Control System: Homework 02 for Unit 3A, 3B, 3C, 3D: Dynamic Response

Assigned: Sep 30, 2022

Due: Oct 6, 2022 (23:59)

Please read the following problems and their solutions. Then, choose one of them and edit your solution for the selected problem. Submit your homework file in PDF format to the NTU-Cool website.

1. (Laplace Transform)

3.12 Consider the standard second-order system

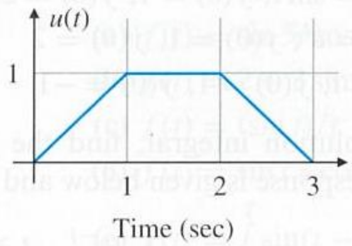
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

(a) Write the Laplace transform of the signal in Fig. 3.47.

(b) What is the transform of the output if this signal is applied to $G(s)$?

(c) Find the output of the system for the input shown in Fig. 3.47.

Figure 3.47
Plot of input signal for Problem 3.12



Solution:

(a) The input signal may be written as:

$$u(t) = t - (t-1) * 1(t-1) - (t-2) * 1(t-2) + (t-3) * 1(t-3),$$

where $1(t-\tau)$ denotes a delayed unit step. The Laplace transform of the input signal is:

$$U(s) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

We can verify this in MATLAB:

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>> ilaplace(1/s^2*(1-exp(-s)-exp(-2*s)+exp(-3*s)))

ans =

t-heaviside(t-1)*(t-1)-heaviside(t-2)*(t-2)+heaviside(t-3)*(t-3)
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(b) The Laplace transform of the output if this input signal is applied is:

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right))(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

(c) However to make the mathematical manipulation easier, consider only the response of the system to a (unit) ramp input:

$$Y_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right)).$$

Partial fractions yields the following:

$$Y_1(s) = \frac{1}{s^2} - \frac{\frac{2\zeta}{\omega_n}}{s} + \frac{\frac{2\zeta}{\omega_n}(s + 2\zeta\omega_n - \frac{\omega_n}{2\zeta})}{(s + \omega_n\zeta)^2 + (\omega_n\sqrt{1-\zeta^2})^2}.$$

Use the following Laplace transform pairs for the case $0 \leq \zeta < 1$:

$$L^{-1}\{\left\{\frac{s + z_1}{(s+a)^2 + \omega^2}\right\}\} = \sqrt{\frac{(z_1 - a)^2 + \omega^2}{\omega^2}} e^{-at}\sin(\omega t + \phi),$$

where

$$\phi \equiv \tan^{-1}(\left(\frac{\omega}{z_1 - a}\right)).$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step}$$

and the following Laplace transform pairs for the case $\zeta = 1$:

$$L^{-1}\{\left\{\frac{1}{(s+a)^2}\right\}\} = te^{-at}.$$

$$L^{-1}\{\left\{\frac{s}{(s+a)^2}\right\}\} = (1 - at)e^{-at}.$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp,}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step,}$$

the following is derived:

$$y_1(t) = \{ \begin{array}{ll} t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin(\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}) & \begin{array}{c} 0 \le \zeta < 1 \\ t \ge 0 \end{array} \\ t - \frac{2}{\omega_n} + \frac{2}{\omega_n}e^{-\omega_n t}(\frac{\omega_n}{2}t + 1) & \begin{array}{c} \zeta = 1 \\ t \ge 0 \end{array} \end{array} .$$

Since $u(t)$ consists of a ramp and three delayed ramp signals, using superposition (the system is linear), then:

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3) \qquad t \ge 0.$$

2. (Laplace Transform)

3.16 For a second-order system with transfer function

$$G(s) = \frac{5}{s^2 + s + 4},$$

Determine the following:

(a) The DC gain and whether the system is stable.

(b) The final value of the output if the input is applied with a step of 2 units or $R(s) = \frac{2}{s}$.

Solution:

(a) The system is stable and therefore the Final Value Theorem is applicable, and the DC gain is

$$G(0) = \frac{5}{4} = 1.25$$

By finding the roots of $s^2 + s + 4 = 0$, the poles are located at $s = -0.5 \pm j1.94$. Since the poles are at the LHS of s-plane, the system is stable.

(b) With a step input of 2 units or $R(s) = \frac{2}{s}$, the final value can be calculated as

$$\lim_{s \to 0} sG(s)R(s) = 2G(0) = 2.5 \text{ units.}$$

3. (Block diagram)

3.20 Find the transfer functions for the block diagrams in Fig. 3.50.

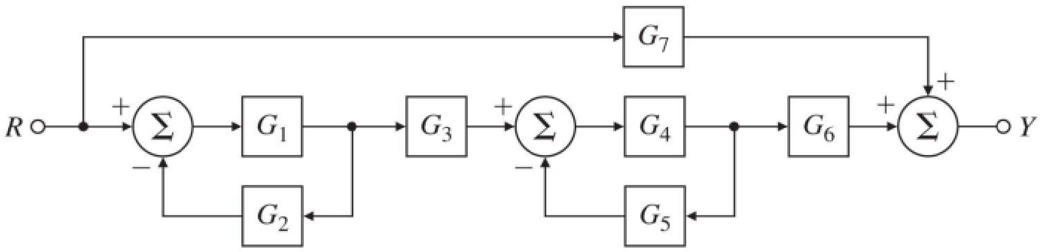
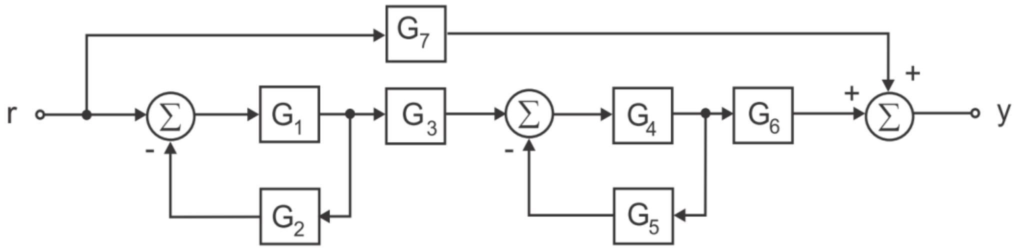
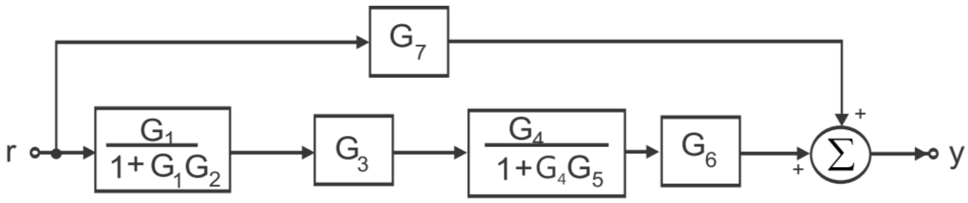


Figure 3.50 Block diagrams for Problem 3.20

Solution:

(b)





$$\frac{Y}{R} = G_7 + \frac{G_1G_3G_4G_6}{(1+G_1G_2)(1+G_4G_5)}.$$

4. (Time domain specification)

26. For the unity feedback system shown in Fig. 3.55, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using MATLAB.

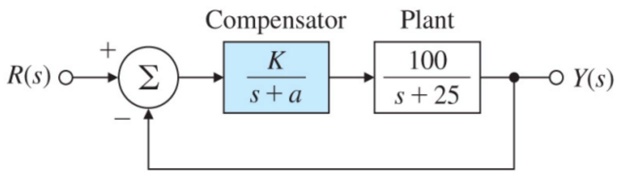


Figure 3.55: Unity feedback system for Problem 3.26

Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25 + a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step,} \\ M_p &\leq 25\%, \\ t_s &\leq 0.1\,\text{sec}. \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037. \end{aligned}$$

Solve for ω_n:

$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time.}$$

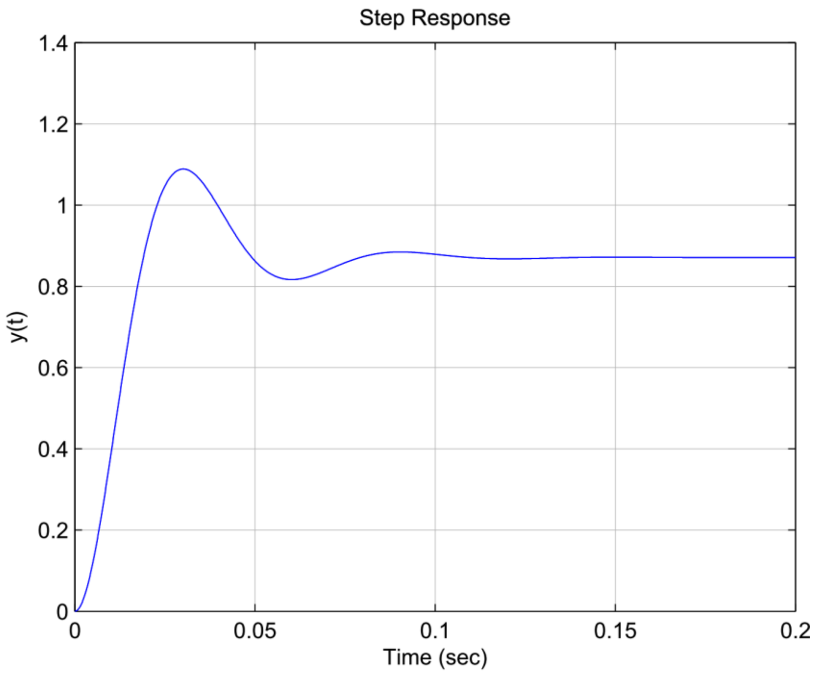
$$t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1,$$

$$\Longrightarrow \omega_n \approx 114.07.$$

Now find a and K :

$$
\begin{aligned}
2\zeta\omega_n &= (25 + a), \\
a = 2\zeta\omega_n - 25 &= 92.10 - 25 = 67.10, \\
\omega_n^2 &= (25a + 100K), \\
K &= \frac{\omega_n^2 - 25a}{100} \approx 113.34.
\end{aligned}
$$

The step response of the system using MATLAB is shown next.



Step response for Problem 3.26.