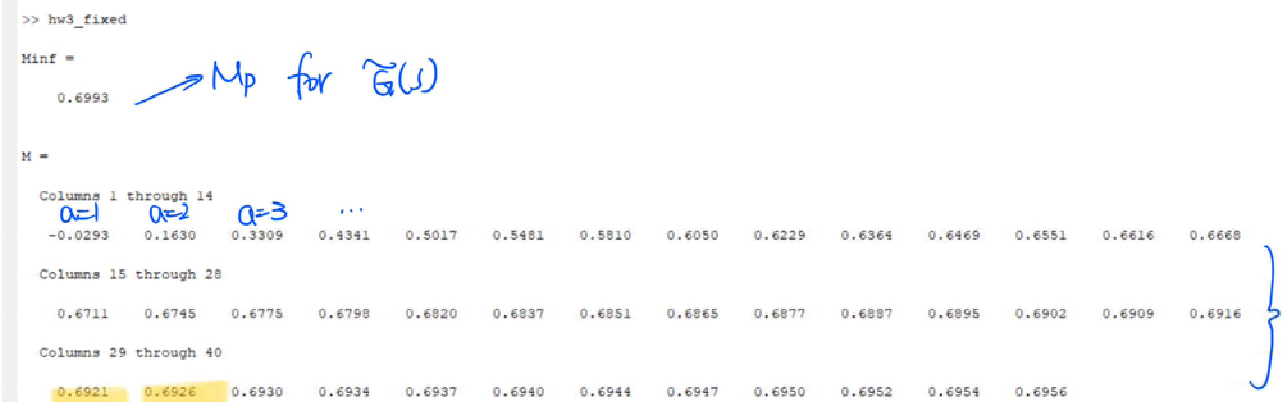
Mp



larger a leads to larger Mp, smaller error with $\tilde{G}(s)$.

a=1, the step response has no maximum value, and therefore don't have a reasonable overshoot.

- a versus Mp. Note that the overshoot Mp for the 2-order approximation is 0.6993, which is shown in the next figure. bigger than any Mp shown on this plot



```
>> hw3_fixed

Minf =

    0.6993


M =

  Columns 1 through 14

   -0.0293    0.1630    0.3309    0.4341    0.5017    0.5481    0.5810    0.6050    0.6229    0.6364    0.6469    0.6551    0.6616    0.6668

  Columns 15 through 28

    0.6711    0.6745    0.6775    0.6798    0.6820    0.6837    0.6851    0.6865    0.6877    0.6887    0.6895    0.6902    0.6909    0.6916

  Columns 29 through 40

    0.6921    0.6926    0.6930    0.6934    0.6937    0.6940    0.6944    0.6947    0.6950    0.6952    0.6954    0.6956
```

→ Mp for $\tilde{G}(s)$

a=1 a=2 a=3 ...

} Mp for Ga(s)

a=29 a=30 a=40

↪ error= $\frac{0.6993-0.6926}{0.6926}$ $=0.97\% < 1\%$

error= $\frac{0.6993-0.6921}{0.6921}$ $= 1.04\% > 1\%$

參考觀摩的作業

2. (Effect of Zeros and Additional Poles)

作者：b08901085，施彥宇

理由：詳細討論 c 小題 undershoot 的定義以及不同形式的 undershoot

作者：b09502033，朱本毅

理由：詳細討論步階響應的暫態，討論不同 zero 對系統的影響

HW03 – Unit 3, Dynamic Response

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- **Question :**

Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}.$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}.$$

(a). Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, playing close attention to the transient part of the response.

(b). Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c). Consider a stable, strictly proper system (that is, m zeros and n poles, when $m < n$.) Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an odd number of real RHP zeros.

- **Solution for part (a). :**

For $G_1(s) : Y_1(s) = \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)}$ (1).

Thus, the transfer function can be rewritten as : $H(s) = k\frac{\prod^j(s-z_j)}{\prod^l(s-p_l)}$ (2).

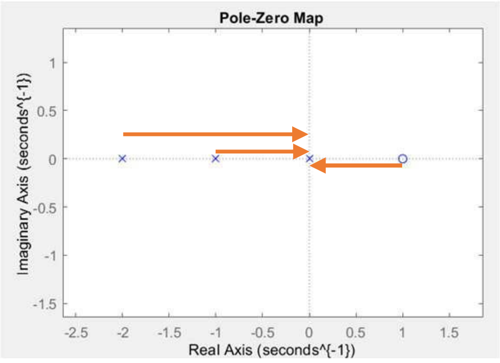
To calculate the step response in time domain, we can translate $Y(s)$ into :

$$Y_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{s+1} + \frac{R_{-2}}{s+2} \quad (3).$$

using partial fraction, which the residues are defined as :

$$R_i = \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j(s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j(p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)} \quad (4).$$

Explaining equation (4). in words is that the residues can be calculated by multiplying the gain and the product of the vectors coming from the zeros and then dividing by the product of the vectors coming from the poles. Thus, the residues are :



$$R_0 = -2 \times \frac{-1}{1\times 2} = 1 \quad (5).$$

$$R_{-1} = -2 \times \frac{-4}{-1\times 1} = -4 \quad (6).$$

$$R_{-2} = -2 \times \frac{-3}{-1\times -2} = 3 \quad (7).$$

Using equation (3). to (7). , we can get the step response in time domain to be :

$$y(t) = 1 - 4e^{-t} + 3e^{-2} \quad (8).$$

and the figure of step response drawn by MATLAB is :

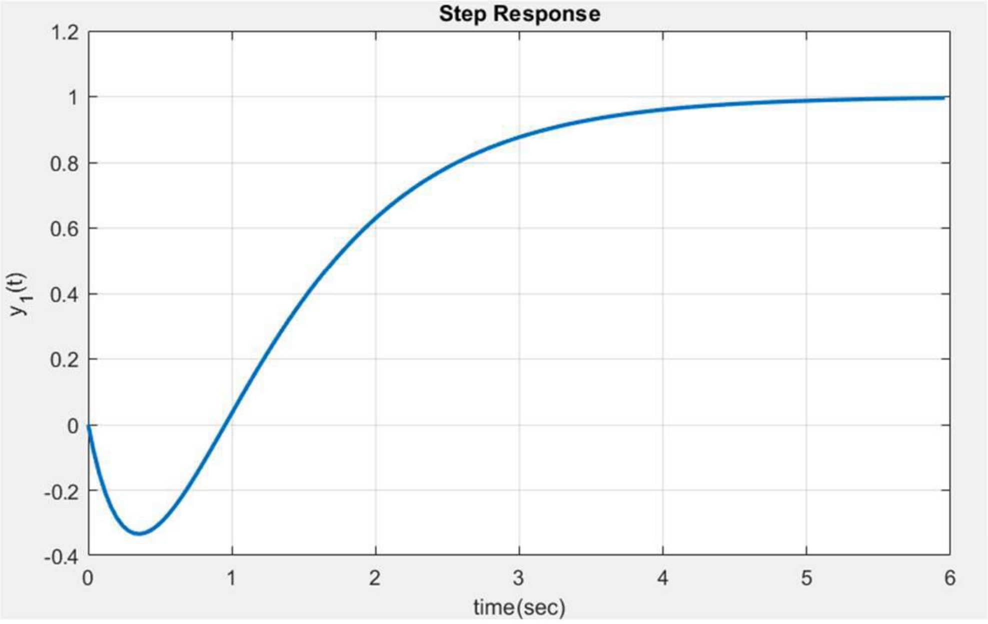


Figure 1. Step Response for $y_1(t)$

Likewise, running the same process to $G_2(s)$, we can get the partial fraction of $Y_2(s)$ to be :

$$Y_2(s) = \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{s+1} + \frac{18}{s+2} + \frac{-10}{s+3} \quad (9).$$

and the step response $y_2(t)$ in time domain to be :

$$y_2(t) = 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t} \quad (10).$$

and the figure of step response drawn by MATLAB is shown below :

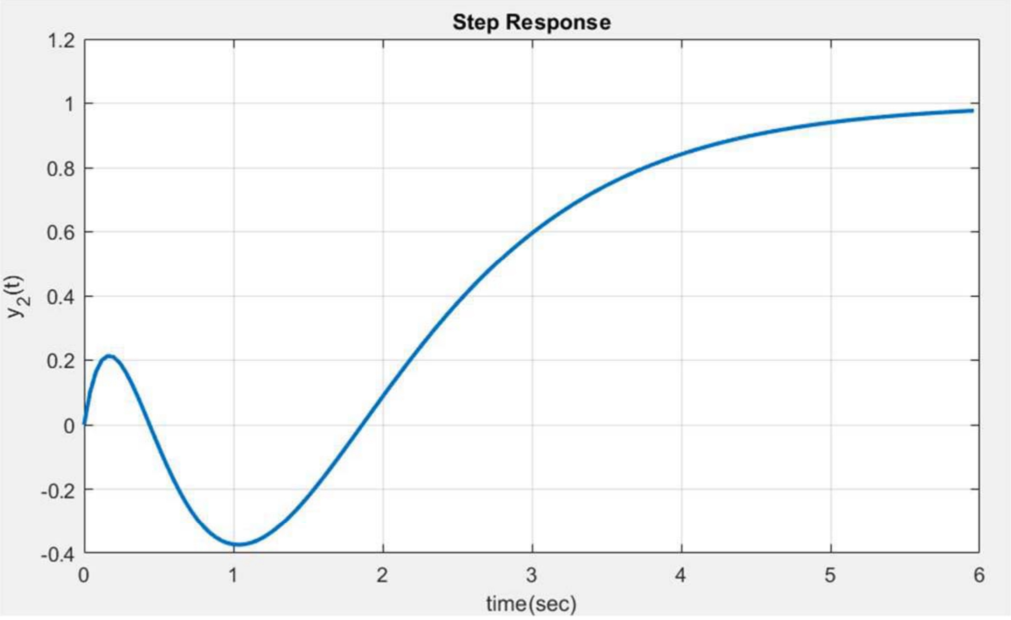


Figure 2. Step Response for $y_2(t)$

- **Solution for part (b). :**

The first system presents an “undershoot”. The second system, on the other hand, starts off in the right direction.

In $y_1(t)$, dominant at $t = 0$ is the term $-4e^{-t}$.

In $y_2(t)$, dominant at $t = 0$ is the term $18e^{-2t}$.

The reasons for this initial behavior of the step response will be analyzed in the next part.

- **Solution for part (c). :**

The proof is found from Ref[1]. Then by stability, the limitation $\lim_{t\to\infty} y(t) = H(0) \neq 0$. Let r denote the relative degree of $H(s)$. Then $y(t)$ and its first $(r-1)$ derivatives are zero at $t = 0$, and $y^{(r)}(0)$ is the first nonzero derivative.

In common words, the step response experiences an “undershoot” when it starts off in the wrong direction, initially. And the precise definition of undershoot written in Ref[1] is “If the steady-state value of the system has a sign opposite from that of its first nonzero derivative at time $t = 0$.” and it can be performed as :

$$y^{(r)}(0)y(\infty) < 0 \qquad (11).$$

Proposition : The system exhibits undershoot if and only if the transfer function has an odd number of real RHP zeros.

We can assume that $p(0) = 1$ without losing generality. As for $y^{(r)}(0)$, from initial value theorem can we get that

$$y^{(r)}(0) = \lim_{s\to\infty} s^r H(s) \qquad (12).$$

and the transfer function can be rewriting as :

$$H(s) = \frac{\prod_{i=1}^{n}(1-s/z_i)}{\prod_{i=1}^{n+r}(1-s/p_i)} \qquad (13).$$

The numerator can be grouped into three types :

1. The positive zeros : $1 - \alpha_i s\ ,\ \alpha_i > 0$.
2. The negative zeros : $1 + \alpha_i s\ ,\ \alpha_i > 0$.
3. The complex zeros : $1 + \beta_i s + \alpha_i s^2\ ,\ \alpha_i > 0$ and $\beta_i^2 < 4\alpha_i$.

And all the denominators are in type 2. and 3.

From equation (12). , the sign of $y^{(r)}(0)$ is determined solely by the number of type 3. As a result, if their number is odd, then $y^{(r)}(0)$ is negative, whereas if their number is even, then $y^{(r)}(0)$ is positive.

Since $y(\infty) = H(0) = 1$, then we have the desired result.

- **What we can do more :**

(1). It is said that from initial value theorem, we can get equation (12). And the derivation of this equation can start from :

$$y(t^+) = \lim_{s\to\infty}[sY(s)] \qquad (14).$$

$$L[y(t)] = \int_{t=0^-}^{\infty} e^{-st}f(t)dt \qquad (15).$$

Combining equation (14). and (15)., the transform of first order derivative is :

$$L\left[\frac{d}{dt}y(t)\right] = \int_{t=0^-}^{\infty} e^{-st}y'(t)dt = \int_{t=0^-}^{t=0^+} e^{-st}y'(t)dt + \int_{t=0^+}^{\infty} e^{-st}y'(t)dt$$

$$= [y(t)]_{0^-}^{0^+} + \int_{t=0^+}^{\infty} e^{-st}y'(t)dt = sY(s) - y(0^-) \qquad (16).$$

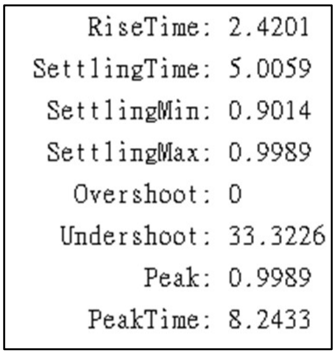
$$L\left[\frac{d^n}{dt^n}y(t)\right] = s^nY(s) - s^{n-1}y(0^-) - s^{n-2}y'^{(0^-)} - \cdots - y^{(n-1)}(0^-) \quad (17).$$

Recalling that in equation (12)., $y^{(r)}(0)$ is the first nonzero term of the derivatives of $y(t)$. And taking the limitation of equation (17).

$$\lim_{s\to\infty} L\left[\frac{d^r}{dt^r}y(t)\right] = y(0^+) - y(0^-) = \lim_{s\to\infty}[s^rY(s)] - y^{(r)}(0) = 0$$

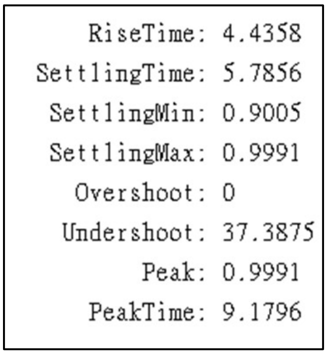
$$\Rightarrow y^{(r)}(0) = \lim_{s\to\infty} s^rY(s) = \lim_{s\to\infty} s^rH(s) \qquad (18).$$

(2). By running MATLAB, I found that the step responses of both transfer functions has undershoot. The results are shown below:



```
     RiseTime: 2.4201
 SettlingTime: 5.0059
  SettlingMin: 0.9014
  SettlingMax: 0.9989
    Overshoot: 0
   Undershoot: 33.3226
         Peak: 0.9989
     PeakTime: 8.2433
```

Fig 3. Stepinfo of $G_1(s)$



```
     RiseTime: 4.4358
 SettlingTime: 5.7856
  SettlingMin: 0.9005
  SettlingMax: 0.9991
    Overshoot: 0
   Undershoot: 37.3875
         Peak: 0.9991
     PeakTime: 9.1796
```

Fig 4. Stepinfo of $G_2(s)$

I was confused at first, so I decided to calculate the conclusion from Ref[1] by myself.

The relative degrees, $r_1$and r_2, of both $G_1(s)$ and $G_2(s)$ are 1.

$$\Rightarrow \quad y_1'(0)y_1(\infty) = (4e^{-t} - 6e^{-2t})|_{t=0} = -2 < 0 \qquad (19).$$

$$y_2'(0)y_2(\infty) = (9e^{-t} - 36e^{-2t} + 30e^{-3t})|_{t=0} = 3 > 0 \quad (20).$$

From (19) and (20), the conclusion seems to be wrong! However, the condition of $G_2(s)$ is considered in Ref[2].

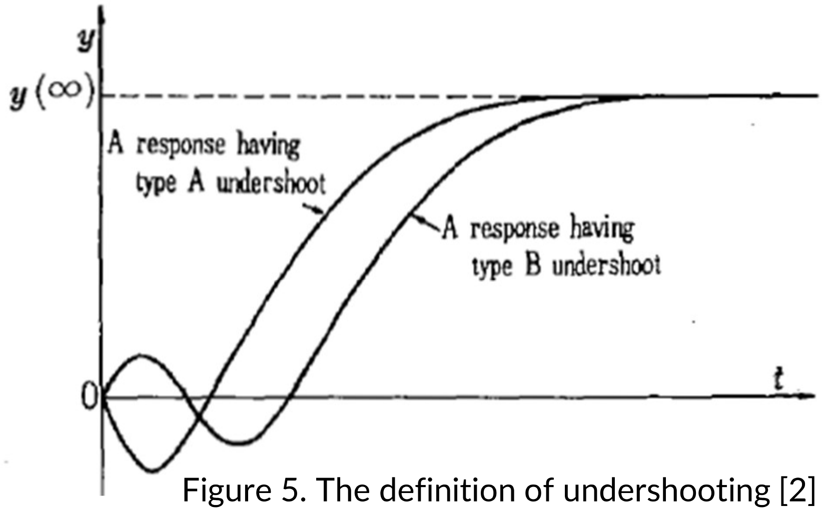


Figure 5. The definition of undershooting [2]

In Ref[2], the undershooting is classified into two different types, which are illustrated in the above figure. The conditions of causing two kinds of undershooting are different and are discussed in the following part.

The conditions of causing undershooting mentioned in Ref[2] :

a. There's a finite positive integer η that satisfies

$$y(0^+) = \cdots = y^{(\eta-1)}(0^+) = 0, \text{ but } y^{(\eta)}(0) \neq 0.$$

b. The steady state value $y(\infty) = \lim_{t\to\infty} y(t) \neq 0$.

c. $y^{(\eta)}(0^+)y(\infty) < 0$.

d. $y(t)y(\infty) < 0$ for each t belongs to an open interval (a, b) if condition c. is not true.

To form a type A undershooting, which is the form of $G_1(s)$, conditions a, b, and c need to be satisfied. This is the conclusion made by Ref[1]. We cam observe that the wrong starting position makes it have an undershoot.

However, to form a type B undershooting, which is the form of $G_2(s)$, conditions a, b, and d must be satisfied. After calculating, the interval of which $y_2(t) < 0$ is $(0.44, 1.84)$.

And if there's no undershooting, condition c and d will fail while condition a and b hold. Another situation is there's no finite positive integer η that satisfies the condition a., or $y(\infty) = 0$.

- **Reference**

[1] "On Undershoot and Nonminimum Phase Zeros", Vidyasagar, May 1986.

[2] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms," pp. 402-407, 1981

[3] https://reurl.cc/kExoLr

[4] https://reurl.cc/KQ79rn

Problem 1

2. (Effect of zeros and additional poles)

46. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \tag{1}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{2}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Answer 1

(a)

先利用 matlab 將 G1(s)和 G2(s)的 pole 和 zero 畫出來。

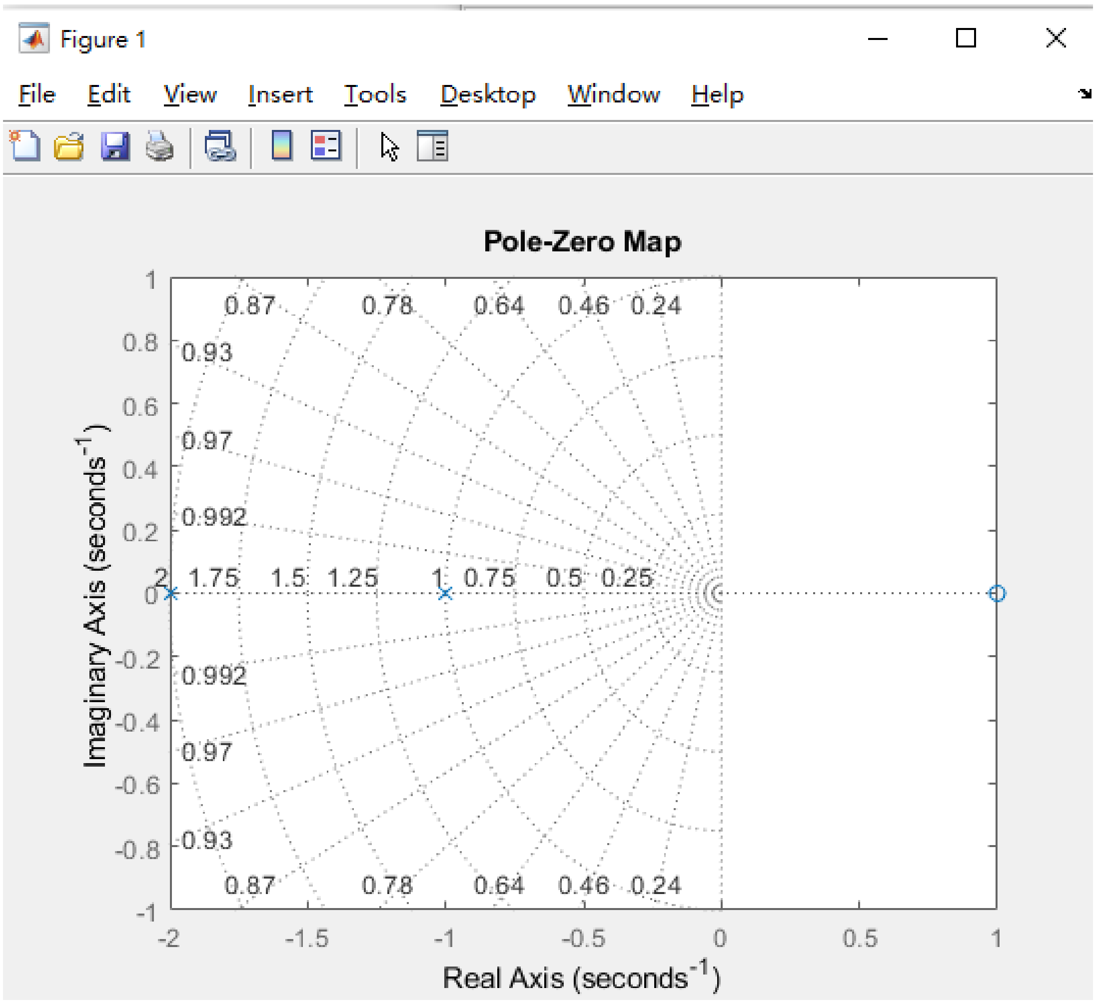


Figure 1-1：Pole and zero map for G1(s)

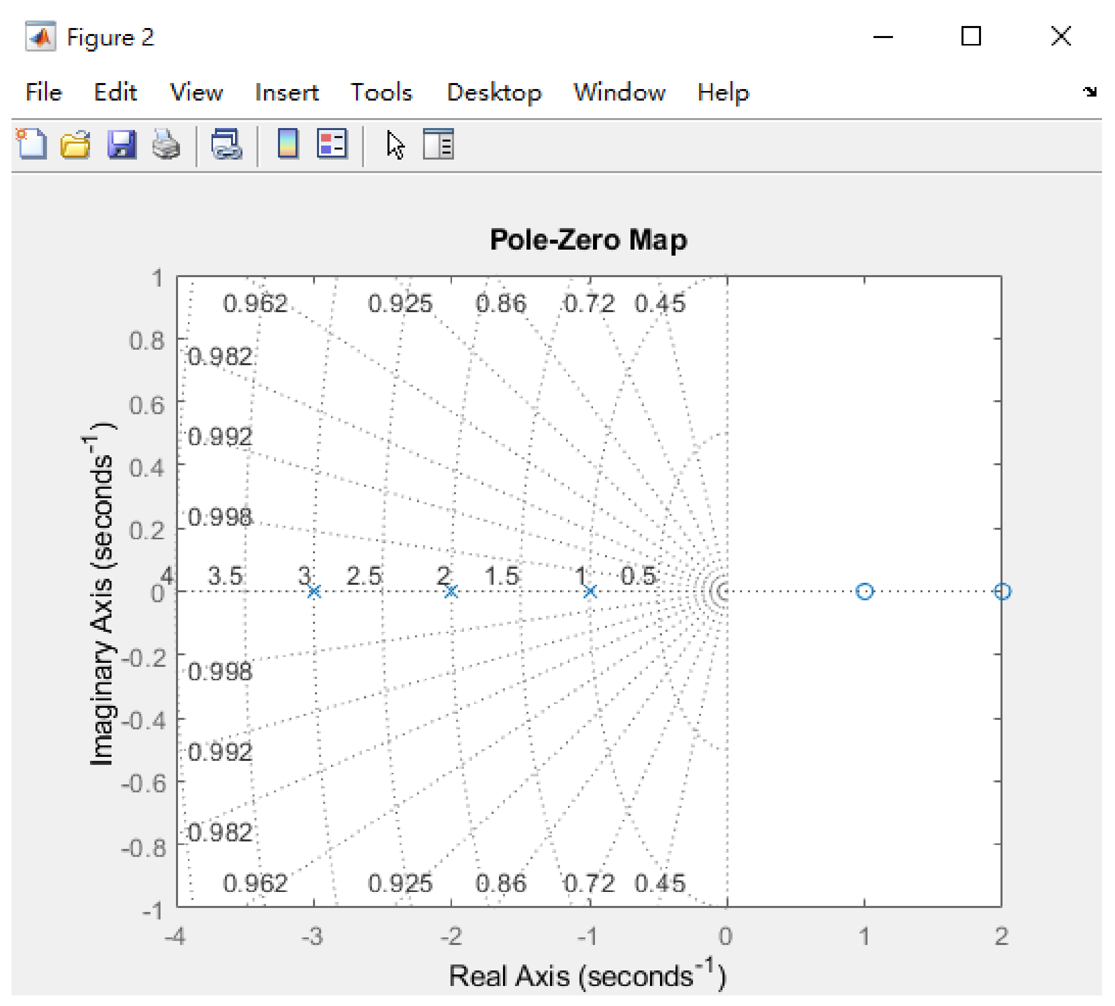


Figure 1-2： Pole and zero map for G2(s)

整理的結果如下

	poles	zeros
G1(s)	-1、-2	1
G2(s)	-1、-2、-3	1、2

而他們的 Step Response 結果如下。

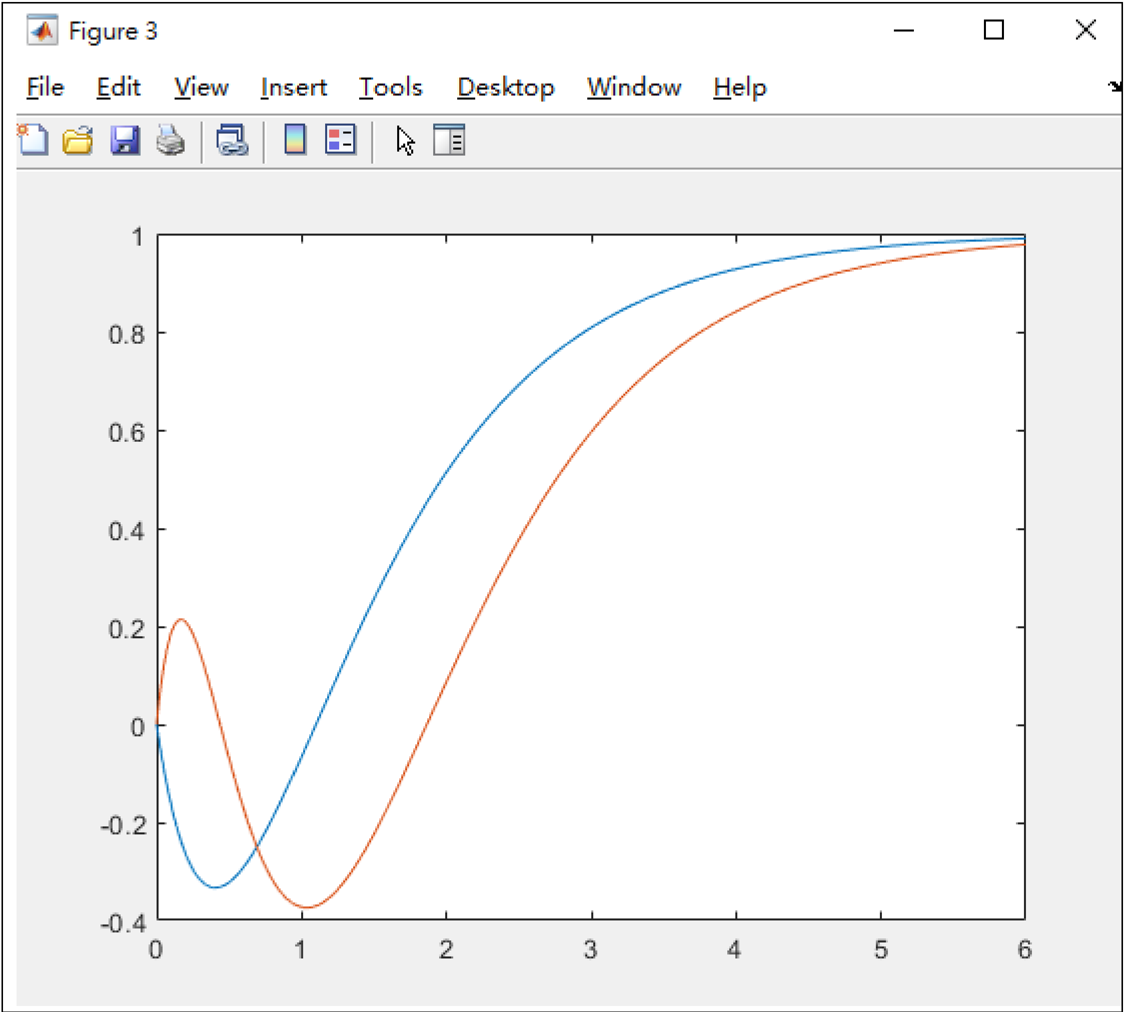


Figure 1-3：Step Response for G1(blue) and G2(red)

(b)

在 G1(s)和 G2(s)中可以明顯看到，由於 zeros 在右半平面、沒有距離 poles 太遠，也沒有與 poles 完全相等(沒有 zero-pole cancelation)，因此會帶來暫態響應。

以 G1(s)來說，zero 在右半平面，$\omega_n = \sqrt{2},\ \zeta = \frac{3}{2\sqrt{2}}$,$a = \frac{4}{3}$。若假設

$$\mathrm{G1(s) * u(t)} = -2\frac{(\mathrm{s}-1)}{s(\mathrm{s}+1)(\mathrm{s}+2)} = \frac{\mathrm{A}}{\mathrm{s}} + \frac{\mathrm{B}}{(\mathrm{s}+1)} + \frac{\mathrm{C}}{(\mathrm{s}+2)}$$

u(t)代表 unit step function，在 s-domain 為 1/s

可解得 A=1、B=-4、C=3，再利用 laplace transform 的幾個常用關係，可以得到 G1(s)的 step response 在數學上的表示：

$$g_1(t) = 1 - 4e^{-1t} + 3e^{-2t} \tag{1}$$

對(1)式做一次微分可以得到：

$$\frac{d}{dt}g_1(t) = 4e^{-1t} - 6e^{-2t} \tag{2}$$

在 t=0 時，斜率為-2，t=1 時，斜率為 0.66 左右，這可以說明為什麼有了這個 zero 的後，一開始 step response 會被往下拉，到了大概一秒左右又開始上升，之後達到穩態。

對 G2(s)做一樣的分析，由於它是三階的函式，我就先不管它的 natural frequency 和 damping ratio 分別是多少，先假設

$$\mathrm{G2(s) * u(t)} = 3\frac{(\mathrm{s}-1)(\mathrm{s}-2)}{(\mathrm{s}+1)(\mathrm{s}+2)(\mathrm{s}+3)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)} + \frac{D}{(s+3)}$$

$$\begin{gathered}\mathrm{A}s^3 + 6As^2 + 11As + 6A \\ +(\mathrm{B}s^3 + 5Bs^2 + 6Bs) \\ +(\mathrm{C}s^3 + 4Cs^2 + 3Cs) \\ +(\mathrm{D}s^3 + 3Ds^2 + 2Ds) = 3\end{gathered}$$

可解得 A=1、B=-9、C=18、D=-10，G2(s)的 step response 為：

$$g_2(t) = 1 - 9e^{-1t} + 18e^{-2t} - 10e^{-3t} \tag{3}$$

做一次微分可以得到：

$$\frac{d}{dt}g_2(t) = 9e^{-1t} - 36e^{-2t} + 30e^{-3t} \tag{4}$$

在 t=0 時，斜率為 3，t=0.5 時，斜率約為-1.1，.t=2 時，斜率為 0.63，因此可以看到它的 step response 會上升、下降、再度上升。

(c)

如果是一個穩定的系統，$\lim_{t\to\infty} y(t)$應該會趨近於一個定值，並且根據 final value theorem，這個值會等於 Y(0)。

此外，當代入 t=0 時，在前(m-n-1)次微分都會得到 0，能得到的第一個非零項會在微分(m-n)次的時候。

不失一般性，假設 y(0)=1，利用 initial value theorem，可以知道

$$y(0) = \lim_{s\to\infty} Y(s) \tag{5}$$

$$y^{\{m-n\}}(0) = \lim_{s\to\infty} s^{m-n}\,Y(s) \tag{6}$$

此外，一個 Transfer Function 的標準形式可以表示為：

$$Y(s) = \frac{\prod_{i=1}^{n}\left(1-\frac{s}{z_i}\right)}{\prod_{i=1}^{m}\left(1-\frac{s}{p_i}\right)} \tag{7}$$

這個級數的每一項 zero 或 pole 可能有以下三種形式

1. 為正，$1 - \alpha_i s = 0$，$\alpha_i > 0$
2. 為負，$1 + \alpha_i s = 0$，$\alpha_i > 0$
3. 為複數，$1 + \beta_i s + \alpha_i s^2 = 0$，$\alpha_i > 0$，$\beta_i^2 - 4\alpha_i < 0$

由於這是一個 stable system，Y(s)的每個分母都應該是第二或第三種的情況，至於分子的部分，每多一個正的 zero，在微分時就能使$y^{\{m-n\}}$多乘上一個負號，因此當 Y(s)有奇數個在 RHP 的 zero 時，會出現 undershoot 的情況。

(4)

針對 G2(s)，我試著調整了兩個 zero 的位置，和上課提到的概念做對照

	G2(s)	Test1	Test2	Test3	Tes4	Test5	Test6	Test7
Zero1	1	-1	2	1	1.5	1	3	10
Zero2	2	-2	3	3	2.5	1	3	11

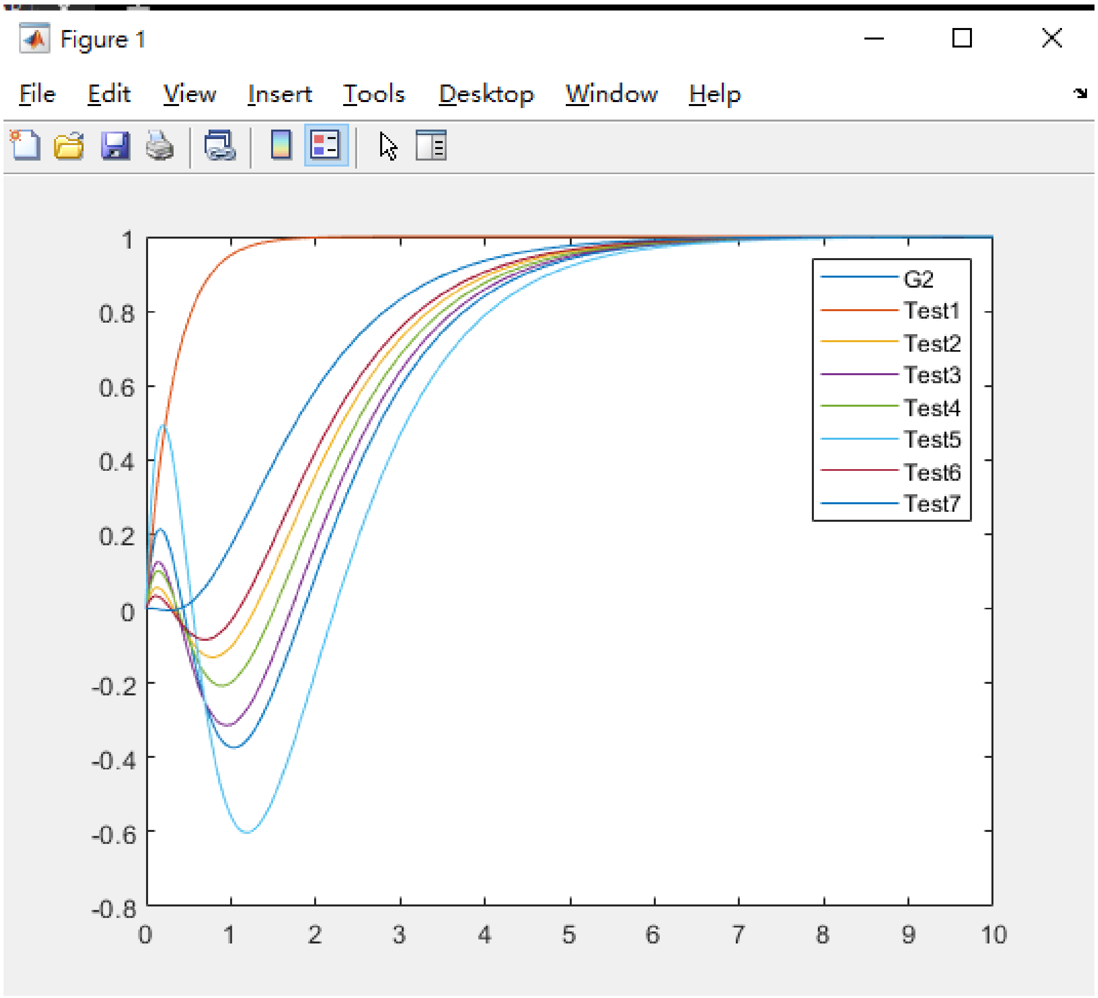


Figure 1-4： Step Response

Test1：Pole-Zero Cancelation，變回一階響應

Test2：影響力第四

Test3：影響力其次

Test4：影響力第三

Test5：兩個 zero 離 dominant pole 最近，影響最明顯

Test6：影響力第五

Test7：zero 距離太遠，影響十分微弱

參考觀摩的作業

3. (Stability)

無

參考觀摩的作業

4. (Stability)

作者：b08901138，王昱淇

理由：改寫轉移函數並討論不同 loop gain 的步階響應

作者：b09901152，施文崴

理由：使用 matlab 畫出系統步階響應在不同穩定 loop gain 下的情況

Control Systems

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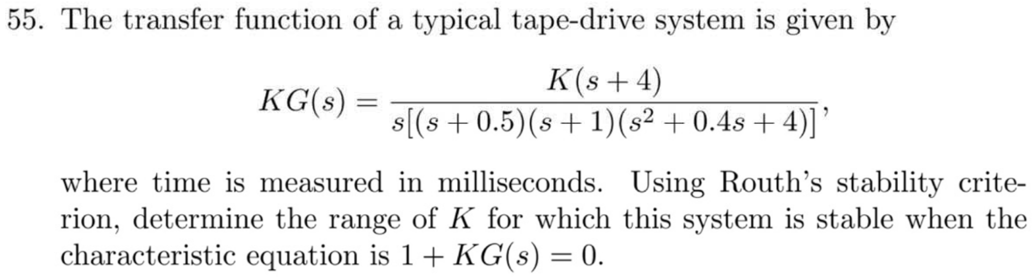
4. (Stability)



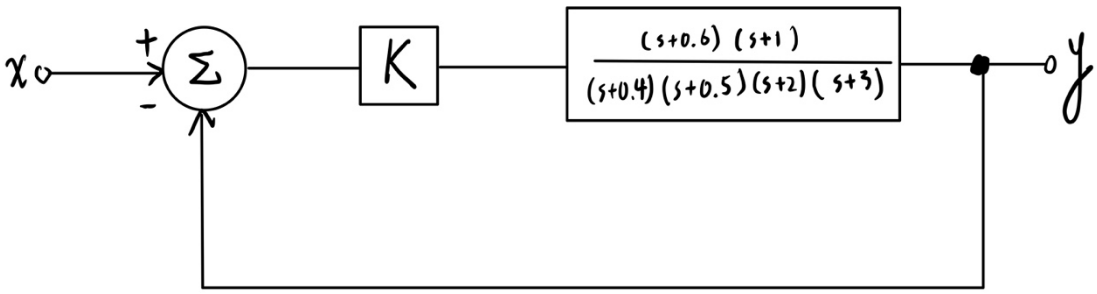
55. The transfer function of a typical tape-drive system is given by

$$KG(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + KG(s) = 0$.

G(s) is changed to : $\dfrac{(s+1)\ (s+0.6)}{(s+0.4)\ (s+0.5)(s+2)\ (s+3)}$

- Block diagram



⇒ <Sol>

$1+KG(s)=0$

$\Rightarrow s^4+5.9s^3+(10.7+K)s^2+(6.4+1.6K)s+(1.2+0.6K)=0$

s^4: 1 $(10.7+K)$ $(1.2+0.6K)$

s^3: 5.9 $(6.4+1.6K)$

s^2: a_1 $1.2+0.6K$

s^1: b_1

s^0: $1.2+0.6K$

$$a_1 = \frac{(6.4+1.6K)-(63.13+5.9K)}{-5.9} = \frac{56.73+4.3K}{5.9}$$

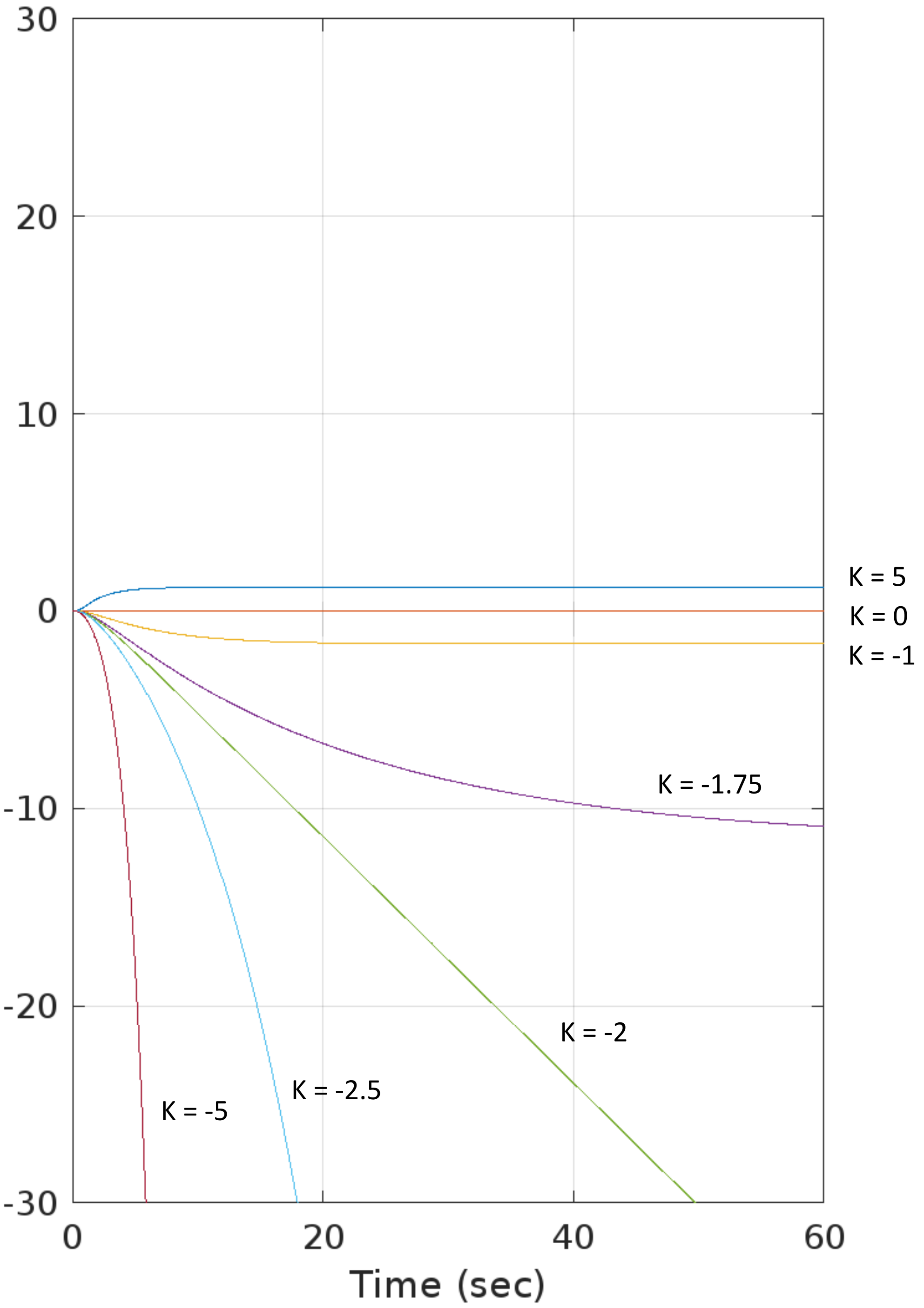
$$b_1 = \frac{(7.08+3.54K)-\left(\frac{56.73+4.3K}{5.9}\right)\times(6.4+1.6K)}{-\left(\frac{56.73+4.3K}{5.9}\right)} = \frac{363.072+118.288K+27.52K^2}{56.73+4.3K} - \frac{20.886K+41.772}{56.73+4.3K} = \frac{27.52K^2+97.402K+321.3}{56.73+4.3K}$$

$$\Rightarrow \begin{cases} \dfrac{56.73+4.3K}{5.9} > 0 \Rightarrow K > -\dfrac{5673}{430} \\ \dfrac{27.52K^2+97.402K+321.3}{56.73+4.3K} > 0 \Rightarrow 27.52K^2+93.102K+264.57 > 0 \\ \qquad \Rightarrow K \in \mathbb{R} \text{ 皆可} \\ 1.2+0.6K > 0 \Rightarrow K > -2 \end{cases}$$

$\Rightarrow$ The system is stable for $K > -2$

- Verification by Matlab:

可發現 K>-2 時曲線最終都會收斂，而在 K=-2 開始便發散了。



HW3 电机三 B09901152 施文葳

4. (Stability)

55. The transfer function of a typical tape-drive system is given by

$$KG(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + KG(s) = 0$.

$$\Rightarrow 1 + \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]} = 0$$

$$K(s+4) + \underline{s(s+0.5)(s+1)(s^2+0.4s+4)} = 0$$

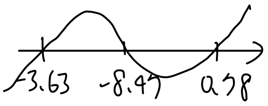
$$= (s^3+1.5s^2+0.5s)(s^2+0.4s+4)$$

$$= s^5+1.9s^4+5.1s^3+6.2s^2+2s$$

$$\Rightarrow s^5+1.9s^4+5.1s^3+6.2s^2+(K+2)s+4K=0$$

s^5: 1	5.1	$K+2$
s^4: 1.9	6.2	$4K$
s^3: 1.837	$2-1.1K$	
s^2: $1.138(K+3.63)$	$4K$	
s^1: $\frac{1.138(K+3.63)(2-1.1K)-7.348K}{1.138(K+3.63)} = \frac{-1.25K^2-9.61K+8.26}{1.138(K+3.63)} = \frac{(K+8.47)(K-0.78)}{-0.91(K+0.363)}$		
s^0: $4K$		

for stability : $\begin{cases} (K+3.63) > 0 \\ \dfrac{(K+8.47)(K-0.78)}{K+3.63} < 0 \\ 4K > 0 \end{cases}$



$$\Rightarrow \begin{cases} K > -3.63 \\ K < -3.63 \text{ or } -8.47 < K < 0.78 \\ K > 0 \end{cases} \Rightarrow 0 < K < 0.78 \quad \#$$

Matlab: