Final Exam, Control Systems, 110-1 (2021) 姓名：
Date: Friday, January 7, 2022. Time: 9:10am-11:10am. 學號：
Closed books, closed notes, no calculators. 系級：
Only pens and erasers are allowed.

[100%] Write down proper description for the following problems.

(1) (20%=5%+5%+10%), (By B07901004 陳恩庭)

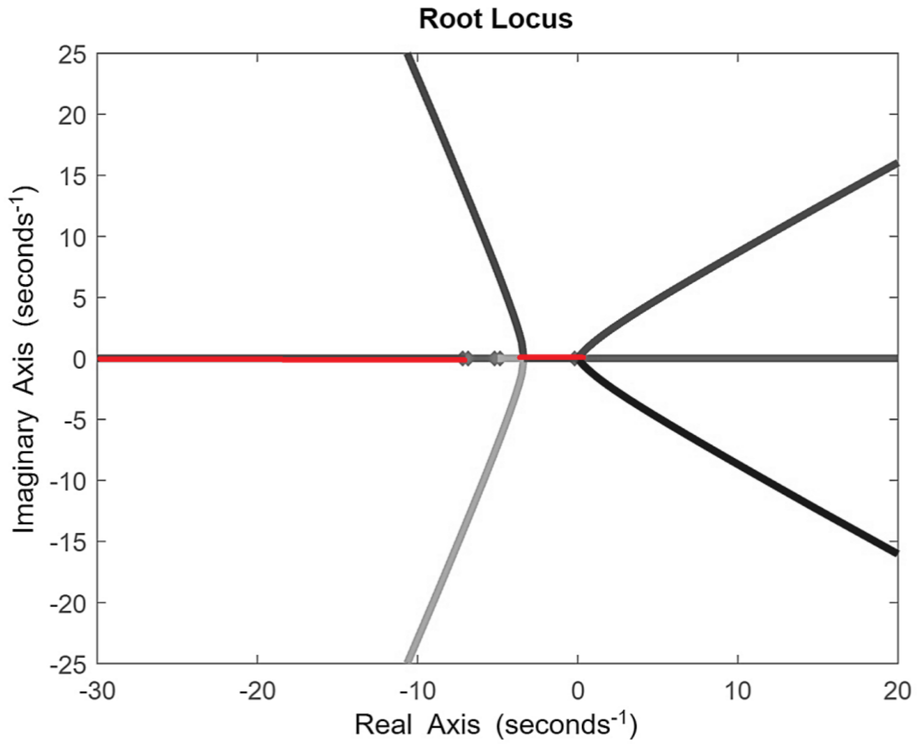
For the characteristic equation:

$$1 + \frac{K}{s^3(s+5)(s+7)} = 0$$

(a) Draw the real-axis segments of the corresponding root locus.
(b) Sketch the asymptotes of the locus for $K \to \infty$.
(c) Sketch the locus.

Solution:

(a)
The characteristic function has five poles: s = 0, 0, 0, -5, -7 and no zeros.
For Rule 2, the locus is on the real axis to the left of an odd number of poles and zeros.
That is, at -5 < s < 0 and s < -7, as the two red lines shown in the following figure.



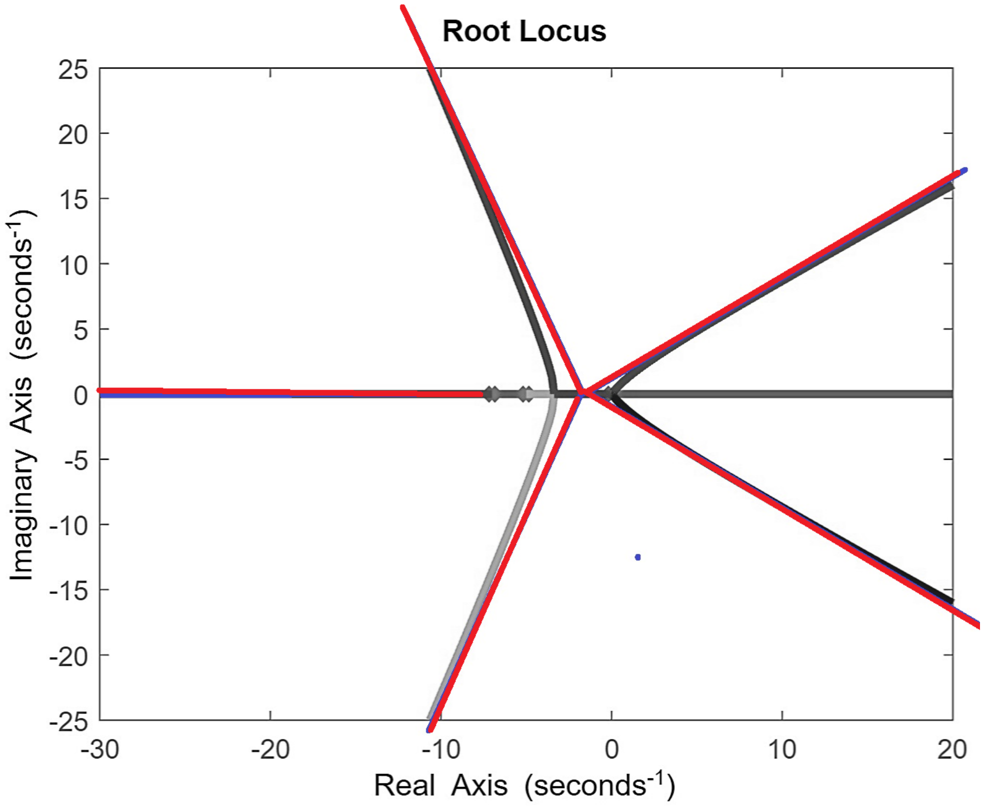
(b)
For Rule 3, n = 5, m = 0.
Thus,

$$\phi_l = \frac{180^o + 360^o (l - 1)}{n - m} = \frac{180^o + 360^o (l - 1)}{5}$$

$$= \pm 36^o, \ \pm 108^o, \ 180^o,$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n - m} = \frac{0 + 0 + 0 + (-5) + (-7) - 0}{5} = -2.4$$

There are five asymptotes centered at s = -2.4 and at the angles $\pm 36^o, \ \pm 108^o, \ 180^o$.
As the five red lines shown in the following figure.



(c)

For Rule 4, the branches depart from the pole at s = 0 (multiplicity = 3) at the angles:

$$q\ \phi_{l,dep} = \sum \psi_i - \sum_{i \neq l,dep} \phi_i - 180^o - 360^o(l-1)$$

$$3\ \phi_{l,dep} = -180^o - 360^o(l-1)$$

$$\phi_{l,dep} = \pm 60^o,\ 180^o$$

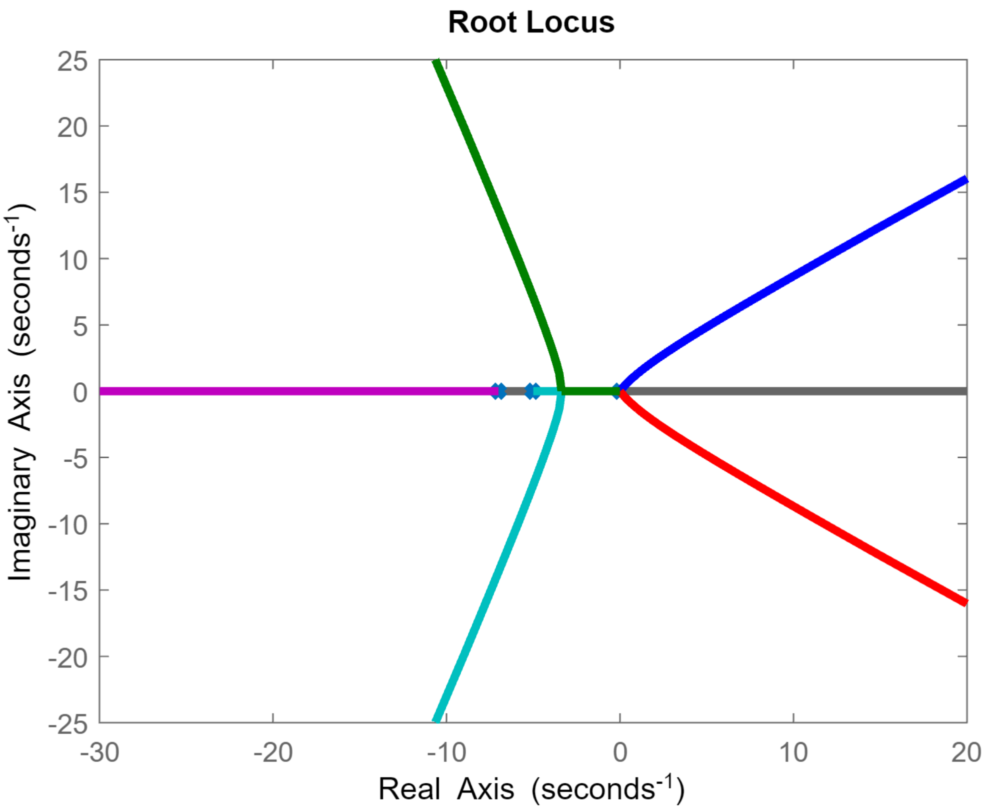
Another branch departs from s = -5 (multiplicity = 1) at the angle:

$$\phi_{l,dep} = 3 \times 180^o - 180^o - 360^o(l-1)$$
$$= 0^o,$$

The other branch departs from s = -7 (multiplicity = 1) at the angle:

$$\phi_{l,dep} = 4 \times 180^o - 180^o - 360^o(l-1)$$
$$= 180^o,$$

The locus is shown in the following figure.



(2) (20%=4%+4%+4%+8%), (By B08901120 鍾銀香)

Consider the following four functions:

$$D_1(s) = (s+10) \qquad D_2(s) = (s+1000)$$

$$D_3(s) = (s^2+0.8s+3600) \qquad D_4(s) = (s+60)^2$$

(a) Please sketch the Bode plots of $D_1(s)$ and $D_2(s)$.

(b) Please sketch the Bode plot of $D_a(s) = \dfrac{D_1(s)}{D_2(s)}$.

Is it a lead or lad compensator? Justify your answer.

(c) Please sketch the Bode plot of $D_b(s) = \dfrac{D_2(s)}{D_1(s)}$.

Is it a lead or lad compensator? Justify your answer.

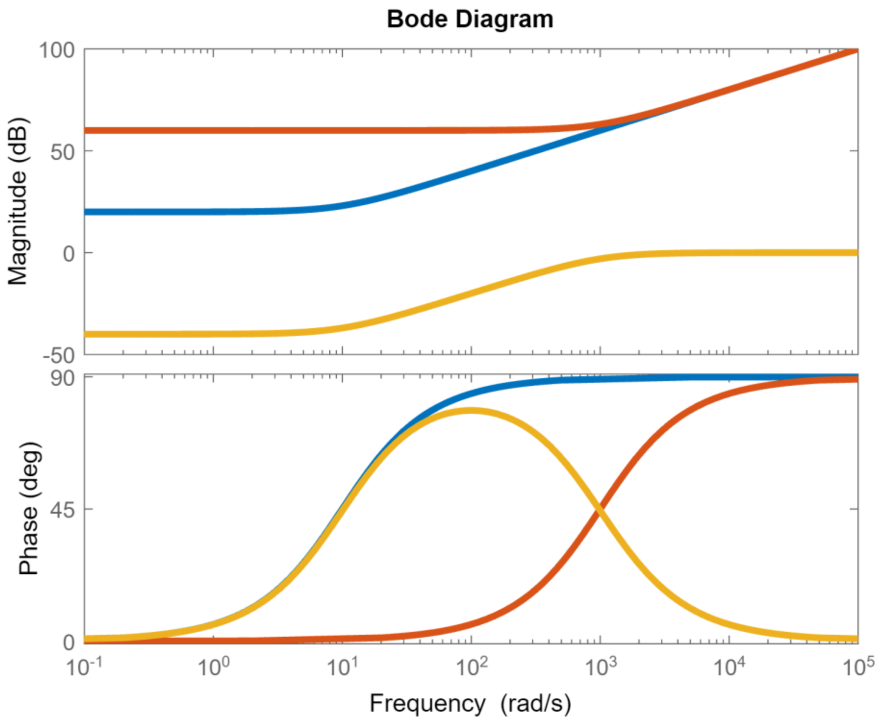
(d) Please plot the Bode plots of $D_3(s)$ $D_4(s)$ and $D_c(s) = \dfrac{D_3(s)}{D_4(s)}$.

What is the special name for the $D_c(s)$ compensator?

Solution:

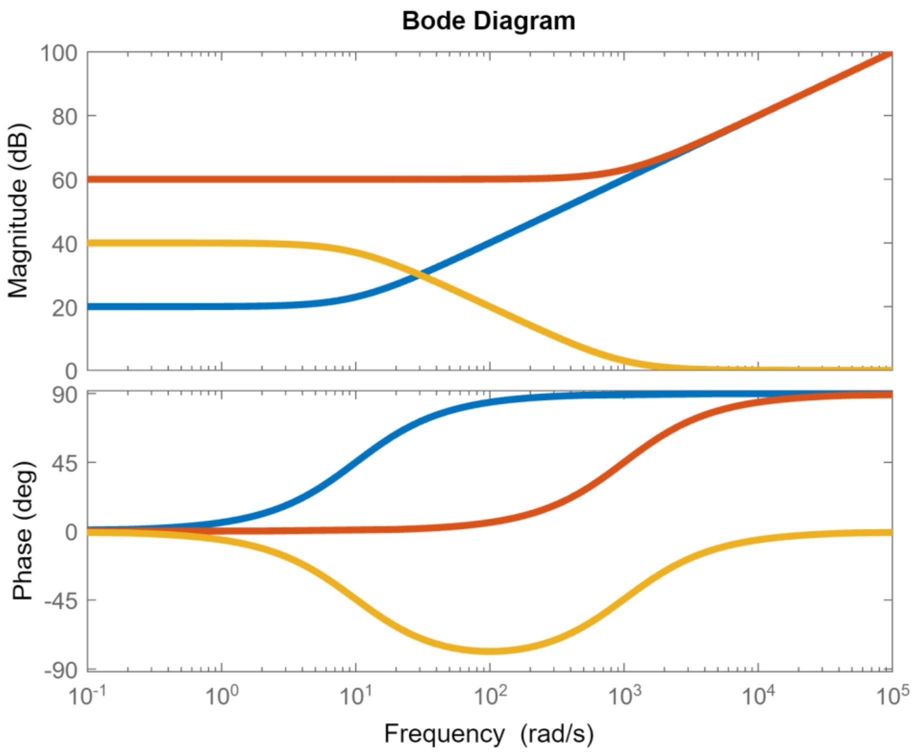
(a) (b)

The Bode plots of *D1(s), D2(s), Da(s)* are shown in the following plots. *Da(s)* is a lead compensator because it can increase phase margin.



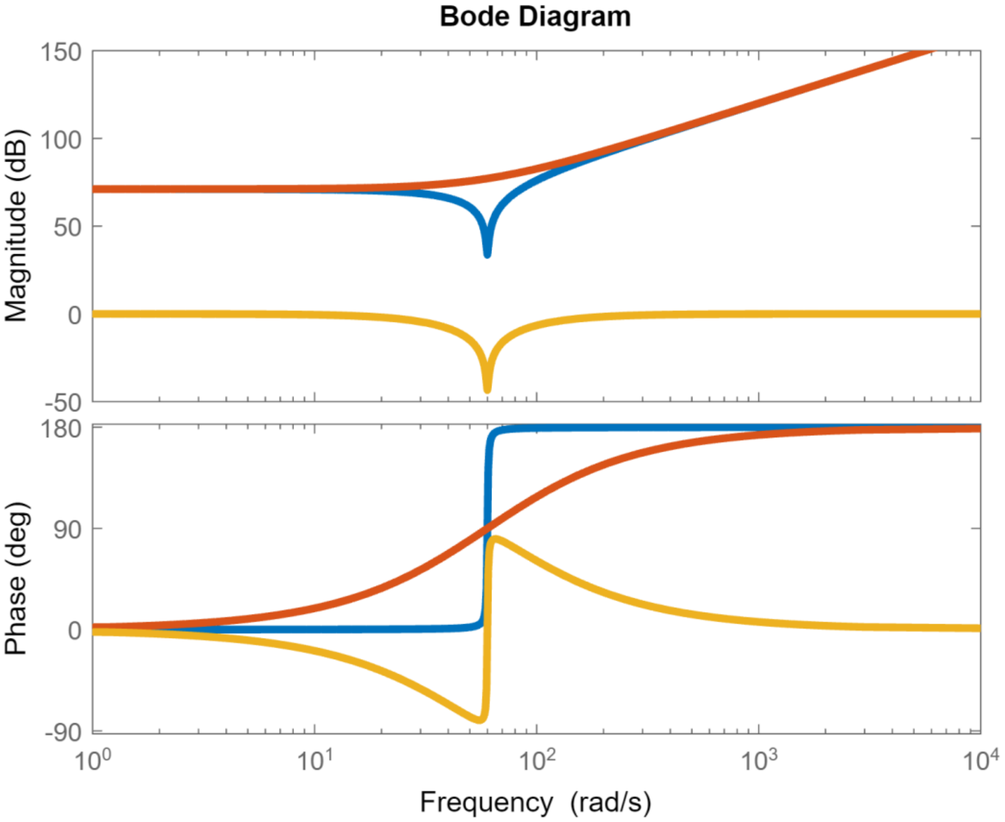
(a) (c)

The Bode plots of *D1(s), D2(s), Db(s)* are shown in the following plots. *Da(s)* is a lag compensator because it can decrease phase margin.



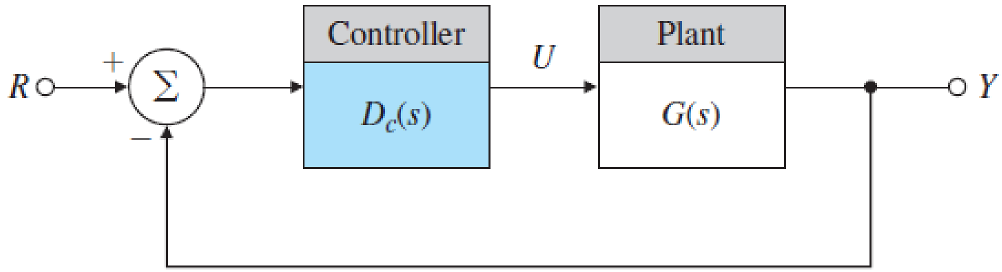
(d)

The Bode plots of *D3(s), D4(s), Dc(s)* are shown in the following plots. *Dc(s)* is a notch compensator because it is cable of filtering one specific frequency of oscillation, i.e., at 60 rad/sec in this case.



(3) (20%=5%+5%+10%), (By B08901111 簡宏哲)

Consider the following block diagram:



where

$$G(s) = \frac{10}{s\,(\,s^2 + 15s + 10\,)}.$$

(a) We design a lead compensator as follows:

$$D_c(s) = K\,\frac{s+z}{s+5z}, \quad K > 0.$$

Please find the minimum value of *K*, such that the velocity constant is more than (or equal) 10, i.e., $K_v \geq 10$.

(b) We design a lag compensator as follows:

$$D_c(s) = K\,\frac{s+5p}{s+p}, \quad K > 0.$$

Please find the minimum value of *K*, such that the velocity constant is more than (or equal) 10, i.e., $K_v \geq 10$.

(c) We design a PD-Type compensator as follows:

$$D_c(s) = K\,(1 + As), \quad A > 1, \; K > 0.$$

Please show that the closed-loop system is always stable using Routh's Stability Criterion.

Solution:

(a)

$$\begin{aligned} K_v &= \lim_{s\to 0} s\, D_c(s)\, G(s) \\ &= \lim_{s\to 0} s \quad K\,\frac{s+z}{s+5z} \quad \frac{10}{s\,(\,s^2+15s+10\,)} \\ &= \frac{K}{5} \quad \geq\ 10 \\ &\Rightarrow\ K \geq 50 \end{aligned}$$

So, the minimum value of K is 50.

(b)

$$\begin{aligned} K_v &= \lim_{s\to 0} s\, D_c(s)\, G(s) \\ &= \lim_{s\to 0} s \quad K\,\frac{s+5p}{s+p} \quad \frac{10}{s\,(\,s^2+15s+10\,)} \\ &= 5K \quad \geq\ 10 \\ &\Rightarrow\ K \geq 2 \end{aligned}$$

So, the minimum value of K is 2.

(c)

The characteristic equation is:

$$1 + D_c(s)\, G(s) = 0$$

$$\Rightarrow\ 1 + K\,(1+As)\,\frac{10}{s\,(\,s^2+15s+10\,)} = 0$$

$$\Rightarrow\ s^3 + 15s^2 + (10+10AK)\,s + 10K = 0$$

Routh's Stability Criterion:

s^3:	1	10 + 10 A K
s^2:	15	10 K
s:	a1	0
1:	a2	0

where

$$a1 = -\frac{1}{15}\begin{vmatrix} 1 & 10+10AK \\ 15 & 10K \end{vmatrix} = \frac{1}{15}(150+K(150A-10)),$$

$$a2 = -\frac{1}{a1}\begin{vmatrix} 15 & 10K \\ a1 & 0 \end{vmatrix} = 10K.$$

Because $K > 0$ and $A > 1$, then, the four numbers: 1, 15, a1, and a2 are all > 0. So, the closed-loop system is stable.

(4) (20%=10%+10%), (By B08901095 邱泓翔)

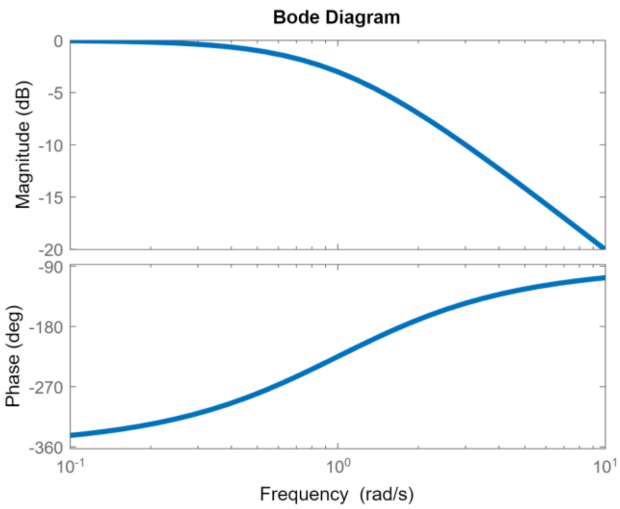
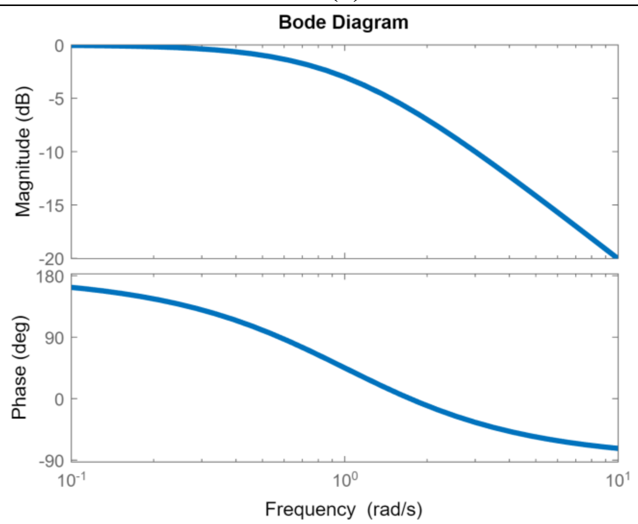
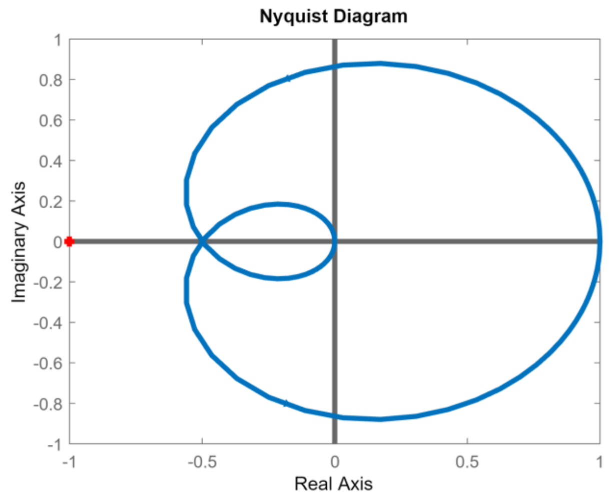
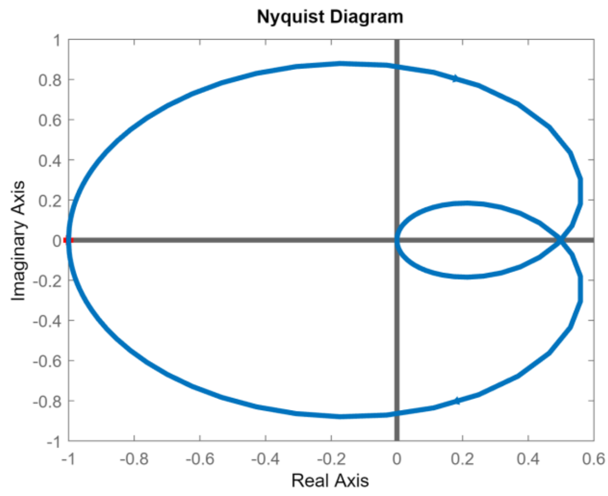
Consider the following two transfer functions:

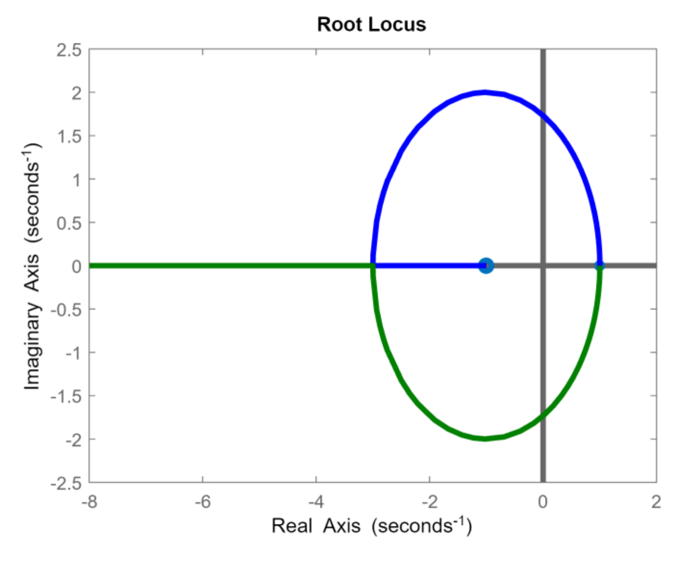
$$G_1(s) = \frac{s+1}{(s-1)^2} \qquad G_2(s) = \frac{s-1}{(s+1)^2}$$

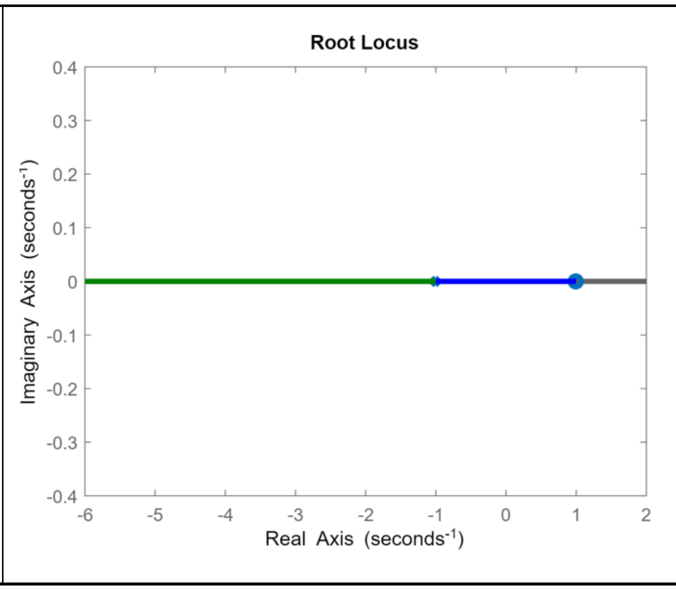
The Bode plot, Nyquist plot, root locus plot of these two transfer functions are shown in the following plots. Please find the detailed answers for the following two questions.

(a) For *G1*(s), please use the above plots to determine the ranges of *K* in *K* >0 for which *KG1*(s) is STABLE or UNSTABLE.

(b) For *G2*(s), please use the above plots to determine the ranges of *K* in *K* >0 for which *KG2*(s) is STABLE or UNSTABLE.

G1(s)	*G2*(s)
	
	





Solution:

(a) For *G1(s)*

The curve is the case when $K = 1$ and it crosses the real axis at -0.5 and 1.
From the Nyquist plot we can observe that

(1) $-\frac{1}{K} < -\frac{1}{2}$

In this case, we have $0 < K < 2, N = 0, P = 2$, so $Z = 2$.
That is, when $0 < K < 2$, the system is unstable and there are two closed-loop roots in RHP.

(2) $-\frac{1}{2} < -\frac{1}{K} < 0$

In this case, we have $K > 2, N = -2, P = 2$, so $Z = 0$.
That is, when $K > 2$, the system is stable and there are no closed-loop roots in RHP.

(3) $0 < -\frac{1}{K} < 1$

In this case, we have $K < -1, N = -1, P = 2$, so $Z = 1$.
That is, when $K < -1$, the system is unstable and there is one closed-loop root in RHP.

(4) $1 < -\frac{1}{K}$

In this case, we have $-1 < K < 0, N = 0, P = 2$, so $Z = 2$.
That is, when $-1 < K < 0$, the system is unstable and there are two closed-loop roots in RHP.

(e) We can use Routh's criterion to verify that the closed-loop system of KG_1 is stable if $K > 2$. The steps are shown in Figure 5.

Closed-loop transfer function:

$$\frac{KG_1(s)}{1+KG_1(s)} = \frac{K\frac{s+1}{(s-1)^2}}{1+K\frac{s+1}{(s-1)^2}} = \frac{K(s+1)}{s^2-(2-K)s+(1+K)}$$

Characteristic equation: $s^2-(2-K)s+(1+K)=0$

Using Routh's criterion:

s^2:	1	$1+K$
s^1:	$-2+K$	0
s^0:	$1+K$	

For stability, $-2+K>0$ and $1+K>0$.

That is, $K>2$ and $K>-1$.

$\therefore$ $K>2 \Rightarrow$ stable, otherwise $\Rightarrow$ unstable.

Figure 5. The steps of using Routh's criterion to verify the condition for stability of $KG_1(s)$.

(b) For *G2(s)*

The curve is the case when $K = 1$ and it crosses the real axis at 0.5 and -1.

From the Nyquist plot we can observe that

(1) $0 < K < 1$

In this case, we have $N = 0, P = 0$, so $Z = 0$.

That is, the system is stable and there are no closed-loop roots in RHP.

(2) $K > 1$

In this case, we have $N = 1, P = 0$, so $Z = 1$.

That is, the system is unstable and there is one closed-loop root in RHP.

(3) $K < -2$

In this case, $N = 2, P = 0$, so $Z = 2$.

That is, the system is unstable and there are two closed-loop roots in RHP.

(4) $-2 < K < 0$

In this case, we have $N = 0, P = 0$, so $Z = 0$.

That is, the system is stable and there are no closed-loop roots in RHP.

(e) We can use Routh's criterion to verify that the closed-loop system of KG_2 is stable if $-2 < K < 1$. The steps are shown in Figure 10.

Closed-loop transfer function:

$$\frac{KG_2(s)}{1+KG_2(s)} = \frac{K\frac{s-1}{(s+1)^2}}{1+K\frac{s-1}{(s+1)^2}} = \frac{K(s-1)}{s^2+(2+K)s+(1-K)}$$

Characteristic equation: $s^2+(2+K)s+(1-K)=0$

Using Routh's criterion:

$$\begin{array}{lll} s^2: & 1 & 1-K \\ s^1: & 2+K & 0 \\ s^0: & 1-K & \end{array}$$

For stability, $2+K>0$ and $1-K>0$.

That is, $-2<K<1$.

$\therefore$ $-2<K<1 \Rightarrow$ stable, otherwise $\Rightarrow$ unstable

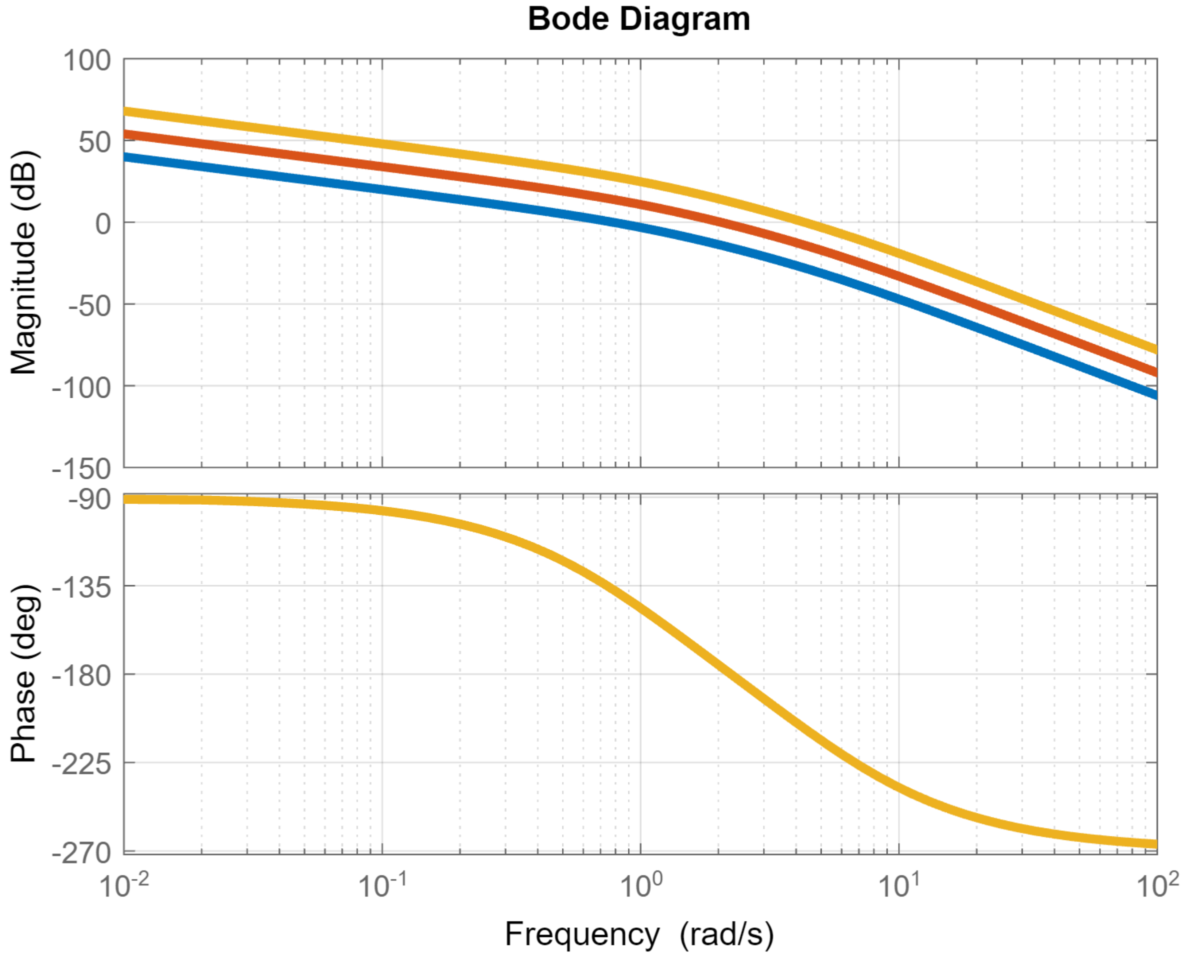
Figure 10. The steps of using Routh's criterion to verify the condition for stability of $KG_2(s)$.

(5) (20%=10%+10%), (By B08502048 陳杰伸)

For the unity feedback system with the following function:

$$K\,G(s) = \frac{K}{s\,(s+1)\,(\frac{s}{5}+1)}.$$

The Bode plots of the above function with *K=1, K=5* and *K=25* are shown as follows:



(a) Please design a lead compensator such that $K_v = 5$ and $\text{PM} \geq 40^o$.

(b) Please design a lag compensator such that $K_v = 5$ and $\text{PM} \geq 40^o$.

Solution:

(a)

For $K_v = 5$, we have $K_v = \lim_{s\to 0} s\,K\,G(s) = K$.

Therefore, $K = 5$.

From the Bode plots, we can roughly identify the phase margin for the case of $K=5$ is about 5 degrees at $w = 2$ rad/sec.

Determine a phase lead = 40 + 10 + 5 = 55.

Looking at the plot of Max Phase Lead and 1/α,

we choose 1/α=10 and a zero at $w\sqrt{\alpha} = 2 \times \sqrt{0.1} = 0.6$ or around z=1.

Pick the pole at (1/α)*zero = 10*1 = 10.

So, design the lead compensator as follows:

$$D_c(s) = \frac{\frac{s}{1}+1}{\frac{s}{10}+1} = 10\,\frac{s+1}{s+10}.$$

Or, aggressively, we can choose 1/α=20 to have more phase margin. The new lead compensator is as follows:

$$D_c(s) = \frac{\frac{s}{1}+1}{\frac{s}{20}+1} = 20\,\frac{s+1}{s+20}$$

(b)

Looking at the Bode plots, to have Phase Margin = 50, it is at about $w = 0.6$ rad/sec. Also, the Gain at this frequency is about 15dB or we can use Gain = 20dB. The gain is selected as $K = 5/10 = 0.5$ to reduce the gain.

Hence, choose the zero at 0.6/10 = 0.06 rad/sec.

For $K_v = 5$

$$K_v = \lim_{s\to 0} s\,K\,D_c(s)\,G(s)$$

$$= \lim_{s\to 0} s\,K\,\alpha\,\frac{T_I\,s+1}{\alpha\,T_I\,s+1}\,\frac{1}{s\,(s+1)\,(\frac{s}{5}+1)}$$

$$5 = K\,\alpha$$

So,

$$\alpha = \frac{5}{0.5} = 10.$$

Then, choose the pole at (1/α)*zero = 0.1*0.06 = 0.006.

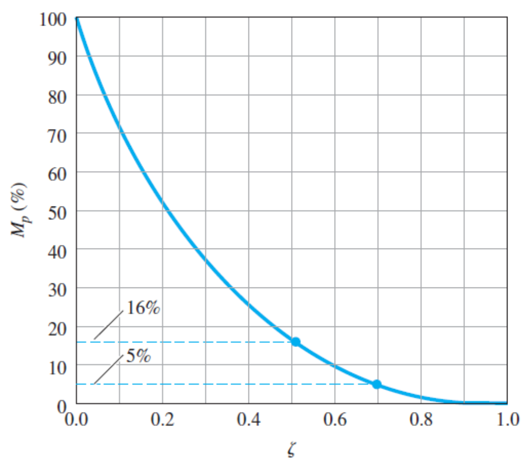
So, design the lag compensator as follows:

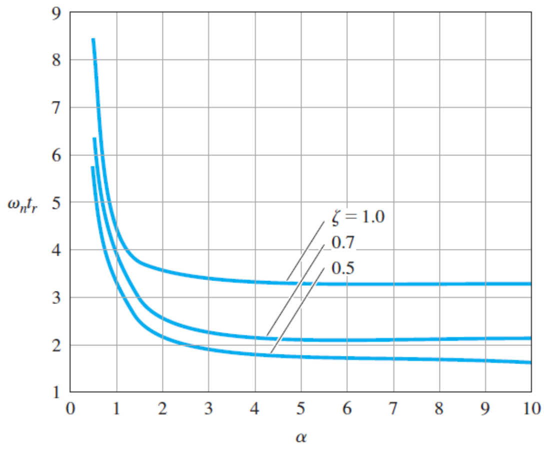
$$D_c(s) = 10\,\frac{\frac{s}{0.06}+1}{\frac{s}{0.006}+1} = 1\,\frac{s+0.06}{s+0.006}.$$

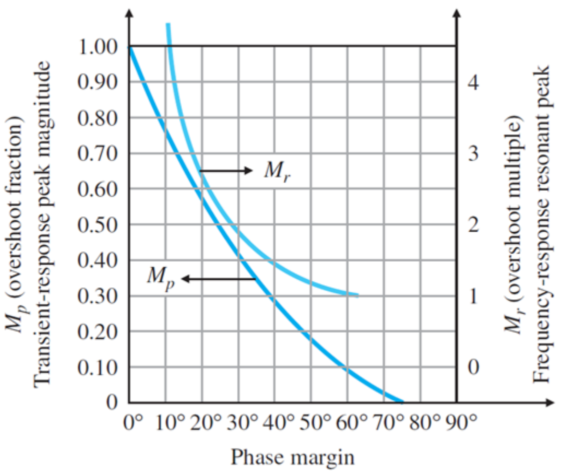
[Helpful Information]

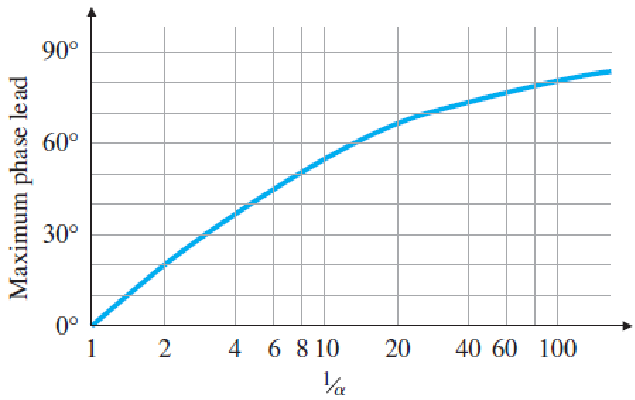
$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem

$$t_r \cong \frac{1.8}{w_n},\quad t_p = \frac{\pi}{w_d},\quad M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}},\quad t_s = \frac{4.6}{\zeta w_n} = \frac{4.6}{\sigma}$$









Errors as a Function of System Type

Type Input	Step (position)	Ramp (velocity)	Parabola (acceleration)
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$