

Fall 2022 (111-1)

控制系統
Control Systems

Unit 5C
Selected Illustrative Root Loci

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Sep 2022 – Dec 2022

Example 0: Double Integrator

$$G(s) = \frac{1}{s^2}$$

$$\Rightarrow 1 + K L(s) = 0$$

$$\Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

- Satellite attitude, hard-disk drive, motor, etc.

With P controller

$$\Rightarrow 1 + K_P \frac{1}{s^2} = 0$$

Rule 1:

- 2 branches start at $s = 0$

Rule 2:

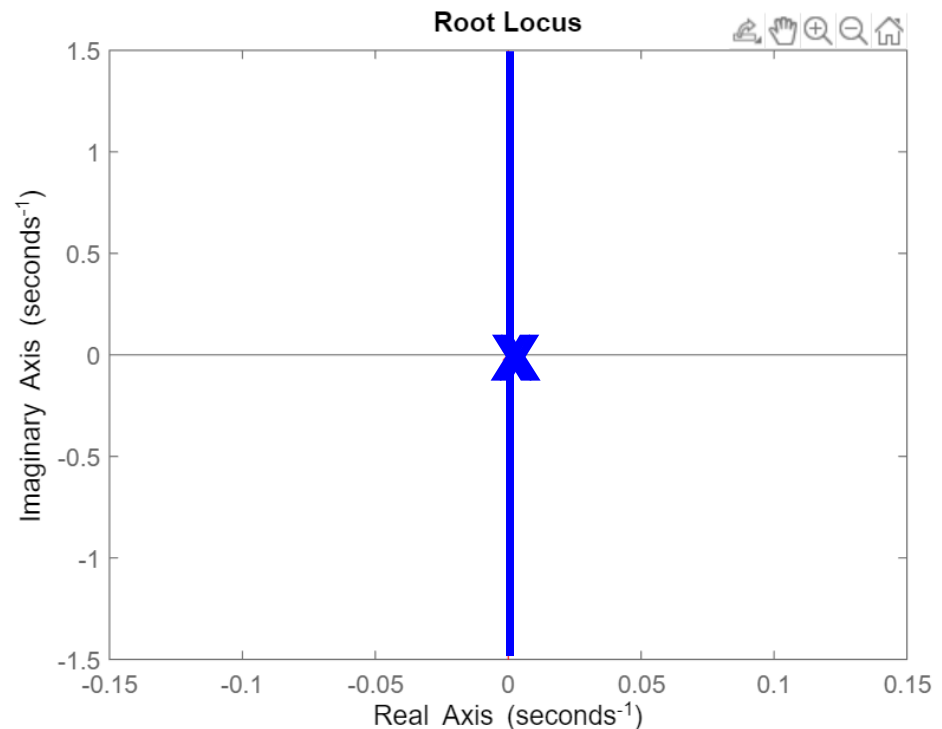
- No locus on real axis

Rule 3:

- Asymptotes: $\pm 90^\circ$

Rule 4:

- Depart at $\pm 90^\circ$

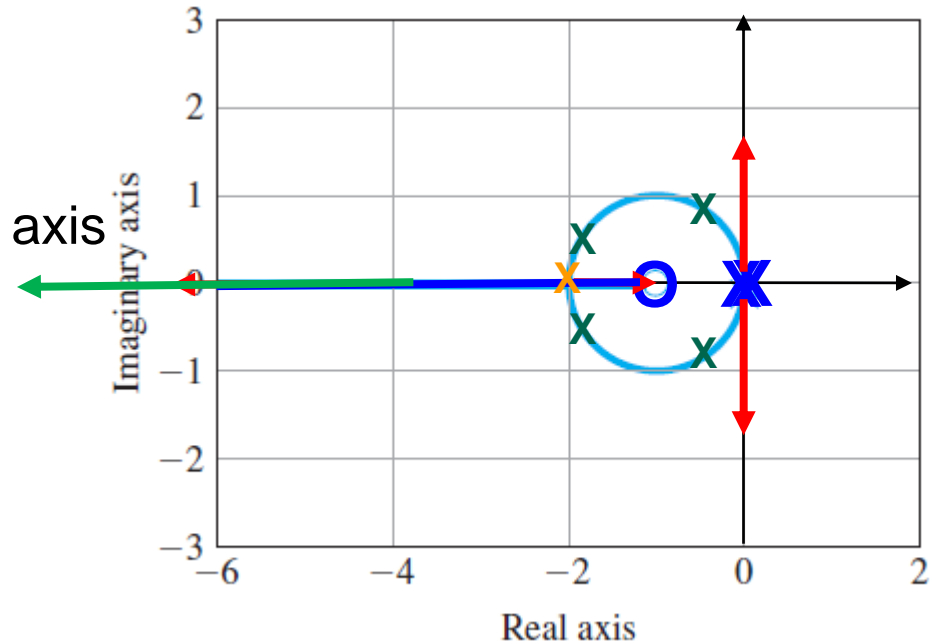


Example 5.3: Satellite Attitude Control w/ PD Control

$$\Rightarrow 1 + [K_P + K_D s] \frac{1}{s^2} = 0 \quad \Rightarrow 1 + K \frac{b(s)}{a(s)} = 0$$

$$\Rightarrow K = K_D \quad \Rightarrow \frac{K_P}{K_D} = 1 \quad \Rightarrow 1 + K \frac{s+1}{s^2} = 0$$

- Rule 1:
 - 2 branches start at $s = 0$, one to $s = -1$, one to ∞
 - Rule 2:
 - Real axis: to the left of $s = -1$
 - Rule 3:
 - Asymptotes: along negative real axis
 - Rule 4:
 - Depart at $\pm 90^\circ$
 - Rule 5:
 - Rejoin and break at $s = -2$
- Additional zero: pull the locus into the LHP



Example 5.3: Satellite Attitude Control w/ PD Control

- Practically, differentiation is not good
- Use the following approximation:

$$\Rightarrow D_c(s) = K_P + \frac{K_D s}{\frac{s}{p} + 1} = K_P + \frac{K_D p s}{s + p}$$

$$= (K_P + pK_D) \frac{s + \frac{pK_P}{K_P + pK_D}}{s + p}$$

$$\Rightarrow 1 + D_c(s) G(s) = 0$$

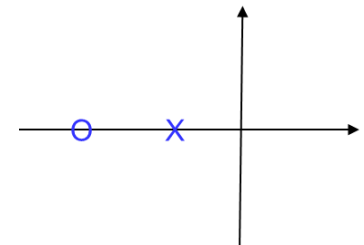
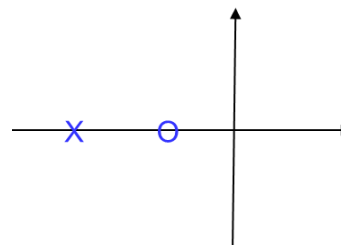
$$\Rightarrow 1 + K \frac{s + z}{s + p} = 0 \quad K = K_P + pK_D$$

$$z = \frac{pK_P}{K_P + pK_D}$$

$$\Rightarrow 1 + K \frac{(s + z)}{(s + p)} \frac{1}{s^2} = 0$$

● Lead compensator: $z < p$

● Lag compensator: $z > p$



Examples

Example 5.4: Satellite Control w/ Modified PD or Lead Compensation

$$z = 1, \quad p = 12 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 12)} = 0$$

● Rule 1:

- 3 branches, two start at $s = 0$, one at $s = -12$

● Rule 2:

- Real axis: $-12 \leq s \leq -1$

● Rule 3:

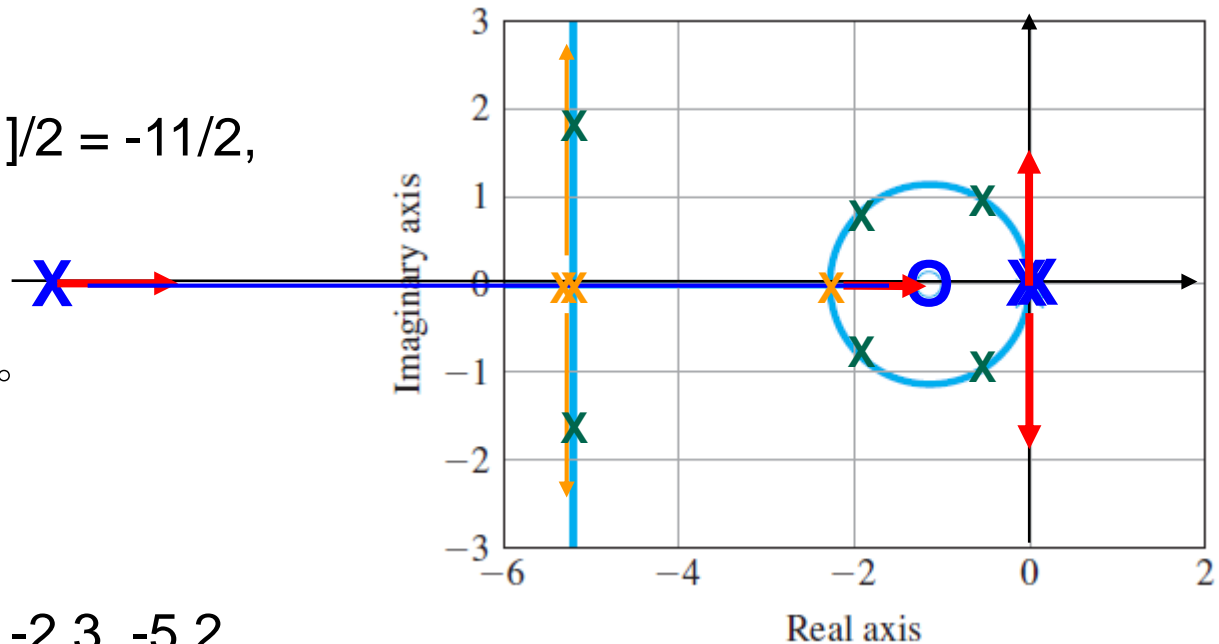
- Asymptotes: $3 - 1 = 2$, centered at $[-12 - (-1)] / 2 = -11/2$, at $\pm 90^\circ$

● Rule 4:

- Depart at $s=0$: $\pm 90^\circ$
- Depart at $s=-12$: 0°

● Rule 5:

- Multiple roots: at $s = -2.3, -5.2$



Example 5.5: Satellite Control w/ Lead Having a Relatively Small Value for the Pole

$$z = 1, \quad p = 4 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 4)} = 0$$

● Rule 1:

- 3 branches, two start at $s = 0$, one at $s = -4$

● Rule 2:

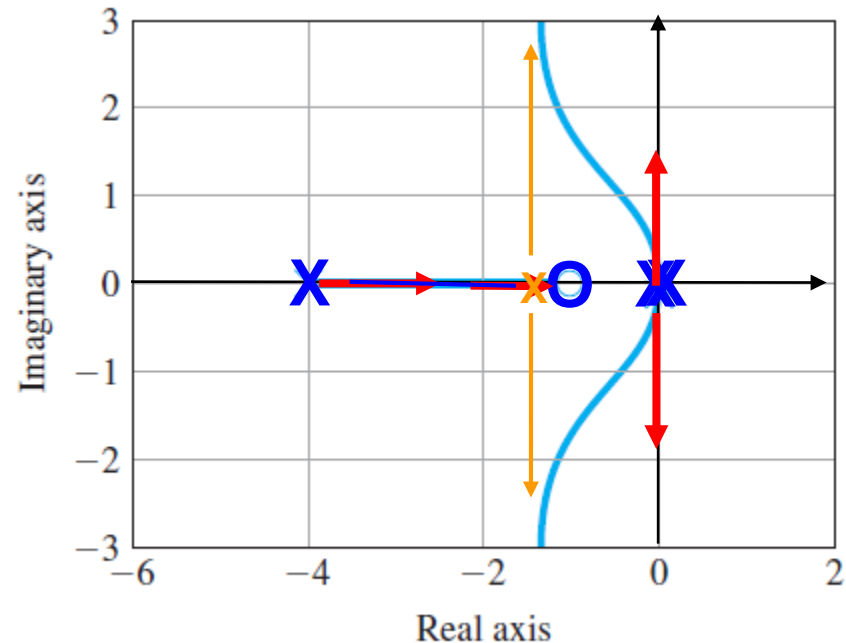
- Real axis: $-4 \leq s \leq -1$

● Rule 3:

- Asymptotes: $3 - 1 = 2$,
centered at $[-4 - (-1)] / 2 = -3/2$,
at $\pm 90^\circ$

● Rule 4:

- Depart at $s=0$: $\pm 90^\circ$



Example 5.6: Satellite w/ Transition Value for the Pole

$$z = 1, \quad p = 9 \quad \Rightarrow \quad 1 + K \frac{(s + 1)}{s^2 (s + 9)} = 0$$

● Rule 1:

- 3 branches, two start at $s = 0$, one at $s = -9$

● Rule 2:

- Real axis: $-9 \leq s \leq -1$

● Rule 3:

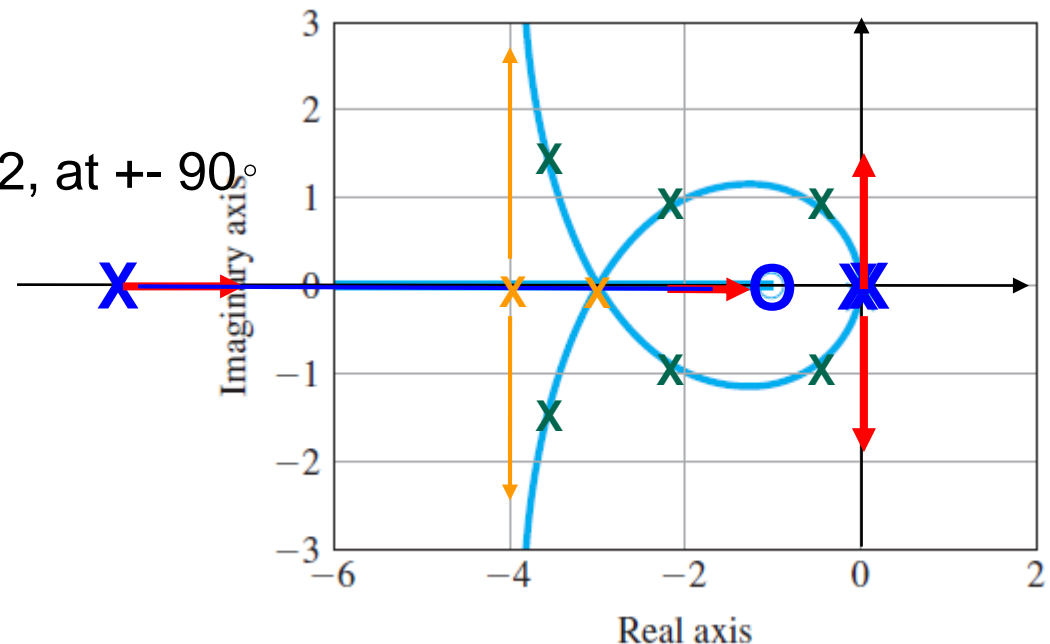
- Asymptotes: $3 - 1 = 2$,
centered at $[-9 - (-1)] / 2 = -8/2$, at $\pm 90^\circ$.

● Rule 4:

- Depart at $s=0$: $\pm 90^\circ$

● At $s = -3$,

- arrival: $+60^\circ, -60^\circ, 0^\circ$
- depart: $+120^\circ, -120^\circ, 0^\circ$



Summary of Examples 5.3 5.4, 5.5, 5.6: Satellite Control

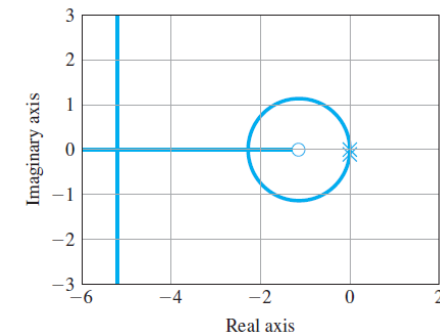
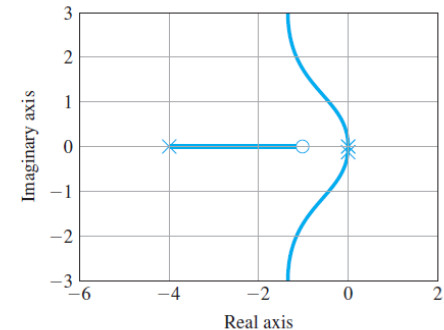
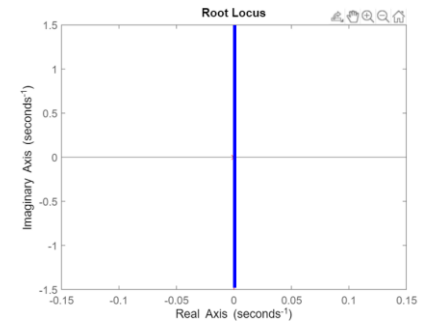
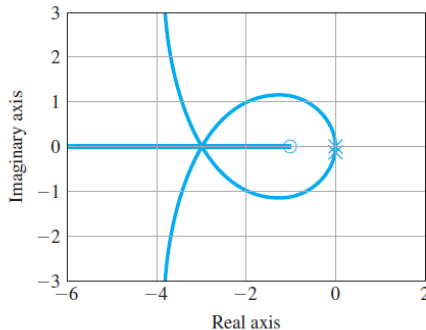
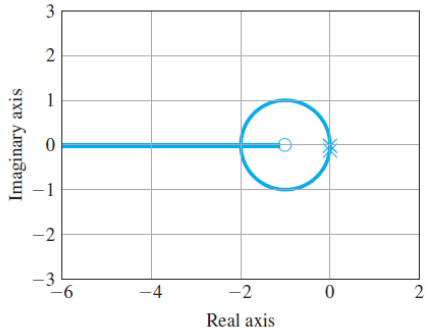
$$\Rightarrow 1 + K_P \frac{1}{s^2} = 0$$

$$\Rightarrow 1 + K \frac{s + 1}{s^2} = 0$$

$$\Rightarrow 1 + K \frac{(s + 1)}{s^2 (s + 4)} = 0$$

$$\Rightarrow 1 + K \frac{(s + 1)}{s^2 (s + 9)} = 0$$

$$\Rightarrow 1 + K \frac{(s + 1)}{s^2 (s + 12)} = 0$$

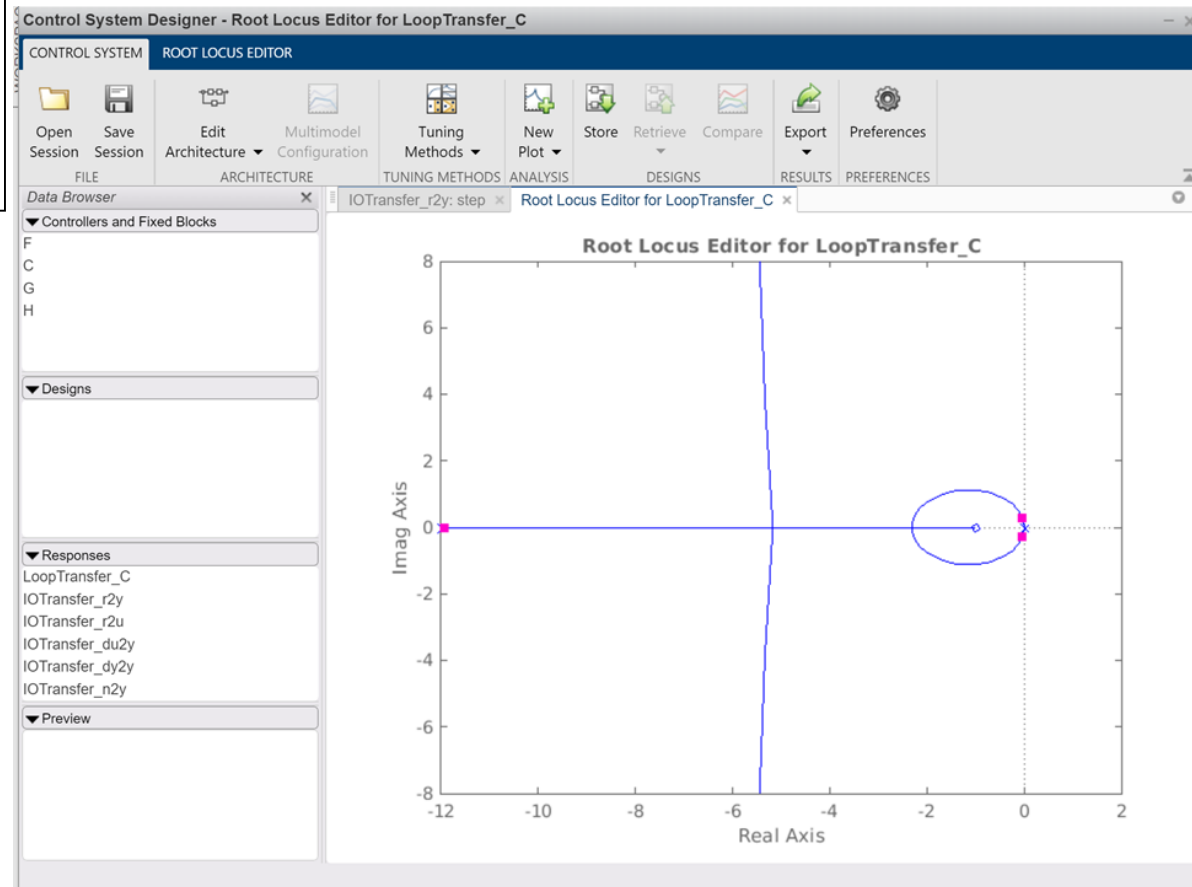


Example 5.7: Exercise to use rtool (Matlab)

```
s = tf( 's' )
```

```
sysL = ( s+1 )/(s^2);  
% sysL = (s+1)/(s^2*(s+12));
```

```
rtool( sysL );  
% sisotool( 'rlocus', sysL );
```



Example 5.8: Satellite Control w/ Collocated Flexibility

$$G(s) = \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\Rightarrow 1 + D_c(s) G(s) = 0$$

$$\Rightarrow D_c(s) = K \frac{(s + 1)}{(s + 12)}$$

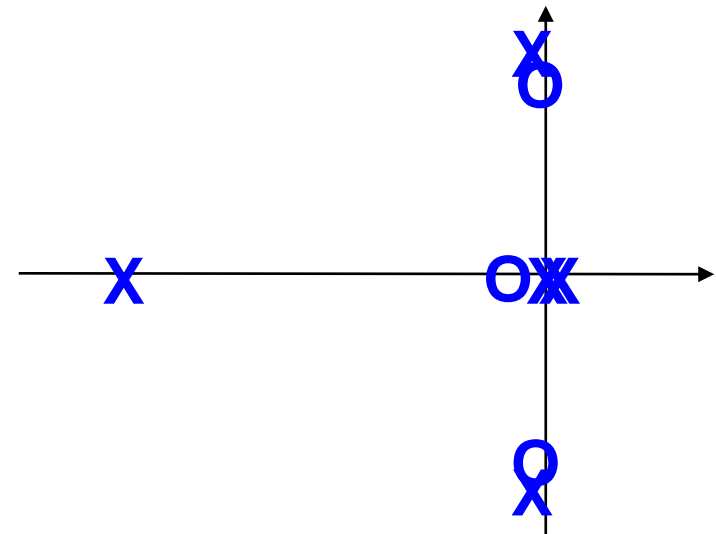
$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

● Rule 1:

- 5 branches,
- 3 -> finite zeros
- 2 -> asymptotes

● Rule 2:

- Real axis: $-12 \leq s \leq -1$



Examples

Example 5.8: Satellite Control w/ Collocated Flexibility

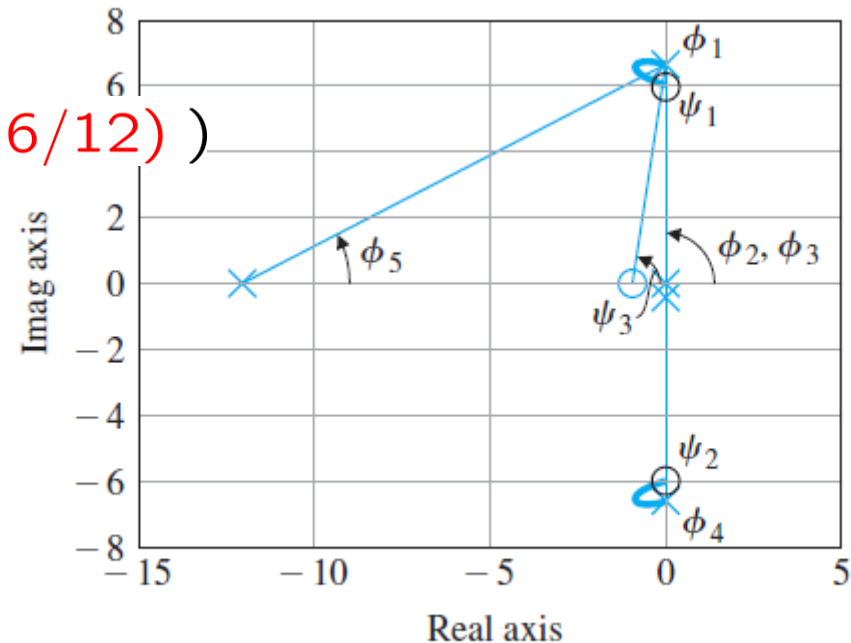
● Rule 3:

- Asymptotes: $5-3 = 2$
- Centered at $[-12 \ -0.1 \ -0.1 - (-0.1 \ -9.1 \ -1)] / 2 = -11/2$, at $\pm 90^\circ$

● Rule 4:

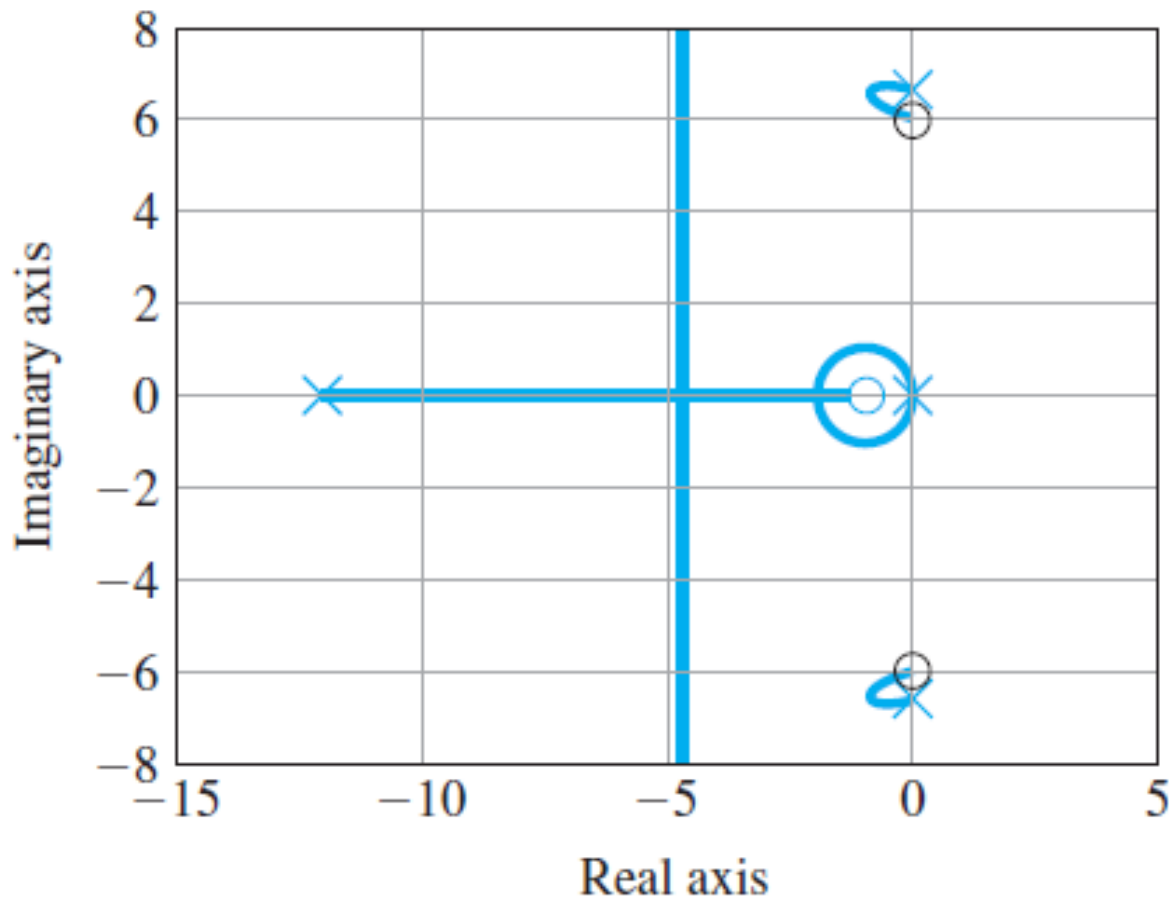
- Depart at $s = -0.1 + j6.6$:

$$\begin{aligned} \phi_1 &= \psi_1 + \psi_2 + \psi_3 - (\phi_2 + \phi_3 + \phi_4 + \phi_5) - 180 \\ &= 90 + 90 + \tan^{-1}(6.6) \\ &\quad - (90 + 90 + 90 + \tan^{-1}(6.6/12)) \\ &\quad - 180 \\ &= 142.6 \end{aligned}$$



- Example 5.8: Satellite Control w/ Collocated Flexibility

$$\Rightarrow L(s) = \frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$



Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

● Rule 1:

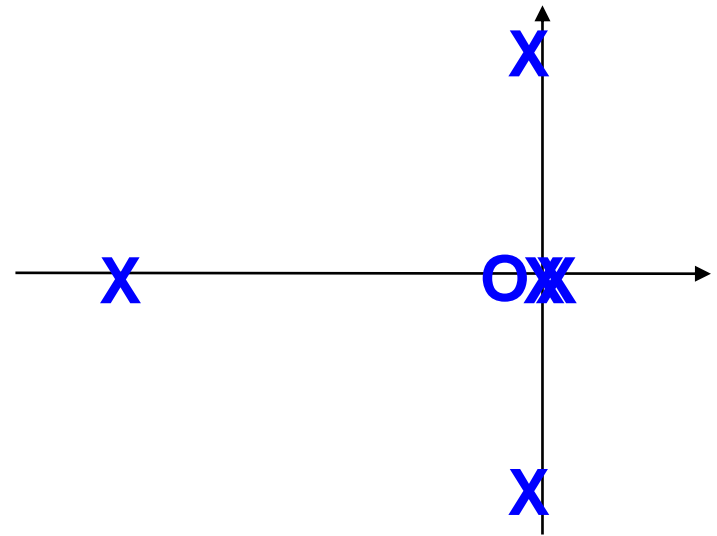
- 5 branches,
- 1 -> finite zero
- 4 -> asymptotes

● Rule 2:

- Real axis: $-12 \leq s \leq -1$

● Rule 3:

- Asymptotes: $5 - 1 = 4$,
- Centered at $[-12 - 0.2 - (-1)] / 4 = -11.2 / 4$,
at $\pm 45^\circ, \pm 135^\circ$



Examples

Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

● Rule 4:

- Depart at $s = -0.1 + j6.6$:

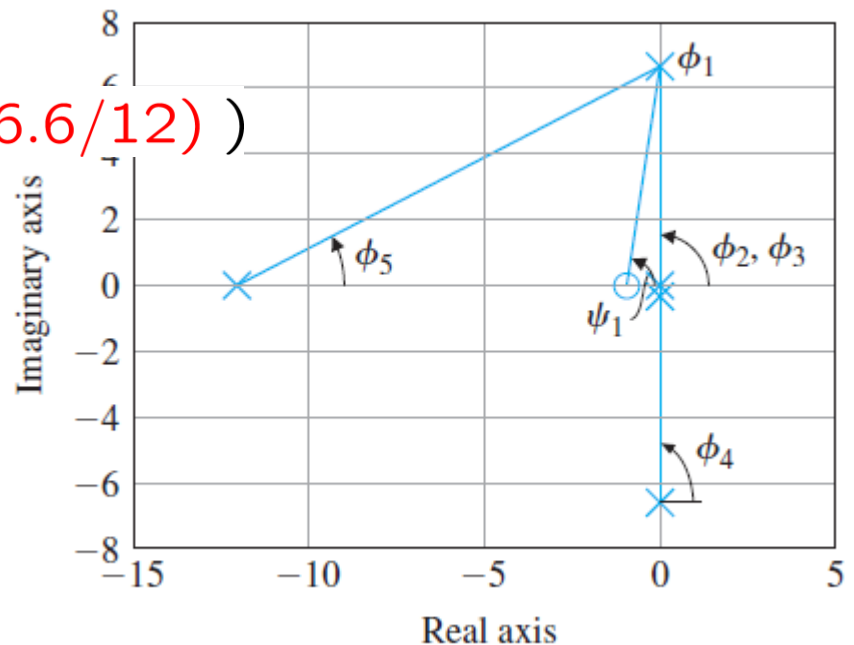
$$\phi_1 = \psi_1 - (\phi_2 + \phi_3 + \phi_4 + \phi_5) - 180$$

$$= \tan^{-1}(6.6)$$

$$- (90 + 90 + 90 + \tan^{-1}(6.6/12))$$

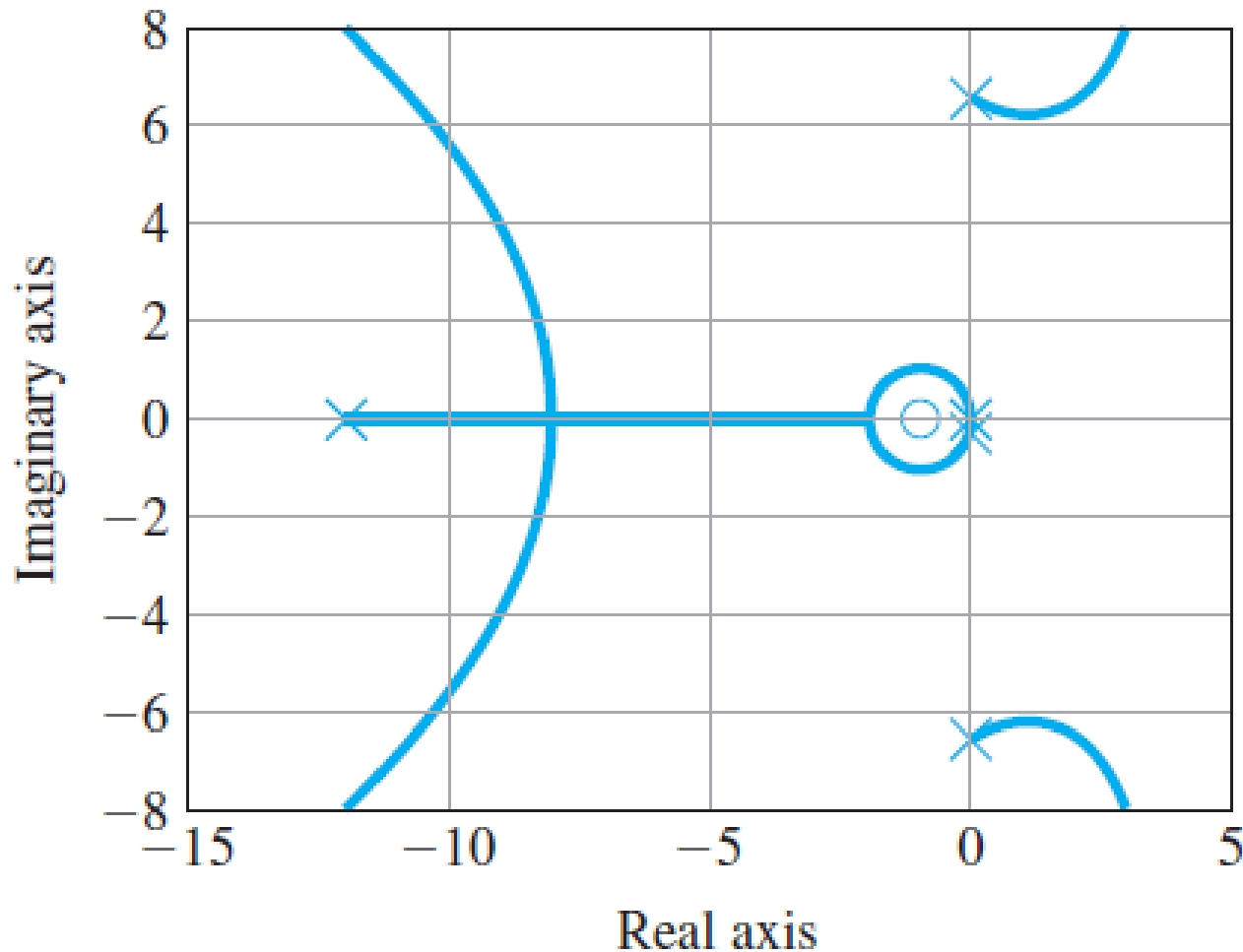
$$- 180$$

$$= -37.4$$



- Example 5.9: For Non-Collocated Case

$$\Rightarrow K L(s) = D_c(s) G(s) = K \frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$



Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

● Rule 1:

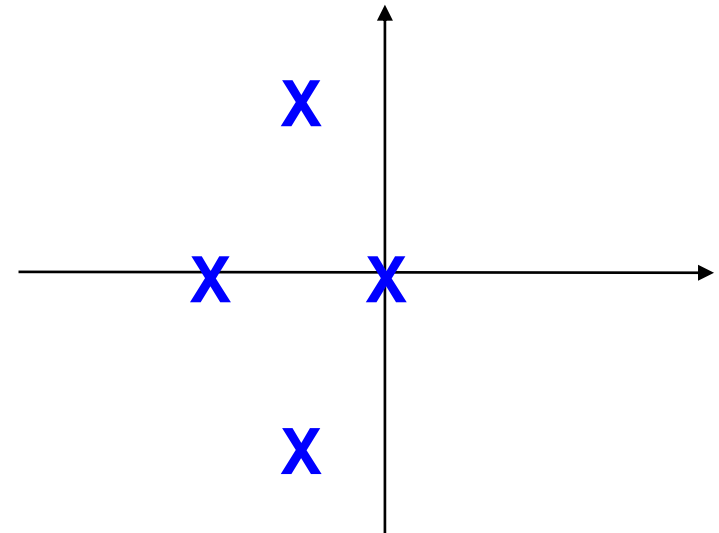
- 4 branches,
- 4 \rightarrow asymptotes

● Rule 2:

- Real axis: $-2 \leq s \leq 0$

● Rule 3:

- Asymptotes: $4 - 0 = 4$,
- Centered at $[-2 - 1 - 1 - 0 + (0)] / 4 = -1$,
at $\pm 45^\circ$, $\pm 135^\circ$



Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

● Rule 4:

- Depart at $s = -1 + j2$:

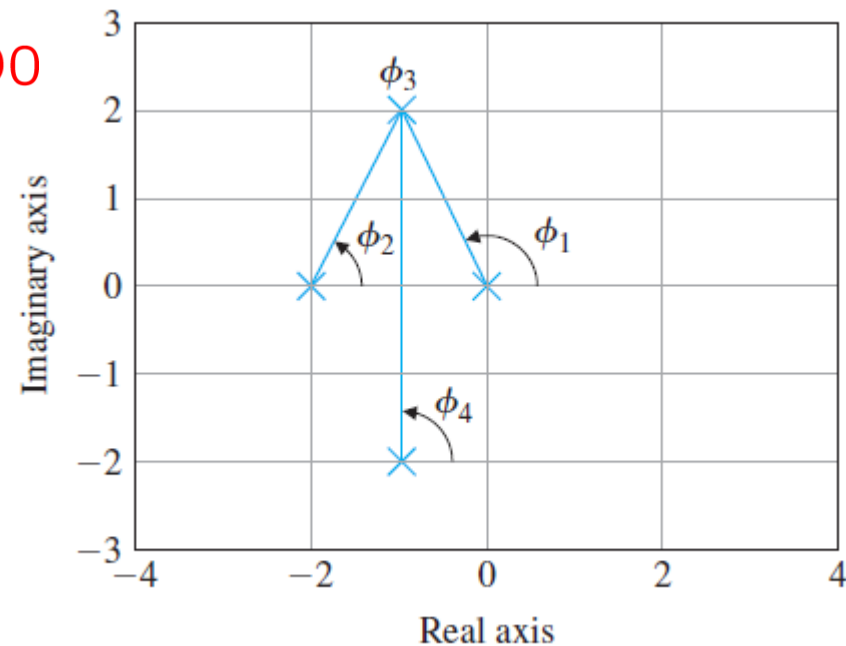
$$\phi_{dep} =$$

$$\phi_3 = -(\phi_1 + \phi_2 + \phi_4) + 180$$

$$= -\tan^{-1}\left(\frac{2}{-1}\right) - \tan^{-1}\left(\frac{2}{1}\right) - 90$$

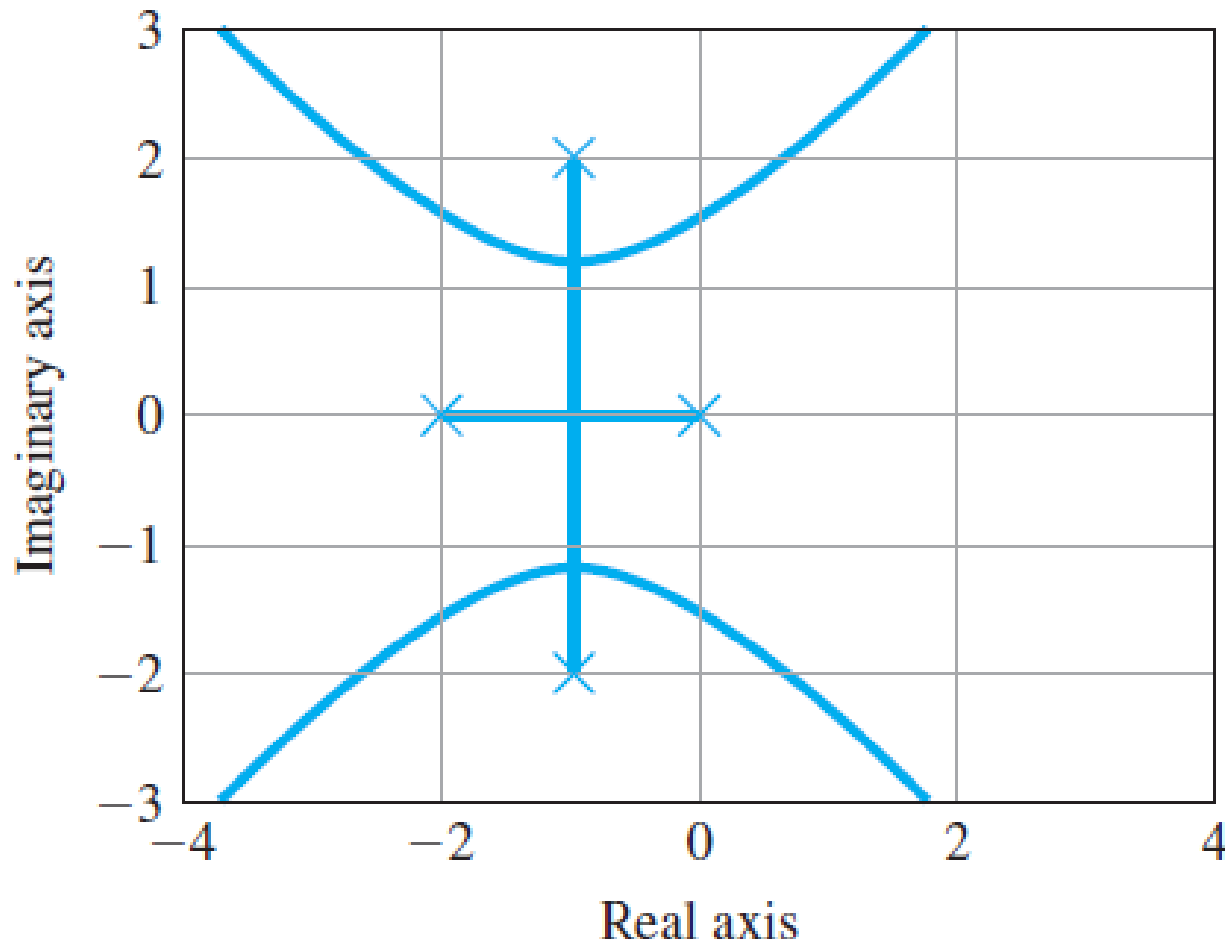
$$+ 180$$

$$= -90$$



- Example 5.10: Having Complex Multiple Roots

$$\Rightarrow 1 + K L(s) = 0 \quad \Rightarrow L(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$



Examples 5.8, 5.9, 5.10

$$\frac{(s + 1)}{(s + 12)} \frac{(s + 0.1)^2 + 6^2}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\frac{(s + 1)}{(s + 12)} \frac{1}{s^2[(s + 0.1)^2 + 6.6^2]}$$

$$\frac{1}{s(s + 2)[(s + 1)^2 + 4]}$$

