Control System: Homework 01 for Unit 2: Dynamic Models

Assigned: Sep 23, 2022

Due: Sep 29, 2022 (23:59)

Please read the following problems and their solutions. Then, choose one of them and edit your solution for the selected problem. Submit your homework file in PDF format to the NTU-Cool website.

1. (Mechanical systems)

2.8 In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Fig. 2.7 where there is flexibility of the solar panels. Figure 2.44 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling between the objects is often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this.

(a) Write the equations of motion governing this system.

(b) Find the transfer function between the control input u and the output y.

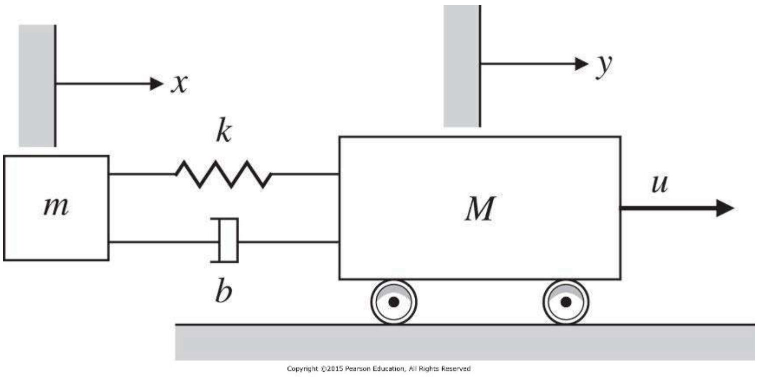
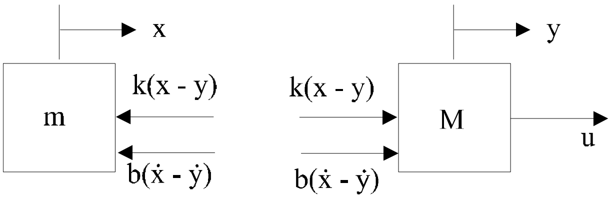


Figure 2.44 Schematic of a system with flexibility

Solution:

(a) The FBD for the system is



Free body diagrams

which results in the equations

$$\begin{aligned} m\ddot{x} &= -k(x-y) - b(\dot{x}-\dot{y}) \\ M\ddot{y} &= u + k(x-y) + b(\dot{x}-\dot{y}) \end{aligned}$$

or

$$\begin{aligned} \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\ -\frac{k}{M}x - \frac{b}{M}\dot{x} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} &= \frac{1}{M}u \end{aligned}$$

(b) If we make Laplace Transform of the equations of motion

$$\begin{aligned} s^2X + \frac{k}{m}X + \frac{b}{m}sX - \frac{k}{m}Y - \frac{b}{m}sY &= 0 \\ -\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY &= \frac{1}{M}U \end{aligned}$$

In matrix form,

$$\begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

From Cramer's Rule,

$$\begin{aligned} Y &= \frac{\det\begin{bmatrix} ms^2+bs+k & 0 \\ -(bs+k) & U \end{bmatrix}}{\det\begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix}} \\ &= \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k)-(bs+k)^2}U \end{aligned}$$

Finally,

$$\begin{aligned} \frac{Y}{U} &= \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k)-(bs+k)^2} \\ &= \frac{ms^2+bs+k}{mMs^4+(m+M)bs^3+(M+m)ks^2} \end{aligned}$$

2. (Electric Circuits)

2.13 A common connection for a motor power amplifier is shown in Fig. 2.49. The idea is to have the motor current follow the input voltage, and the connection is called a current amplifier. Assume that the sense resistor r_s is very small compared with the feedback resistor R, and find the transfer function from V_{in} to I_a. Also show the transfer function when $R_f = \infty$.

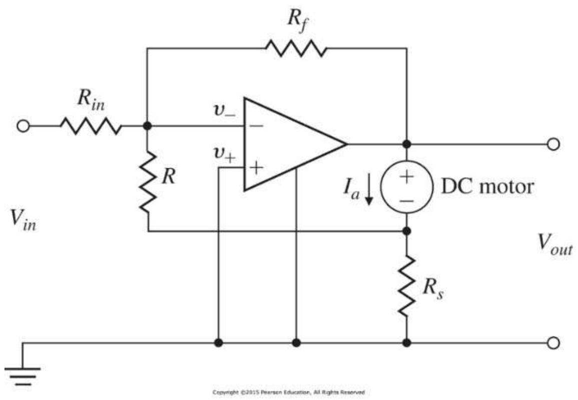


Figure 2.49 Op-amp circuit for Problem 2.13

Solution:

At node A,

$$\frac{V_{in}-0}{R_{in}}+\frac{V_{out}-0}{R_f}+\frac{V_B-0}{R}=0 \tag{93}$$

At node B, with $R_s \ll R$

$$\begin{aligned} I_a+\frac{0-V_B}{R}+\frac{0-V_B}{R_s} &= 0 \\ V_B &= \frac{RR_s}{R+R_s}I_a \\ V_B &\approx R_sI_a \end{aligned} \tag{94}$$

The dynamics of the motor is modeled with negligible inductance as

$$\begin{aligned} J_m\ddot{\theta}_m+b\dot{\theta}_m &= K_tI_a \\ J_ms\Omega+b\Omega &= K_tI_a \end{aligned} \tag{95}$$

At the output, from Eq. 94. Eq. 95 and the motor equation $V_a = I_aR_a + K_es\Omega$

$$\begin{aligned} V_o &= I_aR_s+V_a \\ &= I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b} \end{aligned}$$

Substituting this into Eq.93

$$\frac{V_{in}}{R_{in}}+\frac{1}{R_f}\left[I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b}\right]+\frac{I_aR_s}{R}=0$$

This expression shows that, in the steady state when $s \to 0$, the current is proportional to the input voltage.

If fact, the current amplifier normally has no feedback from the output voltage, in which case $R_f \to \infty$ and we have simply

$$\frac{I_a}{V_{in}}=-\frac{R}{R_{in}R_s}$$

3. (Electromechanical Systems)

2.19 The electromechanical system shown in Fig. 2.53 represents a simplified model of a capacitor microphone. The system consists in part of a parallel plate capacitor connected into an electric circuit. Capacitor plate a is rigidly fastened to the microphone frame. Sound waves pass through the mouthpiece and exert a force $f_s(t)$ on plate b, which has mass M and is connected to the frame by a set of springs and dampers. The capacitance C is a function of the distance x between the plates, as follows:

$$C(x) = \frac{\varepsilon A}{x},$$

where

ε = dielectric constant of the material between the plates,

A = surface area of the plates.

The charge q and the voltage e across the plates are related by

$$q = C(x)e.$$

The electric field in turn produces the following force f_e on the movable plate that opposes its motion:

$$f_e = \frac{q^2}{2\varepsilon A}.$$

(a) Write differential equations that describe the operation of this system. (It is acceptable to leave in nonlinear form.)

(b) Can one get a linear model?

(c) What is the output of the system?

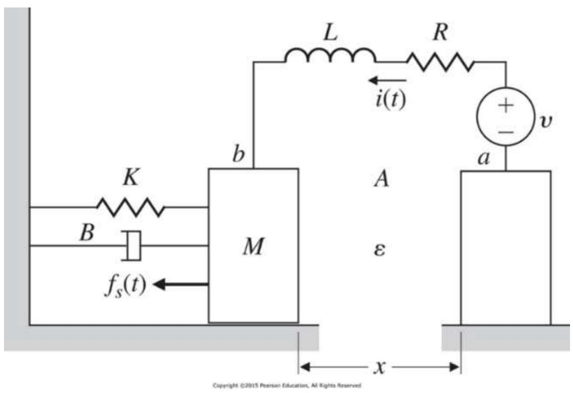
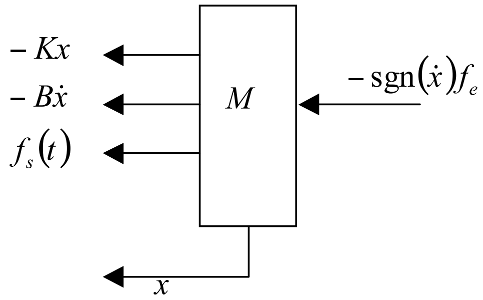


Figure 2.53 Simplified model for capacitor microphone

Solution:

(a) The free body diagram of the capacitor plate b



Free body diagram

So the equation of motion for the plate is

$$M\ddot{x} + B\dot{x} + Kx + f_e sgn(\dot{x}) = f_s(t).$$

The equation of motion for the circuit is

$$v = iR + L\frac{d}{dt}i + e$$

where e is the voltage across the capacitor,

$$e = \frac{1}{C}\int i(t)dt$$

and where $C = \varepsilon A/x$, a variable. Because $i = \frac{d}{dt}q$ and $e = q/C$, we can rewrite the circuit equation as

$$v = R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A}$$

In summary, we have these two, couptled, non-linear differential equation.

$$\begin{aligned} M\ddot{x} + b\dot{x} + kx + \operatorname{sgn}(\dot{x})\frac{q^2}{2\varepsilon A} &= f_s(t) \\ R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A} &= v \end{aligned}$$

(b) The sgn function, q^2, and qx, terms make it impossible to determine a useful linearized version.

(c) The signal representing the voice input is the current, i, or $\dot{q}$.

參考觀摩的作業 1. (Mechanical systems)
作者： b09901026，何式功 **理由：** 使用 Vpython 模擬動態系統，建議可以畫出位置/力/電壓電流對時間變化圖
作者： b10202032，卓 然 **理由：** 使用動摩擦來計算運動方程式，建議假設系統為動態平衡狀態
作者： b10901125，賴禹宏 **理由：** 題目設計很有趣，串聯 N 組相同的方塊，但是從特徵多項式開始推導出(1), (2)結果不太清楚，最後的輸入輸出轉移函數有點難驗證

HW 01: Dynamics Model	Control Systems, Fall 2022, NTU-EE
Name: 何式功 B09901026	Date: 9/29, 2022

I. Problems

Content: 控制系統 HW1 B09901026 何式功

1. $\begin{cases} m\ddot{x} = -k(x-y) - b(\dot{x}-\dot{y}) \\ M\ddot{y} = u + k(x-y) + b(\dot{x}-\dot{y}) \end{cases}$

Laplace Transform $\Rightarrow$ $\begin{cases} s^2X + \frac{k}{m}X + \frac{b}{m}sX - \frac{k}{m}Y - \frac{b}{m}sY \\ -\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY = \frac{1}{M}U \end{cases}$

解 $\frac{Y}{U} = \frac{1}{U} \frac{\begin{vmatrix} ms^2+bs+k & 0 \\ -(bs+k) & U \end{vmatrix}}{\begin{vmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{vmatrix}} = \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k)-(bs+k)^2}$

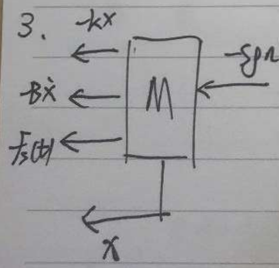
2. node equation $\begin{cases} \frac{V_{in}-0}{R_{in}} + \frac{V_{out}-0}{R_f} + \frac{V_B-0}{R} = 0 \\ I_a + \frac{0-V_B}{R} + \frac{0-V_B}{R_s} = 0 \Rightarrow V_B = \frac{RR_s}{R+R_s} I_a \cong R_s I_a \end{cases}$

Motor dynamics: $V_0 = I_a R_s + V_a = I_a R_s + I_a R_a + K_e \frac{K_t I_a}{J_m s + b}$

$\Rightarrow \frac{V_{in}}{R_{in}} + \frac{1}{R_f}\left[I_a R_s + I_a R_a + K_e \frac{K_t I_a}{J_m s+b} \right] + \frac{I_a R_s}{R} = 0$

當 $R_f \to \infty$ 時，$\frac{I_a}{V_{in}} = \frac{-R}{R_{in}+R_s}$

3.



(a) equation of the motion: (plate)

$M\ddot{x} + B\dot{x} + kx + f_e\, sgn(\dot{x}) = f_s(t)$

(circuit)

$\begin{cases} v = iR + L\frac{di}{dt} + e \\ e = \frac{1}{c}\int i(t)dt \end{cases} \Rightarrow v = R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A}$

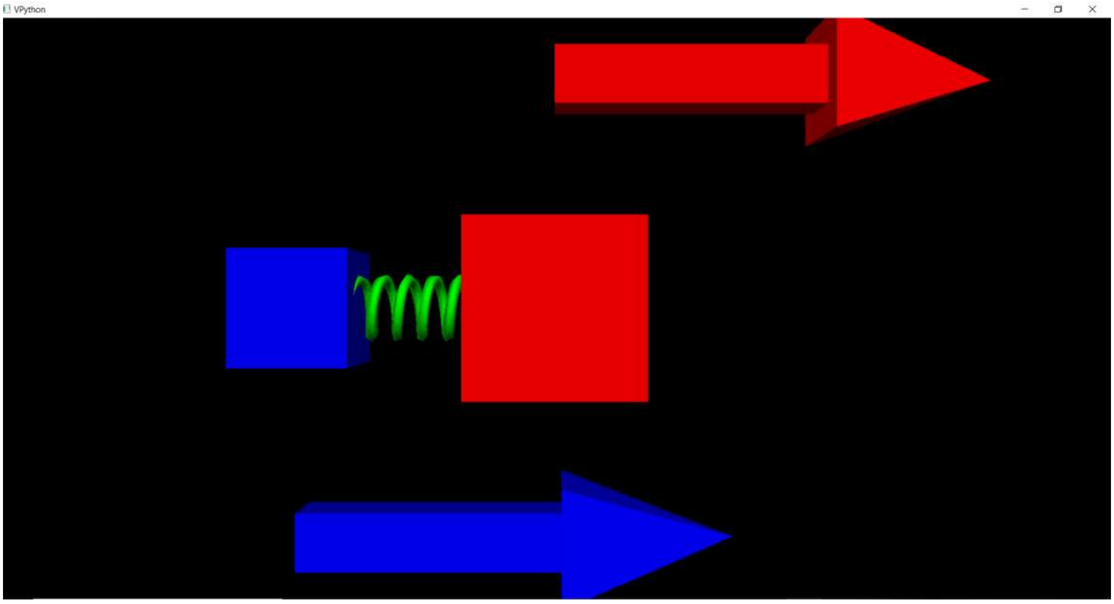
$\Rightarrow \begin{cases} M\ddot{x} + b\dot{x} + kx + sgn(\dot{x})\frac{q^2}{2\varepsilon A} = f_s(t) \\ R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A} = v \end{cases}$

(b) $sgn(\dot{x})$, q^2, qx make it impossible to determine a useful linear version

(c) the signal representing the input is the current i.

II. Bonus

為了 visualize 這次題目中的物理模型，我使用 python 中的 Vpython 套件模擬了第一題，圖片如下



附上 30 秒 demo 影片 url

https://youtu.be/hucN0r4vkFU

雖然模擬本身並不複雜，但是把物理模型視覺化的技巧，說不定會對我之後在做期末專題建模時有幫助，所以就試著做了這個小 bonus。

Control Systems Homework 1
B10202032 物理二 卓然

1. (Mechanical systems)

2.8 In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Fig. 2.7 where there is flexibility of the solar panels. Figure 2.44 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling between the objects is often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this.

(a) Write the equations of motion governing this system.

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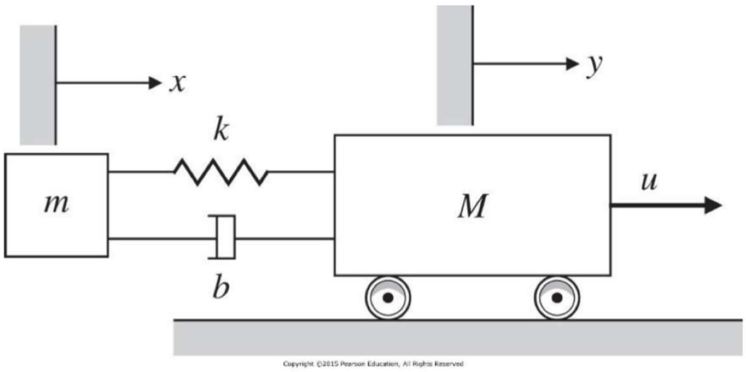


Figure 2.44 Schematic of a system with flexibility

In the original problem, the ground is frictionless. However, practically there must be some frictional force between the masses and the ground. Suppose that the coefficient of kinetic friction of the ground surface is μ_k, and the gravitational acceleration is g. I'll use these parameters to solve this problem.

(a)

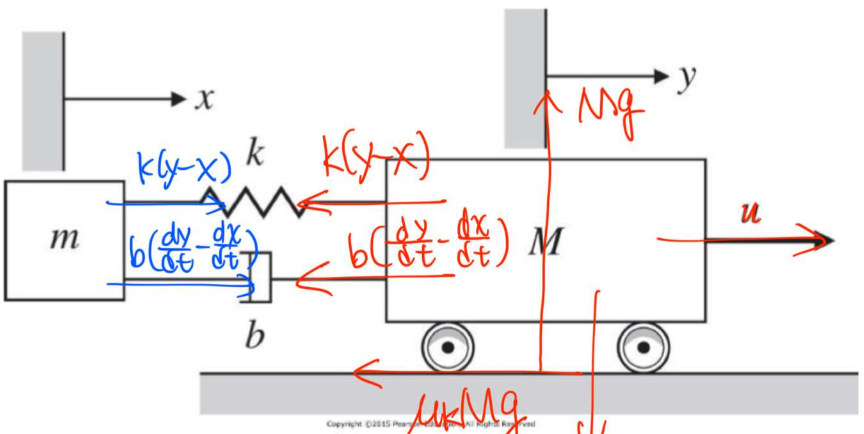


Figure 2.44 Schematic of a system with flexibility

Using Newton's 2nd Law, we get the equations of motion:

$$\begin{cases} \text{For } m, & m\frac{d^2x}{dt^2} = k(y-x) + b\left(\frac{dy}{dt} - \frac{dx}{dt}\right); \\ \text{For } M, & M\frac{d^2y}{dt^2} = u - k(y-x) - b\left(\frac{dy}{dt} - \frac{dx}{dt}\right) - \mu_k Mg. \end{cases}$$

(b) Using the Laplace Transform properties

$\mathcal{L}\{1\} = \frac{1}{s}$ $(s>0)$, $\mathcal{L}\{\frac{d^n x}{dt^n}\} = s^n \mathcal{L}\{x\}$, we can derive the following equations from the equations of motion:

(Denote $\mathcal{L}\{x\} = X$, $\mathcal{L}\{y\} = Y$ and $\mathcal{L}\{u\} = U$)

$$\begin{cases} ms^2X = k(Y-X) + b(sY - sX) \\ Ms^2Y = U - k(Y-X) - b(sY-sX) - \mu_k Mg\frac{1}{s}. \end{cases}$$

Now solve for Y:

$$\Rightarrow \begin{cases} ms^2X - kY + kX - bsY + bsX = 0 \\ Ms^2Y + kY - kX + bsY - bsX = U - \mu_k Mg\frac{1}{s} \end{cases}$$

$$\Rightarrow \begin{pmatrix} ms^2+k+bs & -k-bs \\ -k-bs & Ms^2+k+bs \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ U - \mu_k Mg\frac{1}{s} \end{pmatrix}$$

$$\det\begin{pmatrix} ms^2+k+bs & -k-bs \\ -k-bs & Ms^2+k+bs \end{pmatrix}$$

$$= (ms^2 + (k+bs))(Ms^2 + (k+bs)) - [-(k+bs)]^2$$

$$= mMs^2 + (k+bs)(m+M)s^2 + \cancel{(k+bs)^2} - \cancel{(k+bs)^2}$$

$$= [mM + (m+M)(k+bs)]s^2.$$

$$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{\det\begin{pmatrix} ms^2+k+bs & -k-bs \\ -k-bs & Ms^2+k+bs \end{pmatrix}} \begin{pmatrix} Ms^2+k+bs & k+bs \\ k+bs & ms^2+k+bs \end{pmatrix}\begin{pmatrix} 0 \\ U - \mu_k Mg\frac{1}{s} \end{pmatrix}$$

$$\Rightarrow Y = \frac{1}{(mM+(m+M)(k+bs))s^2} \cdot (ms^2+k+bs)\left(U - \mu_K Mg \frac{1}{s}\right).$$

$$\Rightarrow \frac{Y}{U} = \frac{(ms^2+k+bs)}{[mM+(m+M)(k+bs)]s^2}\left(1 - \frac{\mu_K Mg}{Us}\right).$$

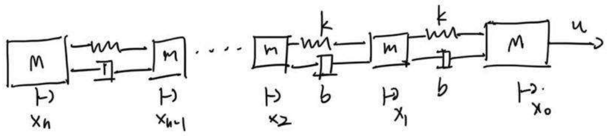
Since there's a term attached to the input term U, we cannot express the transfer function $\frac{Y}{U}$ without the information about the input. We can check that $\frac{Y}{U}$ is independent of the input iff $\mu_K \to 0^+$, which is the case of the original problem.

<u>Conclusion</u>: When there's some force independent of the variables of motion (like the position and velocity), the transfer function may be dependent of the input, and therefore it might be more tricky to deal with such a mechanical system. Unfortunately, I guess that this is usualy the case.

HW 01	**Control Systems, Fall 2022, NTU-EE**
Name: 賴禹宏 B10901125	**Date: 9/29, 2022**

Problem 1

Consider two blocks that are connected by elastic rod, we can discretize the rod and analyze a series of connected small blocks.



- $M\ddot{x}_0 = u + k(x_1-x_0) + b(\dot{x}_1-\dot{x}_0)$

  $m\ddot{x}_1 = k(x_0-x_1) + b(\dot{x}_0-\dot{x}) + k(x_2-x_1) + b(\dot{x}_2-\dot{x}_1)$

  $\vdots$

  $m\ddot{x}_i = k(x_{i-1}-x_i) + b(\dot{x}_{i-1}-\dot{x}_i) + k(x_{i+1}-x_i) + b(\dot{x}_{i+1}-\dot{x}_i)$ $\quad 1 \le i < n$

  $\vdots$

  $M\ddot{x}_n = k(x_{n-1}-x_n) + b(\dot{x}_{n-1}-\dot{x}_n)$

- $(Ms^2+bs+k)X_0 - (bs+k)X_1 = U$ * boundary condition

  $(ms^2+2bs+2k)X_1 - (bs+k)X_0 - (bs+k)X_2 = 0$

  $\vdots$

  $(ms^2+2bs+2k)X_i - (bs+k)X_{i-1} - (bs+k)X_{i+1} = 0$ * recursion relation $(1 \le i < n)$

  $(Ms^2+bs+k)X_n - (bs+k)X_{n-1} = 0$ * boundary condition

- characteristic polynomial: $r^2 - \frac{ms^2+2bs+2k}{bs+k} r + 1 = 0$, let $2\alpha = \frac{ms^2+2bs+2k}{bs+k}$

$\Rightarrow r_\pm = \alpha \pm \sqrt{\alpha^2-1}$, $x_i = Ar_+^i + Br_-^i$

$\Rightarrow$ (1) $(Ms^2+bs+k)(A+B) - (bs+k)(Ar_+ + Br_-) = U$ $\quad$ let $\beta = \frac{Ms^2+bs+k}{bs+k}$

(2) $(Ms^2+bs+k)(Ar_+^n + Br_-^n) - (bs+k)(Ar_+^{n-1} + Br_-^{n-1}) = 0$

$$\begin{pmatrix} \beta - r_+ & \beta - r_- \\ \beta r_+^n - r_+^{n-1} & \beta r_-^n - r_-^{n-1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{U}{bs+k} \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \frac{r_-^{n-1}(\beta r_- - 1)}{(r_-^n - r_+^n)\beta^2 - (r_+r_- + 1)(r_-^{n-1} - r_+^{n-1})\beta + r_+r_-(r_-^{n-2} - r_+^{n-2})} \cdot \frac{U}{bs+k}$$

$$B = \frac{-r_+^{n-1}(\beta r_+ - 1)}{(r_-^n - r_+^n)\beta^2 - (r_+r_- + 1)(r_-^{n-1} - r_+^{n-1})\beta + r_+r_-(r_-^{n-2} - r_+^{n-2})} \cdot \frac{U}{bs+k}$$

$$\Rightarrow \frac{X_0}{U} = \frac{A+B}{U} = \frac{(r_-^n - r_+^n)\beta - (r_-^{n-1} - r_+^{n-1})}{(r_-^n - r_+^n)\beta^2 - (r_+r_- + 1)(r_-^{n-1} - r_+^{n-1})\beta + r_+r_-(r_-^{n-2} - r_+^{n-2})} \cdot \frac{1}{bs+k}$$

when $n=1$, this reduce to the case in problem 1 such that $m = M$.

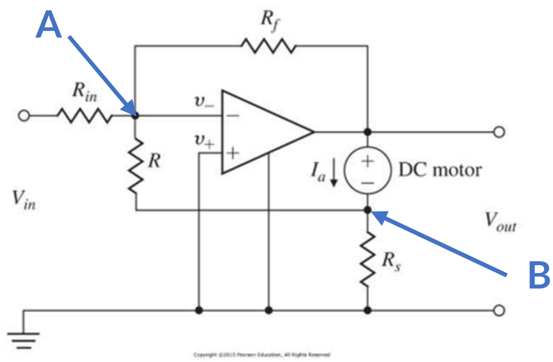
參考觀摩的作業 2. (Electric Circuits)
作者： b08901085，施彥宇 **理由：** 討論完整，建議附上使用的直流馬達模型參考資料來源，(12)式推導來源不是很清楚
作者： b08901115，范博淵 **理由：** 包含電子學與電路學知識於詳細電路推導中

HW01 – Unit 2, Dynamic Model

學號：B08901085　　　系級：電機四　　　姓名：施彥宇

2.13 A common connection for a motor power amplifier is shown in Fig. 2.49. The idea is to have the motor current follow the input voltage, and the connection is called a current amplifier. Assume that the sense resistor r_s is very small compared with the feedback resistor R, and find the transfer function from V_{in} to I_a. Also show the transfer function when $R_f = \infty$.

- **Solution：**



At node A, using KCL, we can get

$$\frac{V_{in}-0}{R_{in}}+\frac{V_{out}-0}{R_f}+\frac{V_B-0}{R}=0 \quad (1).$$

And at node B, using KCL, we can get

$$I_a+\frac{0-V_B}{R}+\frac{0-V_B}{R_S}=0 \quad (2).$$

$$\Rightarrow V_B=\frac{RR_S}{R+R_S}I_a \cong R_S I_a \text{ for } R_S \ll R \quad (3).$$

And then, I found that the dynamics of the motor can be described as the following four equations :

$$v_a(t)=R_a i_a(t)+L_a\frac{di_a(t)}{dt}+e_b(t) \quad (4).$$

$$e_b(t)=K_b\omega(t) \quad (5).$$

$$T_q(t)=K_t i_a(t) \quad (6).$$

$$T_q(t)=J\frac{d\omega(t)}{dt}+B\omega(t) \quad (7).$$

The name of each constant is filled in the following table :

v_a	Armature voltage	R_a	Armature resistance
i_a	Armature current	L_a	Armature inductance
e_b	Induced potential	K_b	Back E.M.F. constant
ω	Angular velocity	T_q	Torque
K_t	Torque constant	J	Moment of inertia
B	Friction coefficient		

Combining eq. (4), (5), (6), (7) and then taking Laplace Transform, we can conclude the following two equations :

$$V_a(s)=R_aI_a(s)+sL_aI_a+E_b(s)=R_aI_a(s)+sL_aI_a+K_b\omega(s) \quad (8).$$

$$K_tI_a(s)=sJ\omega(s)+B\omega(s)\Rightarrow \omega(s)=\frac{K_tI_a(s)}{Js+B} \quad (9).$$

And because in a motor, the armature inductance is often made small, we can neglect the effect caused by inductor. Therefore, from equations (8) and (9), we can write down the equation of output voltage $V_o(s)$:

$$V_o(s) = I_a(s)R_s + V_a(s) = I_a(s)R_s + I_a(s)R_a + K_b\frac{K_t I_a(s)}{Js+B} \quad (10).$$

Substituting equation (10) back to equation (1), then the relationship between V_{in} and I_a can be shown :

$$\frac{V_{in}(s)}{R_{in}} + \frac{1}{R_f}\left[I_a(s)R_s + I_a(s)R_a + K_b\frac{K_t I_a(s)}{Js+B}\right] + \frac{R_s I_a(s)}{R} = 0$$

$$\Rightarrow \frac{I_a(s)}{V_{in}(s)} = \frac{-R_f}{R_{in}}\left(\frac{R_s R_f}{R} + R_s + R_a + \frac{K_b K_t}{Js+B}\right)^{-1} \quad (11).$$

And from equation (11), we can find out that :

1. When $s \to 0$, then $\frac{I_a(s)}{V_{in}(s)} = \frac{-R_f B/R_{in}}{K_t K_b + B(R_s R_f/R + R_a + R_s)}$
2. When $s \to \infty$, then $\frac{I_a(s)}{V_{in}(s)} = \frac{-R_f R}{R_s R_f R_{in} + R_s R_{in} R + R_a R_{in} R}$
3. When $R_f \to \infty$, then $\frac{I_a(s)}{V_{in}(s)} = \frac{-R}{R_{in} R_s}$

And by finding the relationship between input voltage, angular velocity and motor position, we can find the equation below :

$$\frac{\omega(s)}{V_a(s)} = \frac{K_t R_a/J}{s + \frac{K_b K_t R_a B}{R_a J}} = \frac{K_m}{s+T_m} \quad (12).$$

$$\theta(t) = K_H\int_0^t \omega(\tau)d\tau \Rightarrow \frac{\theta(s)}{V_a(s)} = \frac{K_H}{s} \times \frac{K}{s+T_m} = \frac{K_c}{s(s+T_m)} \quad (13).$$

where K_H is the sensing constant of motor.

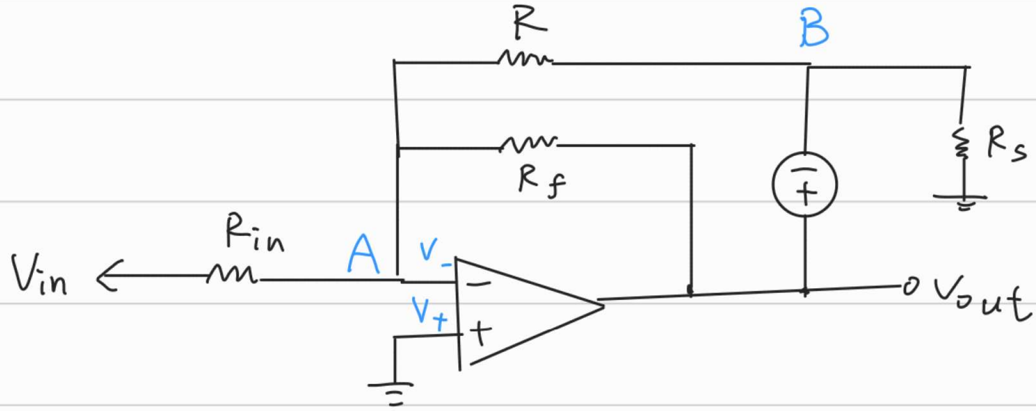
From equation (11), (12), and (13) we can calculate the relationship between input voltage and the output response of motor in this circuit diagram. This skill may be really helpful for us while doing the system identification process.

2. (electric circuits)

范博淵

B08901115

改成 ⇒



本作業會分別解理想 OP 和非理想 OP，理想 OP 接上負回授可使用 virtual short。

非理想 OP 可使用電子學之回授解，(詳情可見 Sedra Smith 10.4 ~ 10.5 or Razavi 8.5) 或是使用米勒定理解。

<理想 OP>：使用 virtual short

$$V_- = V_+ = 0$$

node A → $\dfrac{V_{in}}{R_{in}} + \dfrac{V_{out}}{R_f} + \dfrac{V_B}{R} = 0$ (KCL) the motor

$$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t I_a$$

$$J_m s \Omega + b \Omega = K_t I_a$$

$$\Omega = \frac{K_t I_a}{J_m s + b}$$

$$V_a = I_a R_a + K_e s \Omega = I_a \left(R_a + \frac{K_e K_t}{J_m s + b} \right)$$

node B

$$I_a - \frac{V_B}{R \parallel R_s} = 0 , \quad V_B = (R \parallel R_s) I_a$$

$$R_s \ll R \Rightarrow R \parallel R_s \simeq R_s , \quad V_B \simeq R_s I_a$$

$$V_{out} = V_B + V_a = R_s I_a + I_a \left(R_a + \frac{K_e K_t}{J_m s + b} \right)$$

將 V_B, V_{out} 代回 $\frac{V_{in}}{R_{in}} + \frac{V_{out}}{R_f} + \frac{V_B}{R} = 0$

$$\frac{V_{in}}{R_{in}} + \frac{I_a}{R_f} \cdot \left[R_s + R_a + \frac{K_e K_t}{J_m s + b} \right] + \frac{I_a R_s}{R} = 0$$

$$\therefore \frac{I_a}{V_{in}} = \frac{-\frac{1}{R_{in}}}{\frac{R_s + R_a + \frac{K_e K_t}{J_m s + b}}{R_f} + \frac{R_s}{R}}$$

令

$$H(s) = \frac{I_a}{V_{in}} = \frac{-\frac{1}{R_{in}}}{\frac{R_s + R_a + \frac{K_e K_t}{J_m s + b}}{R_f} + \frac{R_s}{R}}$$

$$= \frac{-R \cdot R_f}{R_{in} \cdot R \cdot \left(R_s + R_a + \frac{K_e K_t}{J_m s + b}\right) + R_s \cdot R_f \cdot R_{in}}$$

$$= \frac{-R \cdot R_f \cdot (J_m s + b)}{R_{in} \cdot R \left[(R_s + R_a) \cdot (J_m s + b) + K_e K_t\right] + R_s \cdot R_f \cdot R_{in} \cdot (J_m s + b)}$$

$$h(t) = \frac{I_a(t)}{V_{in}(t)}$$

The initial-value theorem :

$$\lim_{t \to \infty} h(t) = \lim_{s \to 0} s \cdot H(s) = 0$$

---

dc steady state : (電壓，電流比)

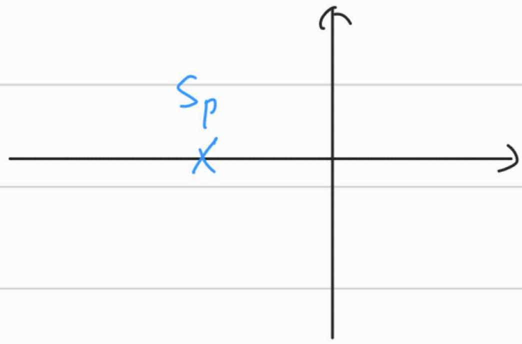
$$\lim_{s \to 0} H(s) = \frac{-\cancel{R} \cdot R_f \cdot b}{R_{in} \cdot \cancel{R} \left[(R_s + R_a) \cdot b + K_e K_t\right] + R_s \cdot R_f \cdot R_{in} \cdot b}$$

此即為 I_a, V_{in} 正比例之比值

dc transient state :

$H(s)$ 有一個 pole (first-order)

設此 pole 為 s_p



令 $S_p = -\frac{1}{\tau}$

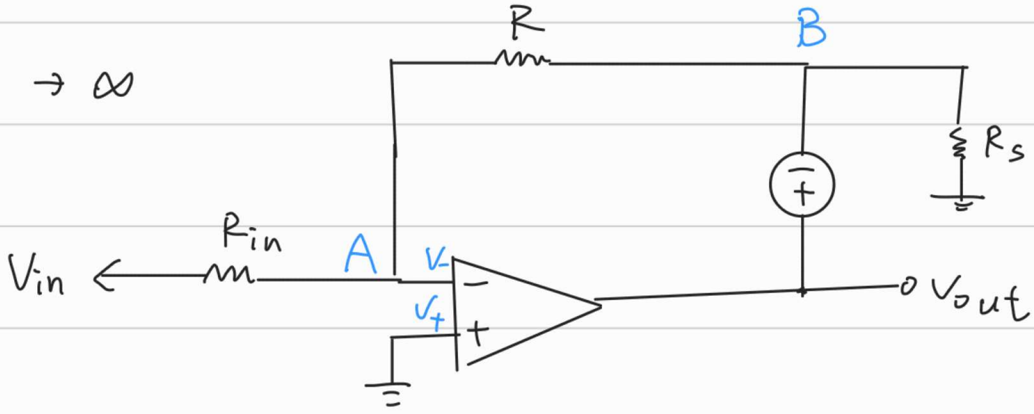
$$I_a(t) = \underbrace{K}_{\text{const}} e^{s_p t} = K e^{-t/\tau}$$

$$R_{in} \cdot R \left[(R_s + R_a) \cdot (J_m s + b) + K_e K_t \right] + R_s \cdot R_f \cdot R_{in} \cdot (J_m s + b) = 0$$

S_p 為方程式之根

$$S_p = -\frac{R_{in} \cdot R \cdot (R_s + R_a) \cdot b + R_{in} \cdot R \cdot K_e \cdot K_t + R_s \cdot R_f \cdot R_{in} \cdot b}{R_{in} \cdot R \cdot (R_s + R_a) \cdot J_m + R_s \cdot R_f \cdot R_{in} \cdot J_m}$$

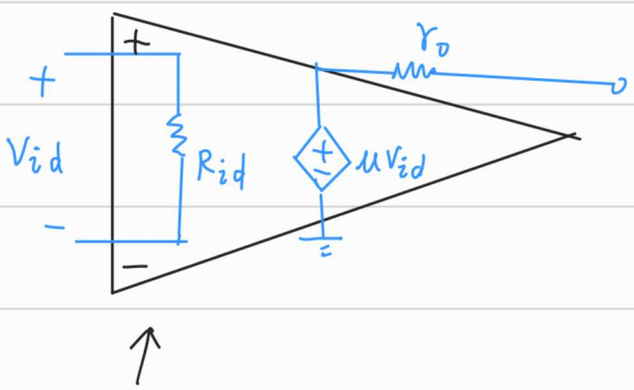
$R_f \to \infty$



$$\left.\frac{I_a}{V_{in}}\right|_{R_f \to \infty} = \frac{-\frac{1}{R_{in}}}{\dfrac{R_s + R_a + \frac{K_e K_t}{J_m s + b}}{R_f} + \dfrac{R_s}{R}} = \frac{-\frac{1}{R_{in}}}{\frac{R_s}{R}}$$

$$= -\frac{R}{R_{in} \cdot R_s}$$

<非理想 op>：

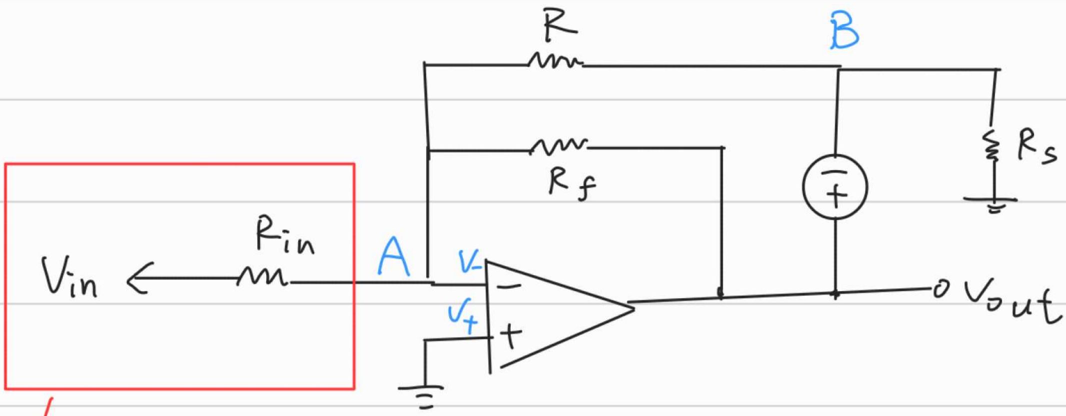


假設輸入阻抗為 R_{id}，
輸出阻抗為 r_o
開路電壓增益為 μ

the motor ⇒

$$V_a = I_a R_a + K_e s \Omega = I_a \left(R_a + \frac{K_e K_t}{J_m s + b} \right)$$

定義其阻抗： $Z_a = \frac{V_a}{I_a} = R_a + \frac{K_e K_t}{J_m s + b}$

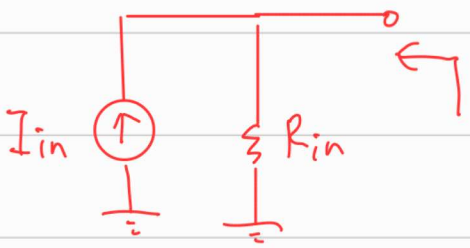


⇓

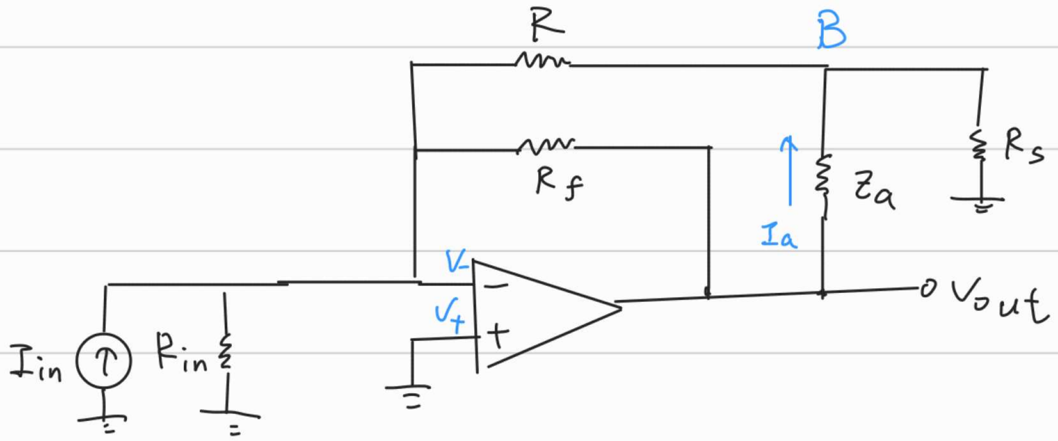
此為 current - voltage feedback

(shunt - shunt)

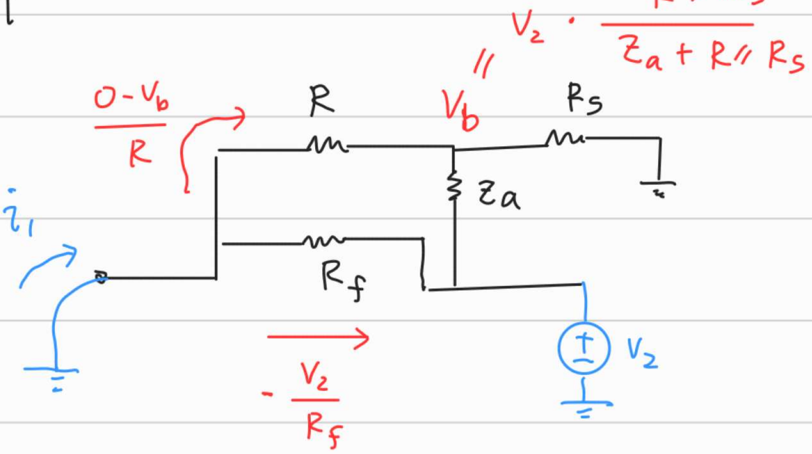
戴維寧改成諾頓等效電路



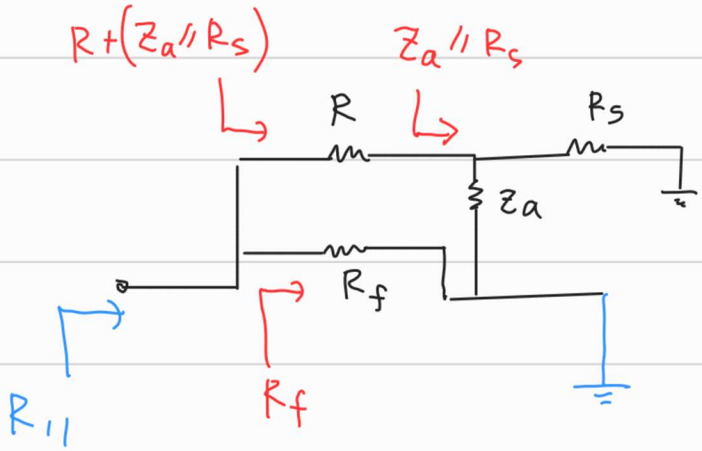
⇐ $I_{in} = \frac{V_{in}}{R_{in}}$



求 β :

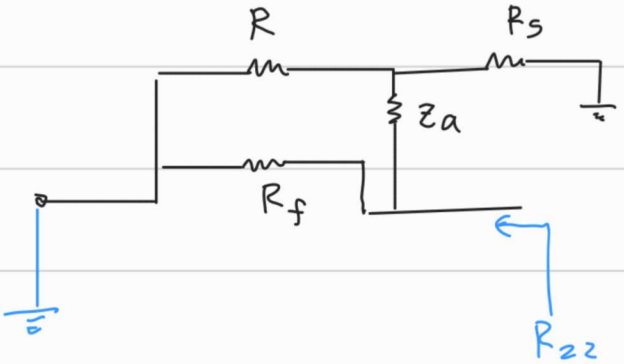


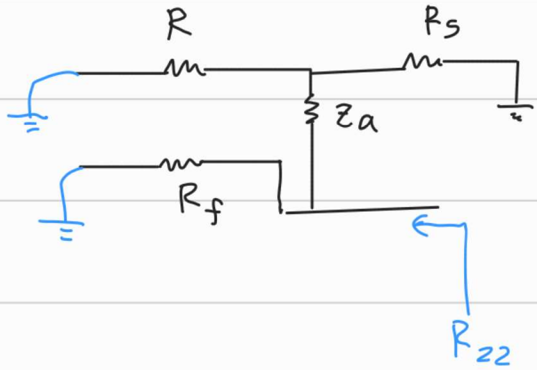
$$\beta = \frac{i_1}{V_2} = -\frac{(R \parallel R_s)}{R(Z_a + R \parallel R_s)} - \frac{1}{R_f}$$



求 R_{11}

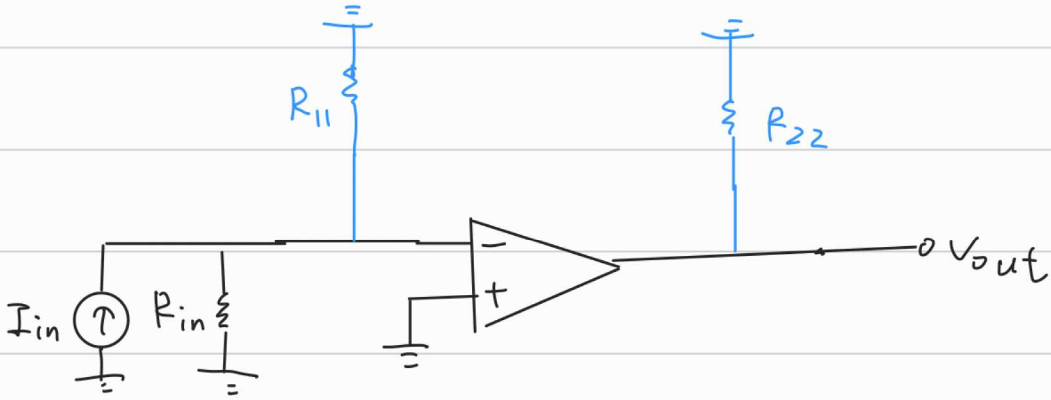
$$R_{11} = [R + (R_s \parallel Z_a)] \parallel R_f$$





求 R_{22}

$$R_{22} = R_f \,//\, (Z_a + R \,//\, R_s)$$



求 A = ?

分流 分壓

$$A = \frac{R_{in} \,//\, R_{11}}{R_{in} \,//\, R_{11} + R_{id}} \cdot (R_{id}) \cdot (-\mu) \cdot \frac{R_{22}}{r_o + R_{22}}$$

A, β 求出皆為負數，Aβ > 0，

因此確定是負回授之電路

求出 A, β 後可得 $\dfrac{V_{out}}{I_{in}} = \dfrac{A}{1+A\beta}$

而題目所求為 $\dfrac{V_{out}}{I_{in}}$

$$\frac{V_{out}}{I_{in}} \simeq \frac{I_a \cdot R_s + I_a \cdot Z_a}{V_{in}/R_{in}} = \frac{I_a}{V_{in}} \frac{R_s + Z_a}{1/R_{in}}$$

改回戴維寧

$$\boxed{\begin{aligned} I_a/V_{in} &= \frac{V_{out}}{I_{in}} \frac{1}{(R_s+Z_a)\cdot R_{in}} \\ &= \frac{A}{1+A\beta} \cdot \frac{1}{(R_s+Z_a)\cdot R_{in}} \end{aligned}}$$

$$\text{where } \begin{cases} A = \dfrac{R_{in} /\!/ R_{11}}{R_{in} /\!/ R_{11} + R_{id}} \cdot (R_{id}) \cdot (-\mu) \cdot \dfrac{R_{22}}{r_o + R_{22}} \\[2ex] \beta = -\dfrac{(R /\!/ R_s)}{R(Z_a + R /\!/ R_s)} - \dfrac{1}{R_f} \end{cases}$$

如果 $\mu \to \infty$ (OP 的 open-loop gain $\to \infty$)

，則 $A \to \infty$

$$\frac{V_{out}}{I_{in}} = \frac{A}{1+A\beta} = \frac{1}{\beta} = \frac{1}{-\frac{(R /\!/ R_s)}{R(Z_a + R /\!/ R_s)} - \frac{1}{R_f}}$$

$$I_a / V_{in} = \frac{1}{-\frac{(R /\!/ R_s)}{R(Z_a + R /\!/ R_s)} - \frac{1}{R_f}} \cdot \frac{1}{(R_s + Z_a) \cdot R_{in}}$$

where $Z_a = R_a + \frac{K_e K_t}{J_m s + b}$

如果 $\mu \to \infty$，$R_{id} \to \infty$，$r_o \to 0$

則為理想 OP，如最前面所解之答案。

參考觀摩的作業 3. (Electromechanical Systems)
無