Fall 2022 (111-1)

控制系統
Control Systems

Unit 6A
Frequency Response

Feng-Li Lian
NTU-EE

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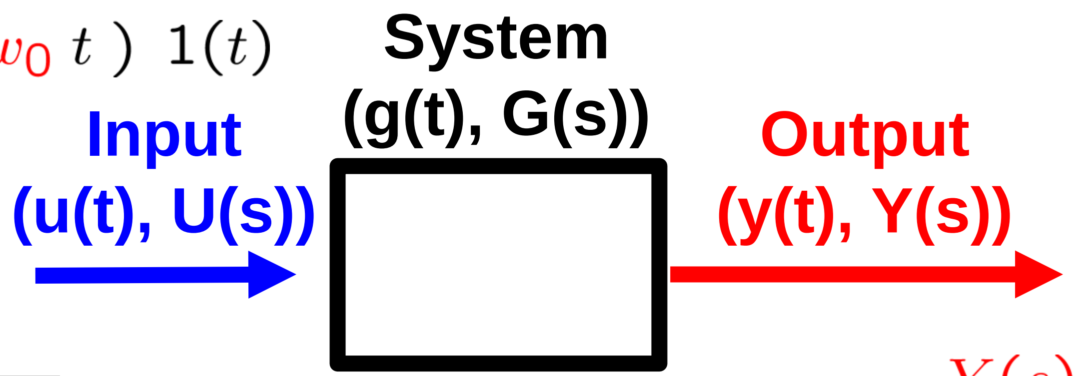
The System's Frequency Response:

- A linear system's response to sinusoidal inputs
- Can be obtained from the knowledge of its pole and zero locations.

$$u(t) = A \sin(w_0 t)\, 1(t)$$



$$U(s) = \frac{A\, w_0}{s^2 + w_0^2}$$

$$\frac{Y(s)}{U(s)} = G(s)$$

$$Y(s) = G(s) \frac{A\, w_0}{s^2 + w_0^2}$$

$$= \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \cdots + \frac{\alpha_n}{s - p_n} + \frac{\alpha_0}{s + j w_0} + \frac{\alpha_0^*}{s - j w_0}$$

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_n e^{p_n t} + 2\,|\alpha_0|\,\cos(\,w_0\,t + \phi\,)$$

for $t \geq 0$

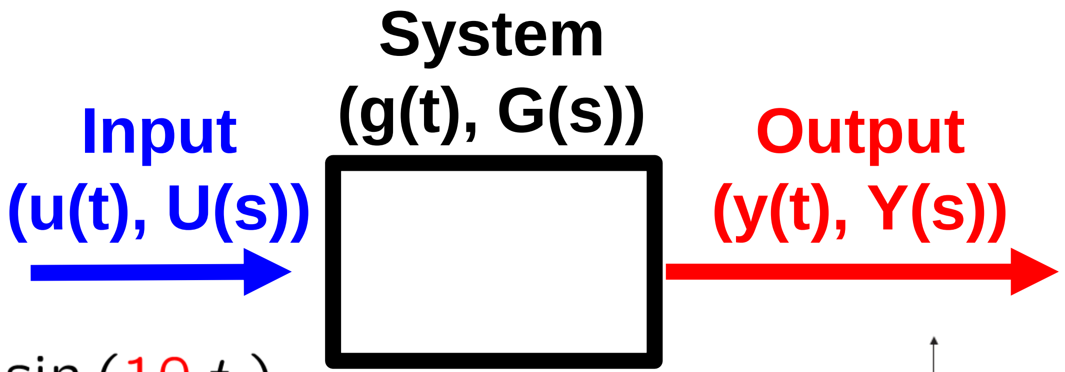
$$\phi = \tan^{-1}\left[\frac{\text{Im}(\alpha_0)}{\text{Re}(\alpha_0)}\right]$$

- If all the poles of the system represent stable behavior

  (the real parts of $p_1, p_2, \ldots, p_n < 0$),
- the natural unforced response will die out eventually,
- and therefore the steady-state response of the system

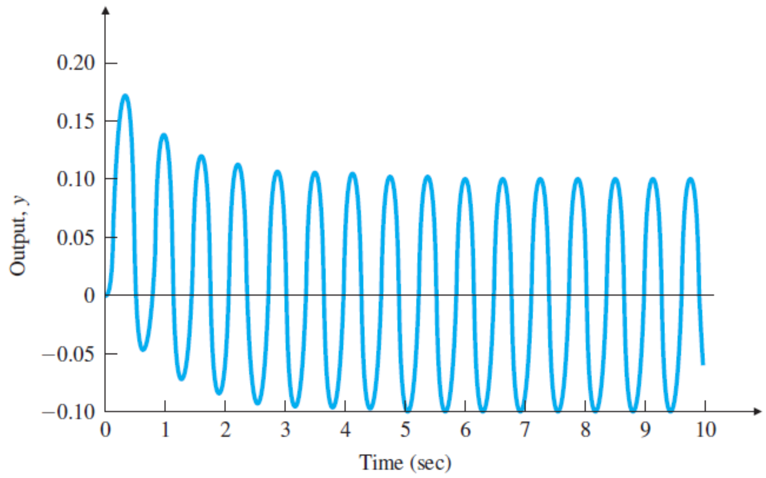
  will be due solely to the sinusoidal term.

System Response and Frequency Response



$$u(t) = \sin(10\,t)$$

$$G(s) = \frac{1}{s+1}$$



$$Y(s) = \frac{1}{s+1}\,\frac{10}{s^2 + 10^2}$$

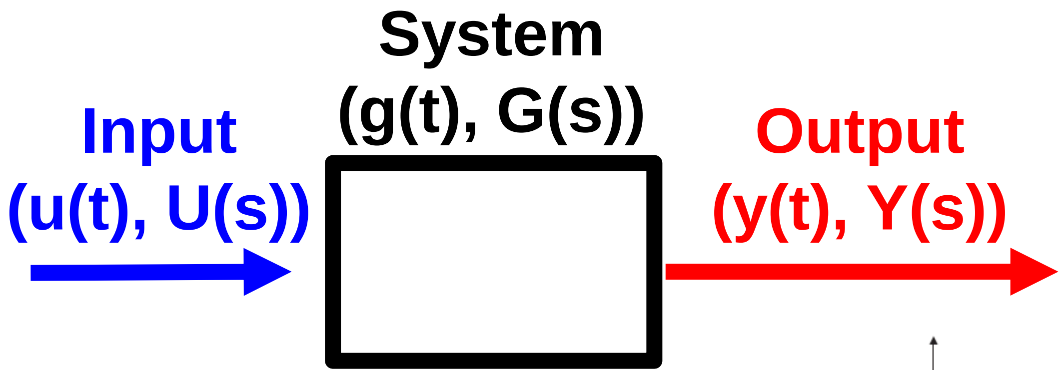
$$= \frac{\frac{10}{101}}{s+1} + \frac{\frac{j}{2(1-j10)}}{s+j10} + \frac{\frac{-j}{2(1+j10)}}{s-j10}$$

$$y(t) = \frac{10}{101}\,e^{-t} + \frac{1}{\sqrt{101}}\,\sin(10t + \phi)$$

$$\phi = \tan^{-1}(-10)$$

$$= -84.2^{o}$$

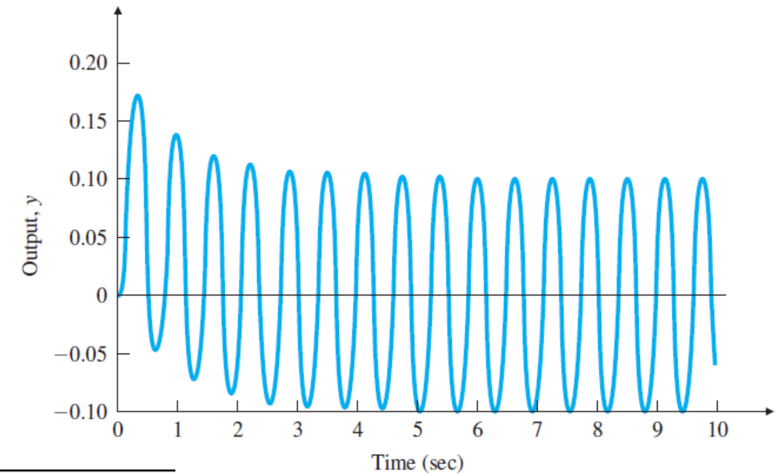
System Response and Frequency Response



$$Y(s) = G(s)\,U(s)$$

$$u(t) = A\,\sin(w_0\,t)$$

$$y(t) \rightarrow A\,M\,\sin(w_0\,t + \phi)$$



$$M = |G(jw_0)| = |G(s)|_{s=jw_0}$$

$$= \sqrt{\{\mathrm{Re}[G(jw_0)]\}^2 + \{\mathrm{Im}[G(jw_0)]\}^2}$$

$$\phi = \tan^{-1}\left[\frac{\mathrm{Im}[G(jw_0)]}{\mathrm{Re}[G(jw_0)]}\right] = \angle G(jw_0)$$

$$G(jw_0) = M\,e^{j\,\phi} = M(w_0)\,e^{j\,\phi(w_0)}$$

- Magnitude
- Phase

- Example 6.1:

  Frequency-Response Characteristics of a Capacitor

$$i = C \frac{dv}{dt}$$

$$G(s) = \frac{I(s)}{V(s)} = C\, s$$

$$G(jw) = C\, j\, w$$

$$M = |\, C\, j\, w\, | = C\, w$$

$$\phi = \angle G(C\, j\, w) = 90^o$$

- Magnitude
- Phase

- Example 6.2:
  Frequency-Response Characteristics of a Lead Compensator

$$D_c(s) = K \frac{Ts + 1}{\alpha T s + 1}, \quad \alpha < 1$$

$$D_c(s) = K \frac{s + z}{s + p}$$

z < p

$$D_c(jw) = K \frac{T(jw) + 1}{\alpha T(jw) + 1}$$

- Frequency: Low vs High

$$M = |D_c| = |K| \frac{\sqrt{1 + (Tw)^2}}{\sqrt{1 + (\alpha T w)^2}} \qquad |K| \qquad |K/\alpha|$$

$$\phi = \angle(1 + jwT) - \angle(1 + j\alpha w T) \qquad 0 \qquad + \qquad 0$$

$$= \tan^{-1}[wT] - \tan^{-1}[\alpha w T]$$

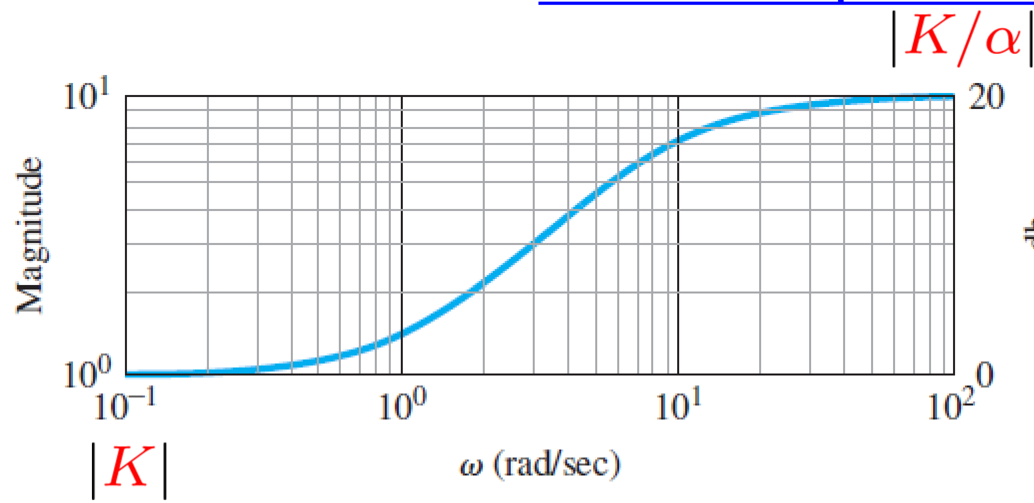
- Example 6.2:
  Frequency-Response Characteristics of a Lead Compensator

$\alpha = 1/10$

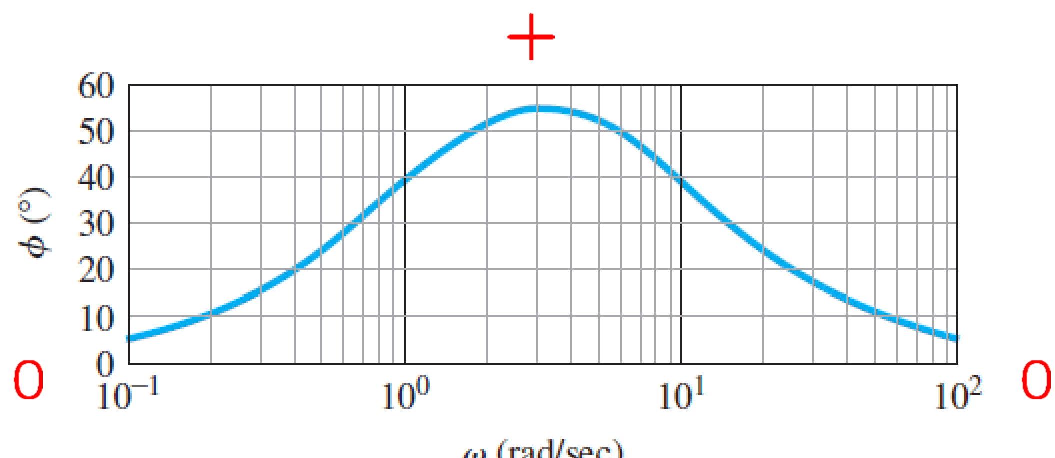
```
sysD = (s+1)/(s/10+1);
w = logspace(-1,2);
[mag,ph] = bode( sysD, w );

loglog( w, squeeze(mag) );
```

$$|K| \, \frac{\sqrt{1+(Tw)^2}}{\sqrt{1+(\alpha T w)^2}}$$

$$\tan^{-1}[wT] \; - \; \tan^{-1}[\alpha w T]$$



(a)



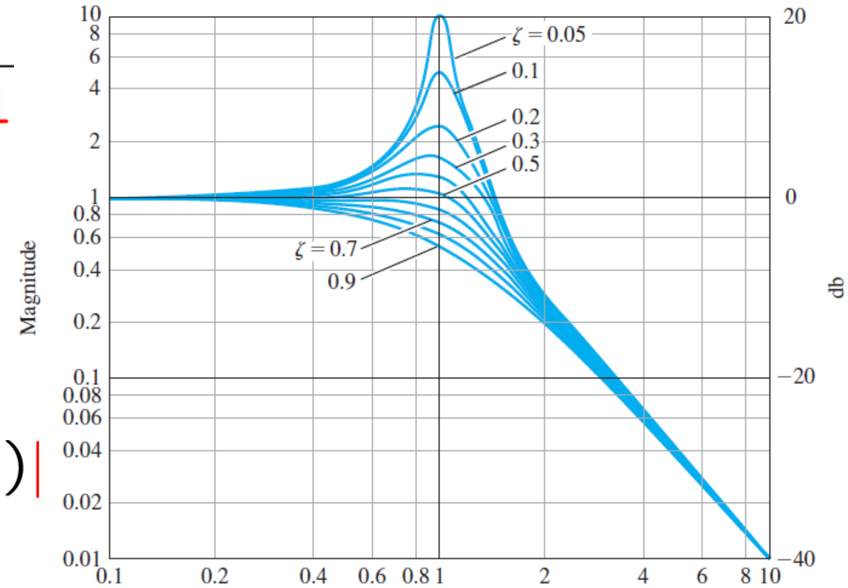
(b)

General Form of 2nd-Order System

$$G(s) = \frac{1}{(s/w_n)^2 + 2\,\zeta\,(s/w_n) + 1}$$

Peak Overshoot $\approx \frac{1}{2\zeta}$,
For $\zeta < 0.5$

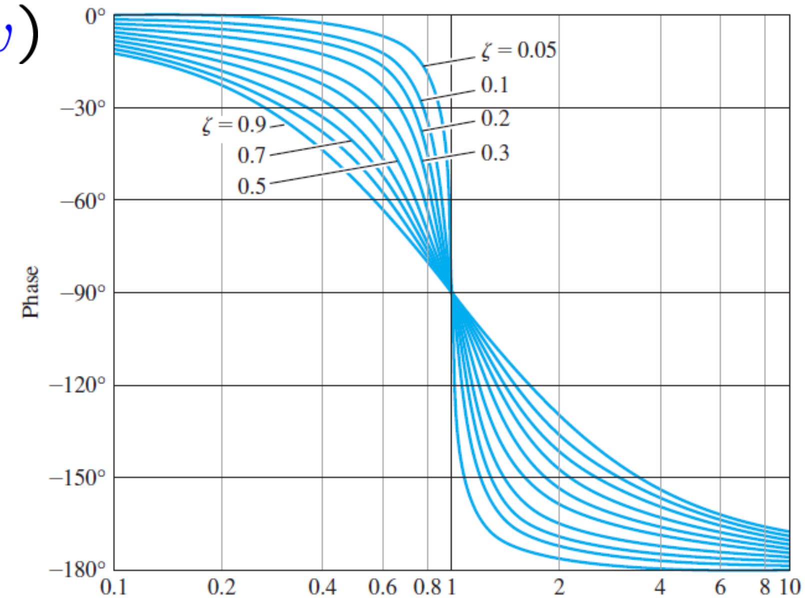
$|\,G(s = jw)\,|$



w/w_n

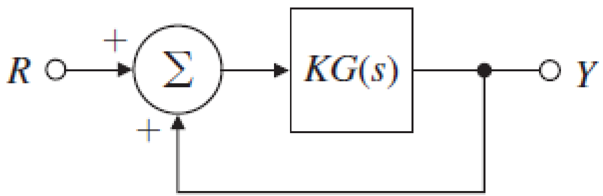
```
w = logspace(-1,1,200);
zeta = 0.05;
% set_zeta: 0.05, 0.1, …, 0.9;

numG = 1;
denG  = [1 2*zeta 1];
[ m, p ] = bode( numG, denG, w) ;

loglog( w, m );
semilogx( w, p );
```

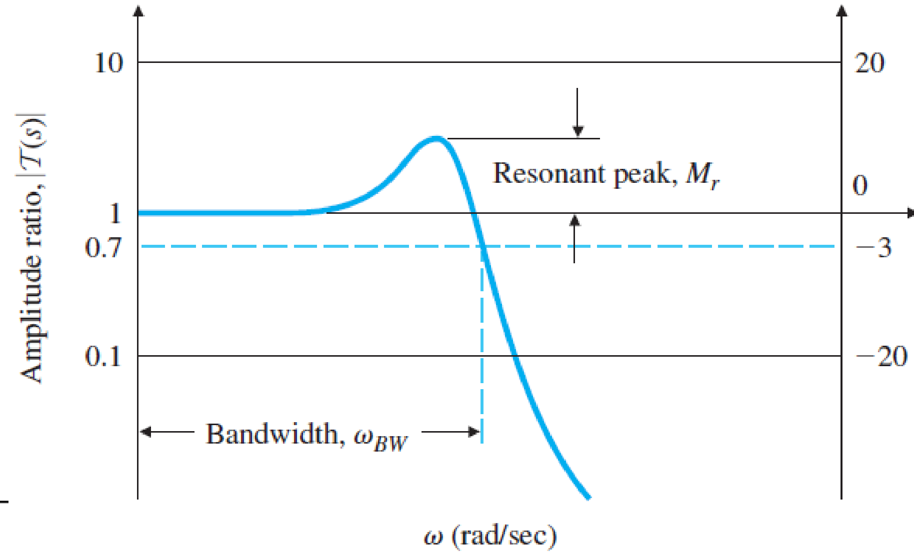
$\angle G(s = jw)$



w/w_n

General Form of 2nd-Order System



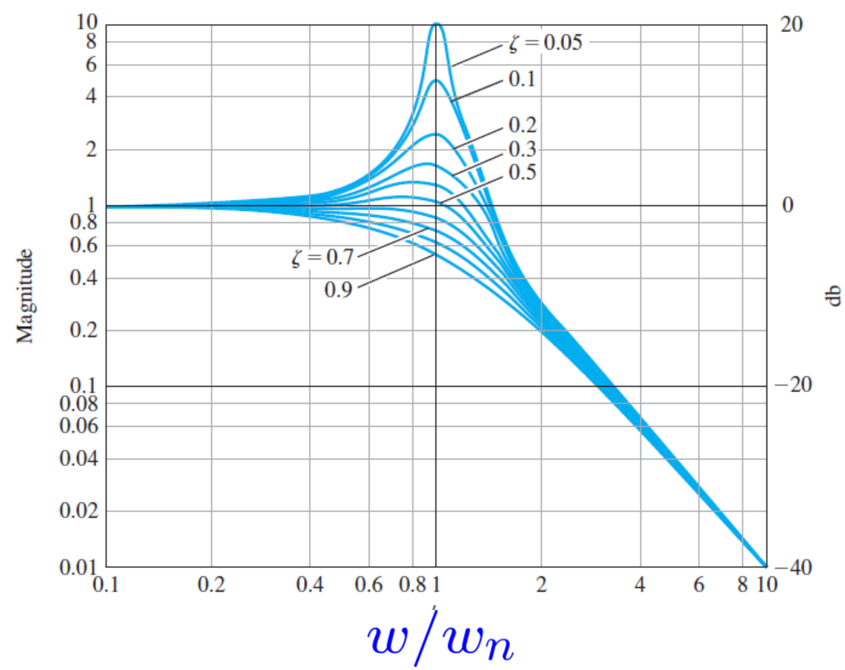


$$\mathcal{T}(s) = \frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1 + K\,G(s)}$$

- Resonant Peak: M_r
- Bandwidth: ω_{BW}

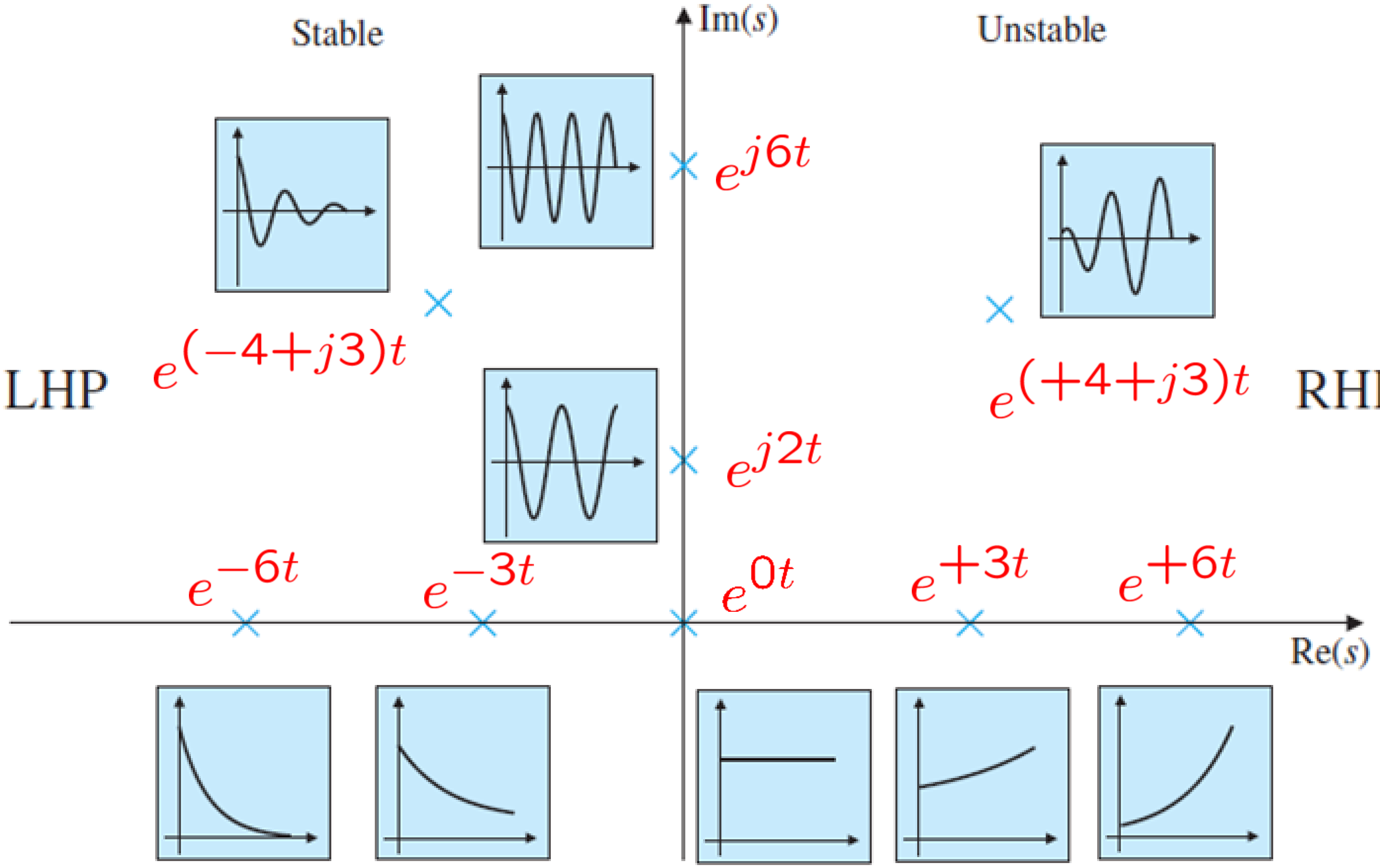
$\Rightarrow$ speed of response

$\Rightarrow$ $w_{BW} = w_n,\ \zeta = 0.7$



Time-Domain and s-Domain

- Time functions associated with points in the s-plane
  (LHP, left half-plane; RHP, right half-plane)



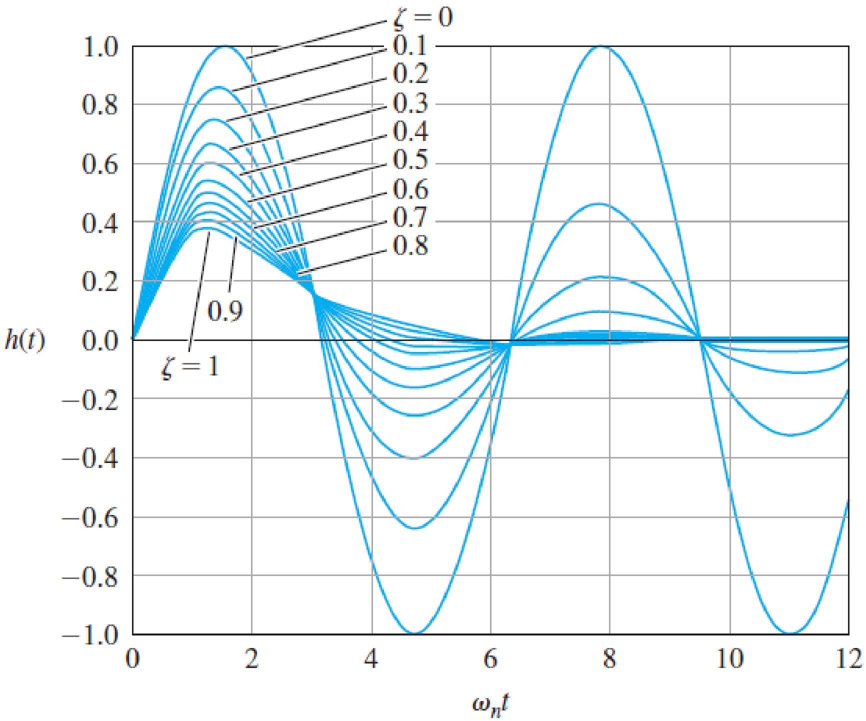
Time-Domain and s-Domain

$$H(s) = \frac{{w_n}^2}{(s+\zeta w_n)^2 + {w_n}^2(1-\zeta^2)} \qquad \sigma = w_n \, \zeta$$

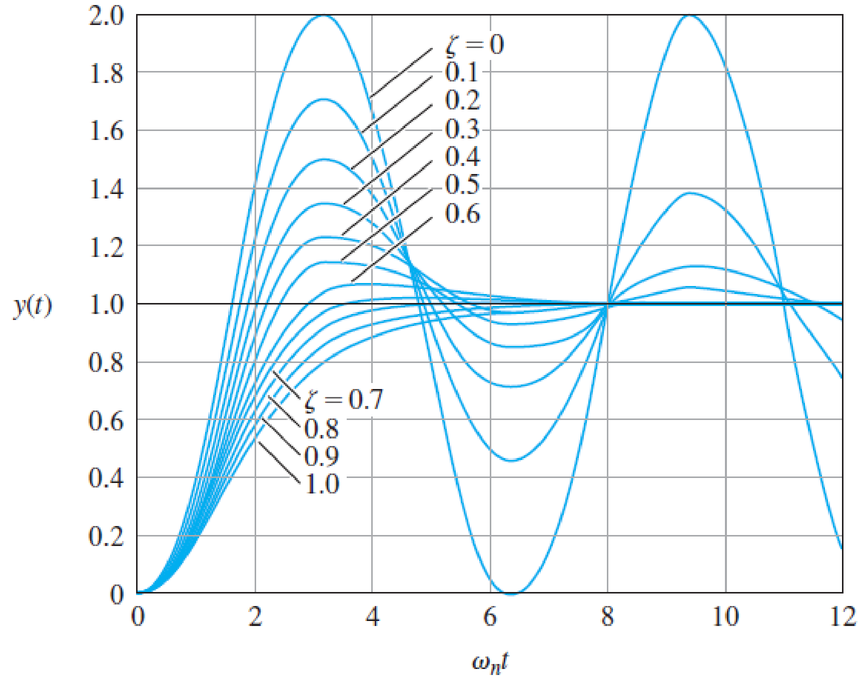
$$h(t) = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} (\sin w_d t) \, 1(t) \qquad w_d = w_n \sqrt{1-\zeta^2}$$

- Responses of second-order systems versus ζ :

(a) Impulse Responses (b) Step Responses



(a)



(b)