Fall 2022 (111-1)

控制系統
Control Systems

Unit 5B
Guidelines for Determining a Root Locus

Feng-Li Lian
NTU-EE

Sep 2022 – Dec 2022

Formal Definition of Root Locus

- Definition I:

$$1 + K\frac{b(s)}{a(s)} = 0$$

- The root locus is
  the set of values of s

$$L(s) = \frac{b(s)}{a(s)}$$

- for which $1 + KL(s) = 0$ is satisfied
  as the real parameter K varies from 0 to + ∞
- Typically, $1 + KL(s) = 0$ is the characteristic equation of the system, and in this case the roots on the locus are the closed-loop poles of that system.

- Definition II:

$$\Rightarrow L(s) = -\frac{1}{K}$$

- The root locus of $L(s)$ is the set of points in the s-plane
  where the phase of $L(s)$ is 180°.
- To test whether a point in the s-plane is on the locus,
  we define the angle to the test point from a zero as ψ and
  the angle to test point from a pole as ϕ as follows:

$$\sum \psi_i - \sum \phi_i = 180^o + 360^o(l - 1)$$

Formal Definition of Root Locus

- K is real and positive,
  - the phase of $L(s)$ is 180°,
  - the positive locus or 180° locus

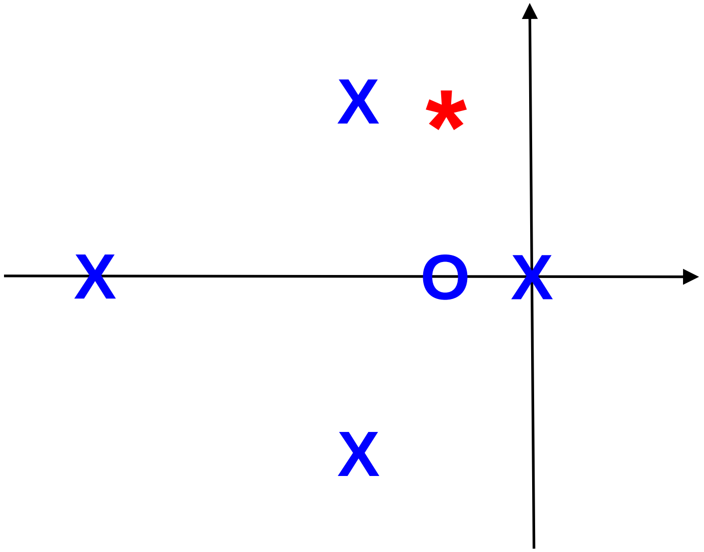
$$\Rightarrow \quad L(s) = -\frac{1}{K}$$

- K is real and negative,
  - the phase of $L(s)$ is 0°,
  - the negative locus or 0° locus

- Illustrative Example: $$L(s) = \frac{s+1}{s\,(s+5)\,[\,(s+2)^2+4\,]}$$

$$s_0 = -1 + 2j$$

$$\angle L(s_0) = 180^o + 360^o\,(\,l - 1\,)$$



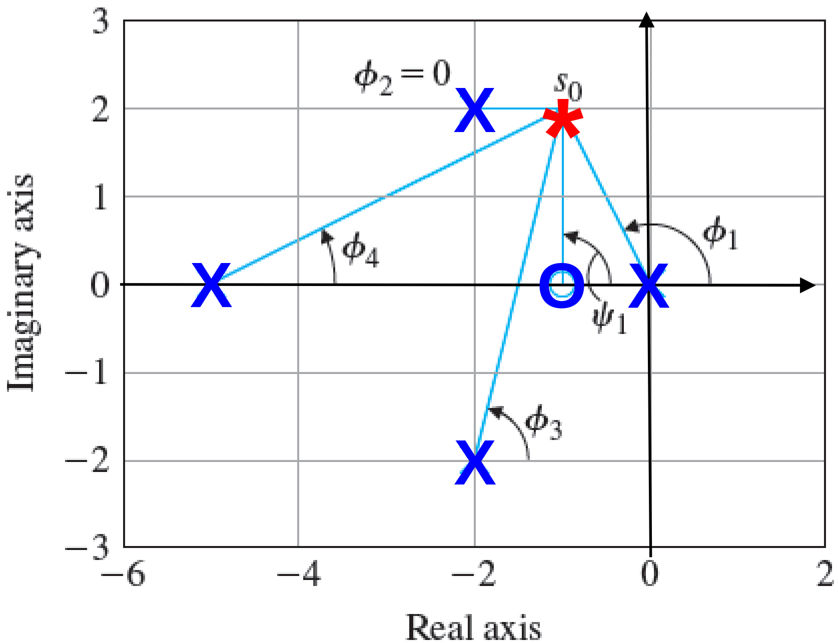
Formal Definition of Root Locus

- Illustrative Example: $L(s) = \dfrac{s+1}{s\,(s+5)\,[\,(s+2)^2+4\,]}$

$$s_0 = -1+2j$$

$$\begin{aligned}\angle L(s_0) &= 180^o + 360^o\,(\,l - 1\,) \\ &= \sum \psi_i - \sum \phi_i \\ &= \angle(s_0+1) \\ &\quad - \angle(s_0) - \angle(s_0+5) \\ &\quad - \angle[(s_0+2)^2+4] \\ &= 90^o - 116.6^o - 0^o \\ &\qquad - 76^o - 26.6^o \\ &= -129.2^o \quad \neq 180^o\end{aligned}$$

$\Rightarrow$ s_0 is not on the root locus

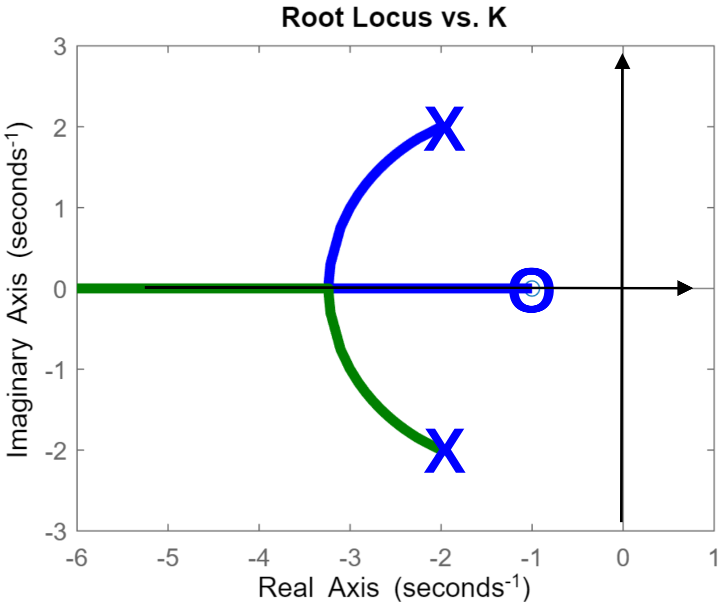


Rules for Determining a Positive (180◦) Root Locus

$$L(s) = \frac{s+1}{s^2+4s+8}$$

$$1 + K\frac{s+1}{s^2+4s+8} = 0$$

$$(s^2+4s+8) + K(s+1) = 0$$



- Rule 1:
- The n branches of the locus start at the poles of L(s) and
- m of these branches end on the zeros of L(s).
- a(s) + K b(s) = 0,
- If K = 0, then a(s) = 0, whose roots are the poles.
- When K → ∞, then b(s) = 0 (m zeros) or s → ∞. (the rest n-m)

Rules for Determining a Positive (180°) Root Locus

$$L(s) = \frac{s+1}{s^2+4s+8}$$

$$1 + K \frac{s+1}{s^2+4s+8} = 0$$

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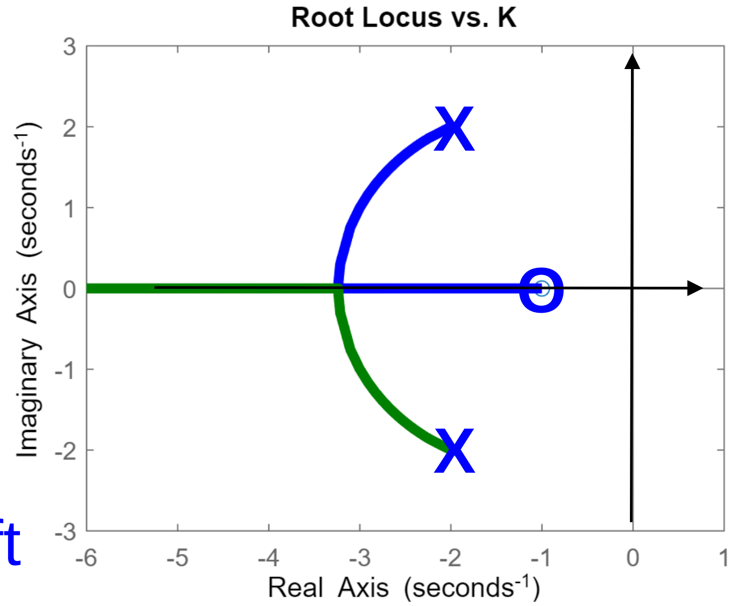
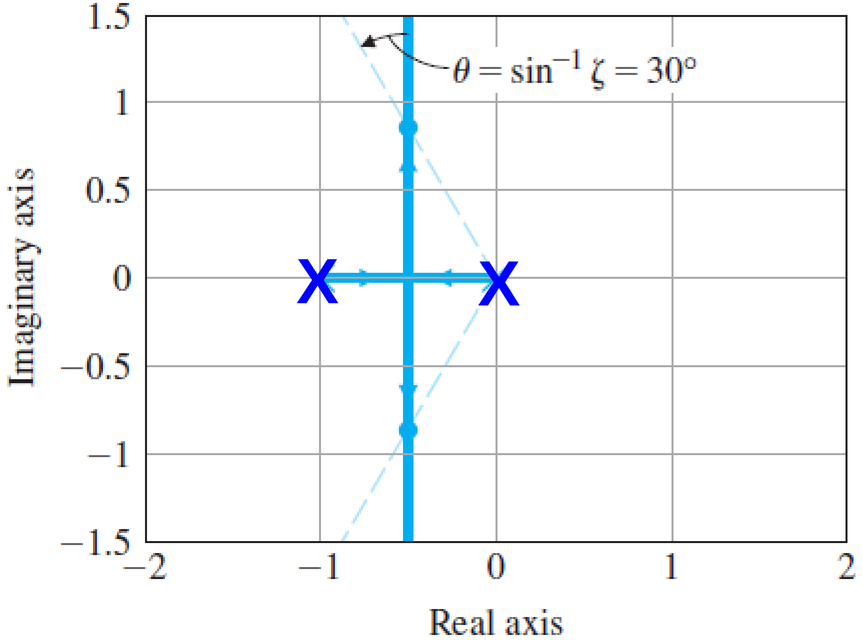


- Rule 2:
- The loci are on the real axis to the left of an odd number of poles and zeros.

$$L(s) = \frac{1}{s\,(s+1)}$$



Rules for Determining a Positive (180°) Root Locus

- Rule 3:
- For large s and K,

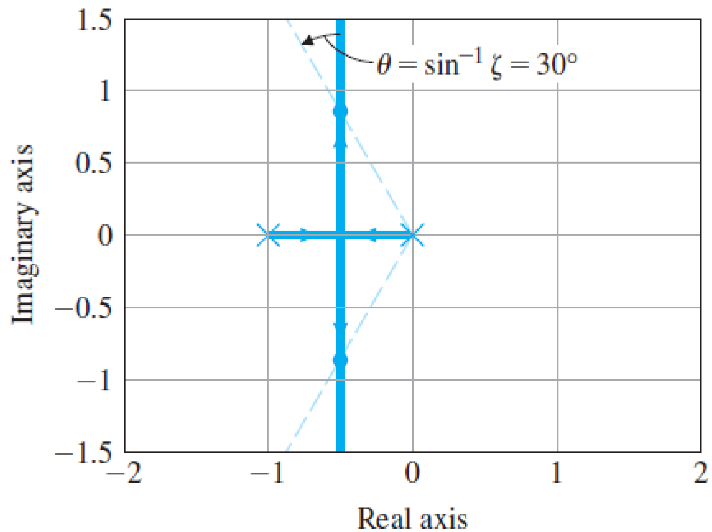
$$L(s) = \frac{s+1}{s^2+4s+8}$$

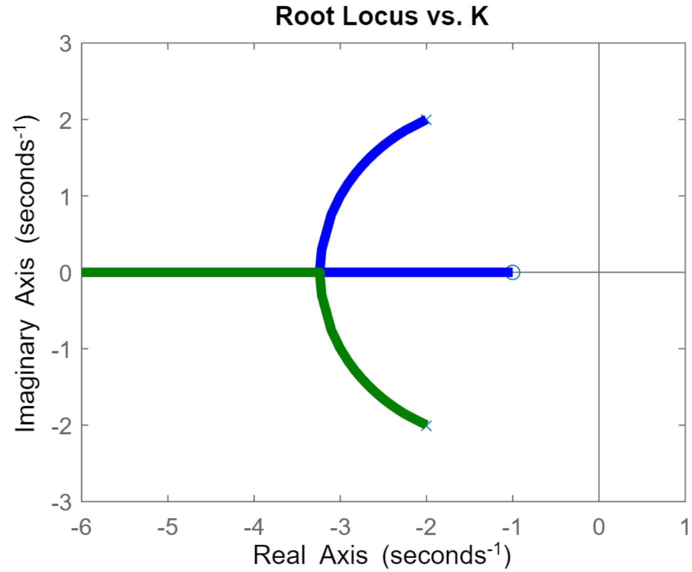
n-m branches of the loci are asymptotic to lines at angles ϕ radiating out from the point $s = \alpha$ on the real axis, where

$$L(s) = \frac{1}{s(s+1)}$$

$$\phi_l = \frac{180^o + 360^o(l-1)}{n-m} \qquad l = 1, 2, \cdots, n-m$$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$





Rules for Determining a Positive (180◦) Root Locus

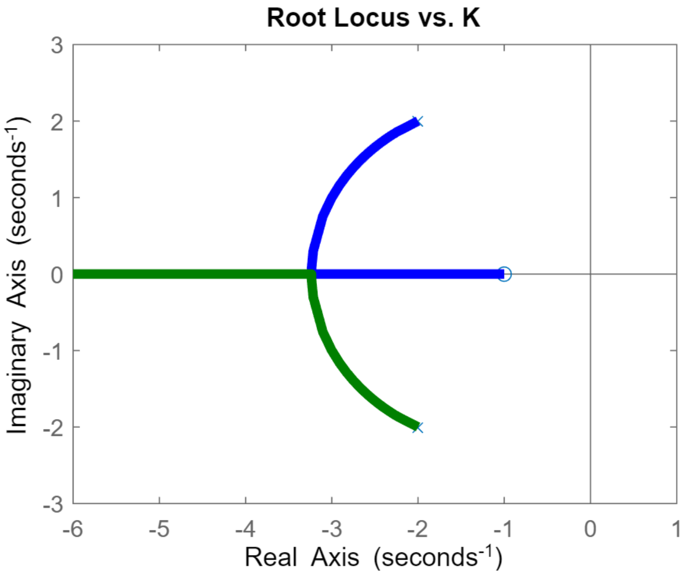
- Rule 3:

$$L(s) = \frac{s+1}{s^2+4s+8}$$

- As $K \to \infty$, $L(s) = -\frac{1}{K} \Rightarrow L(s) = 0$

1) *m* roots will be found to approach the zeros of L(s)

2) $s \to \infty$ because $n >= m$
   that is,
   $n - m$ roots approach $s \to \infty$



$$\Rightarrow 1 + K\frac{b(s)}{a(s)} = 0$$

$$\Rightarrow 1 + K\frac{s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n} = 0$$

- Can be approximated by

$$\Rightarrow 1 + K\frac{1}{(s-\alpha)^{n-m}} = 0$$

Rules for Determining a Positive (180∘) Root Locus

- Rule 3:

$$\Rightarrow 1 + K \frac{1}{(s-\alpha)^{n-m}} = 0$$

- The search point: $s_0 = R\, e^{j\phi}$

$$l = 1, 2, \cdots, (n-m)$$

$$(n-m)\, \phi_l = 180^o + 360^o\,(\, l - 1\,)$$

$$\Rightarrow \phi_l = \frac{180^o + 360^o\,(\, l - 1\,)}{(n-m)}$$

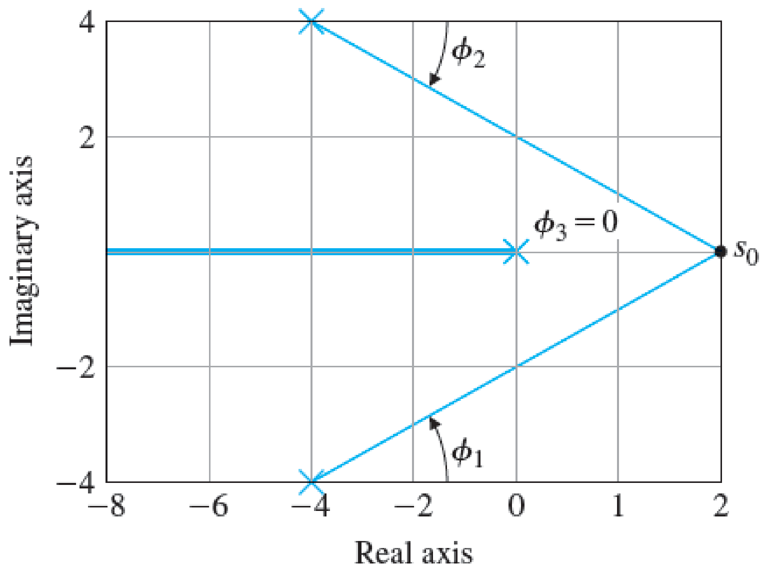
- For this example:

$$L(s) = \frac{1}{s\,[\,(s+4)^2 + 16\,)\,]}$$

$$(n-m) = 3$$

$$\phi_{1,2,3} = 60^o, 180^o, 300^o,$$

$$\text{or } \pm 60^o, 180^o$$



Rules for Determining a Positive (180∘) Root Locus

- Rule 3:
- Determine asymptotic lines:

$$L(s) = \frac{1}{s\,[\,(s+4)^2+16\,]\,} = \frac{1}{s^3+8s^2+32s+0}$$

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

$$= (s-p_1)(s-p_2)\cdots(s-p_{n-1})(s-p_n)$$

$$\Rightarrow a_1 = -p_1 - p_2 \cdots - p_{n-1} - p_n = -\sum p_i$$

$$b(s) = s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1} s + b_m$$

$$= (s-z_1)(s-z_2)\cdots(s-z_{m-1})(s-z_m)$$

$$\Rightarrow b_1 = -z_1 - z_2 \cdots - z_{n-1} - z_n = -\sum z_i$$

$$= \frac{s+1}{(s+2+2j)(s+2-2j)}$$

$$L(s) = \frac{s+1}{s^2+4s+8}$$

Rules for Determining a Positive (180°) Root Locus

- Rule 3:

$$L(s) = \frac{s+1}{s\,[\,(s+4)^2+16\,]\,} = \frac{s+1}{s^3+8s^2+32s+0}$$

- Determine asymptotic lines:

$$\Rightarrow\ a(s) + K\,b(s) = 0$$

$$(s^3+8s^2+32s) + K\,(s+1) = 0$$

$$\Rightarrow\ s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n$$

$$+K\,(s^m + b_1 s^{m-1} + b_2 s^{m-2} + \cdots + b_{m-1}s + b_m) = 0$$

$$= (s-r_1)(s-r_2)\cdots(s-r_{n-1})(s-r_n) = 0$$

- If m < n - 1:

$$\Rightarrow\ a_1 = -r_1 - r_2 \cdots - r_{n-1} - r_n \qquad = -\sum r_i$$

- And this term is independent of *K*
- The open-loop sum and closed-loop sum are the same and are equal to $-\,a_1$

$$\Rightarrow\ -\sum r_i = -\sum p_i$$

Rules for Determining a Positive (180°) Root Locus

- Rule 3:

$$L(s) = \frac{1}{s\,[\,(s+4)^2+16\,]}$$

- For large values of *K*:
  - *m* of the roots r_i approach the zeros z_i
  - *n - m* of the roots r_i approach the branches of the asymptotic system

$$\frac{1}{(s-\alpha)^{n-m}}$$

whose poles add up to $(n-m)\,\alpha$

$$\Rightarrow -\sum r_i = -(n-m)\alpha - \sum z_i = -\sum p_i$$

$$\Rightarrow \alpha = \frac{\sum p_i - \sum z_i}{(n-m)}$$

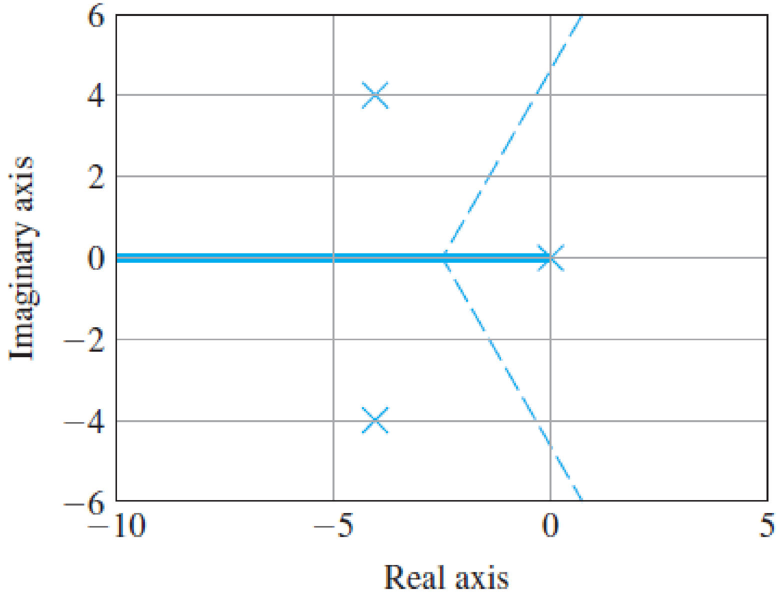
- For this example:

$$\Rightarrow \alpha = \frac{-4-4+0}{3-0}$$

$$= -\frac{8}{3} = -2.67$$

$$\phi_{1,2,3} = \pm 60^o, 180^o$$

$$2.67 \times \sqrt{3} = 4.62$$

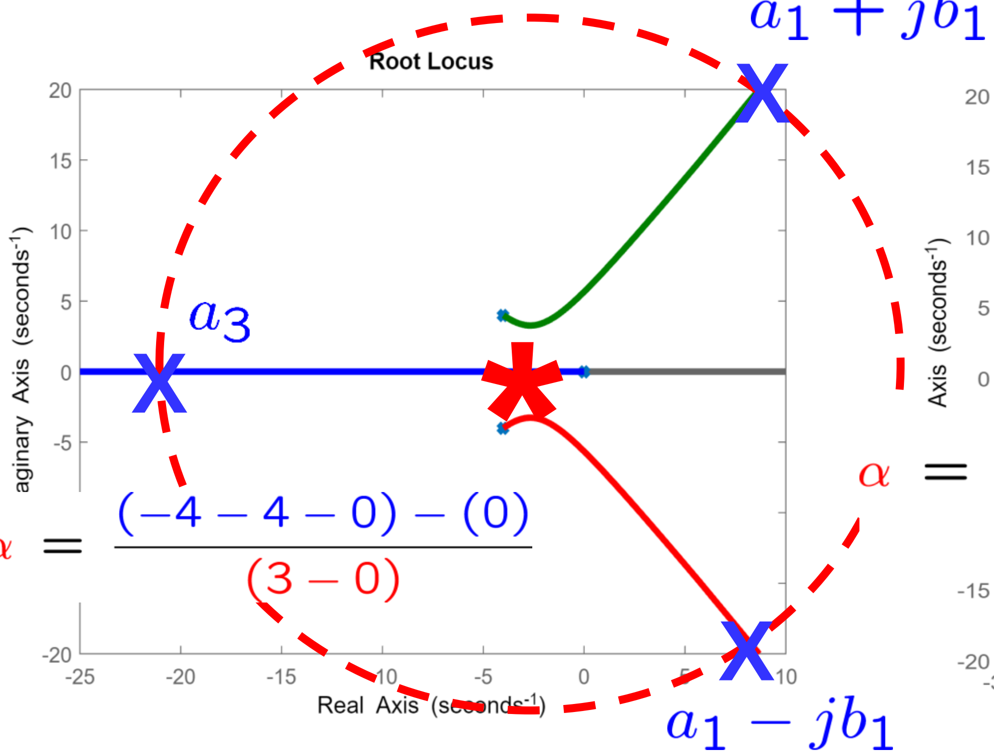


Rules for Determining a Positive (180∘) Root Locus

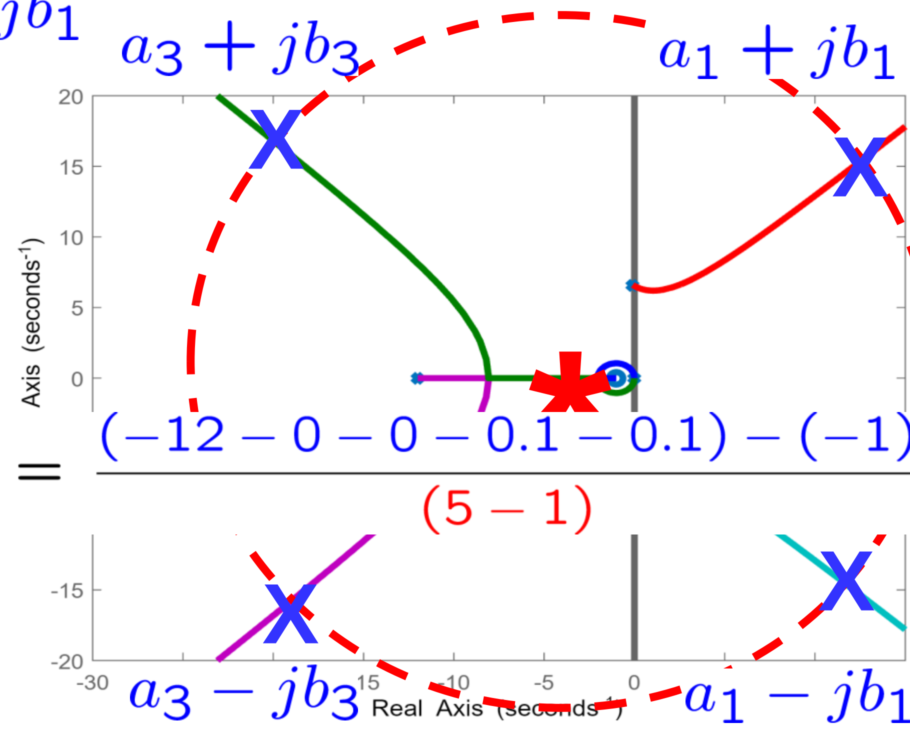
- Rule 3:
- For large values of *K*:
  - *m* of the roots r_i approach the zeros z_i
  - *n - m* of the roots r_i approach the branches of the asymptotic system

$$\Rightarrow \alpha = \frac{\sum p_i - \sum z_i}{(n-m)}$$

$$L(s) = \frac{1}{s\,[\,(s+4)^2 + 16\,)\,]} \qquad \frac{(s+1)}{(s+12)}\,\frac{1}{s^2[(s+0.1)^2+6.6^2]}$$



$$\alpha = \frac{(-4-4-0)-(0)}{(3-0)}$$



$$\alpha = \frac{(-12-0-0-0.1-0.1)-(-1)}{(5-1)}$$

- Rule 4: $\sum \psi_i - \sum \phi_i = 180^o + 360^o (l - 1)$

- The angle of departure of a branch of the locus from a single pole is given by

$$\phi_{dep} = \sum \psi_i - \sum_{i \neq dep} \phi_i - 180^o - 360^o (l - 1)$$

$\sum \phi_i$ the sum of the angles to the remaining poles

$\sum \psi_i$ the sum of the angles to all the zeros

- The angle of departure of a branch of the locus from repeated poles with multiplicity q is given by

$$q \, \phi_{l,dep} = \sum \psi_i - \sum_{i \neq l,dep} \phi_i - 180^o - 360^o(l-1)$$

$$l = 1, 2, \cdots, q$$

- Rule 4: $\sum \psi_i - \sum \phi_i = 180^o + 360^o(l-1)$

- The angle of arrival of a branch at a zero with multiplicity q is given by

$$q\,\psi_{l,arr} = \sum \phi_i - \sum_{i \neq l,arr} \psi_i + 180^o + 360^o(l-1)$$

$$l = 1, 2, \cdots, q$$

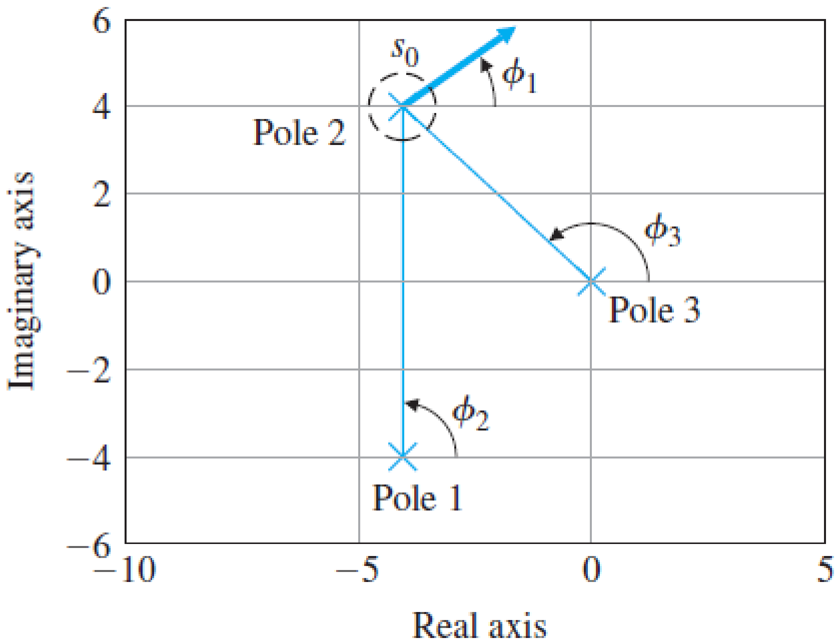
$\sum \phi_i$ the sum of the angles to all the poles

$\sum \psi_i$ the sum of the angles to the remaining zeros

- For this example:

$$\sum \psi_i - \sum \phi_i = 180^o + 360^o(l-1)$$

$$\begin{aligned} \phi_1 &= -90^o - 135^o - 180^o \\ &= -405^o \\ &= -45^o \end{aligned}$$



- Rule 5:
- The locus can have multiple roots at points on the locus and the branches will approach a point of q roots at angles separated by

$$\frac{180^o - 360^o(l-1)}{q}$$

- And will depart at angles with same separation.

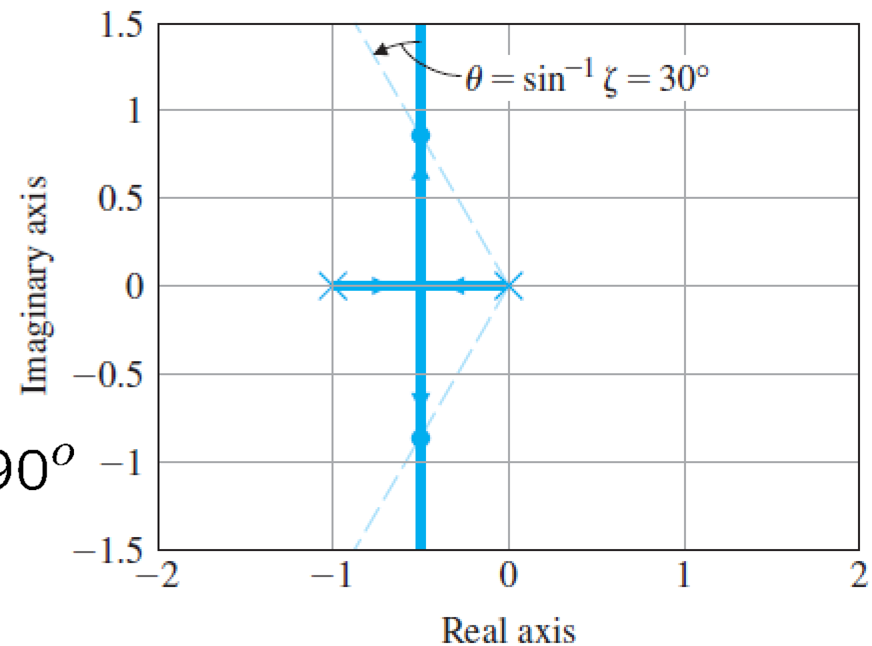
- For example:

$$L(s) = \frac{b(s)}{a(s)} = \frac{1}{s\,(s+1)}$$

$$K = \frac{1}{4} \quad \Rightarrow s_{1,2} = -\frac{1}{2}$$

$$\text{sep.} = \frac{180^o - 360^o(l-1)}{2} = 90^o$$

$$\Rightarrow 0^o,\ 180^o \quad \Rightarrow +90^o,\ -90^o$$



- Rule 5:
- Continuation Locus:

$$L(s) = \frac{b(s)}{a(s)} = \frac{1}{s\,(s+1)}$$

$$K_1 = \frac{1}{4} \quad \Rightarrow K = K_1 + K_2 = \frac{1}{4} + K_2$$

$$\Rightarrow s^2 + s + \frac{1}{4} + K_2 = 0$$

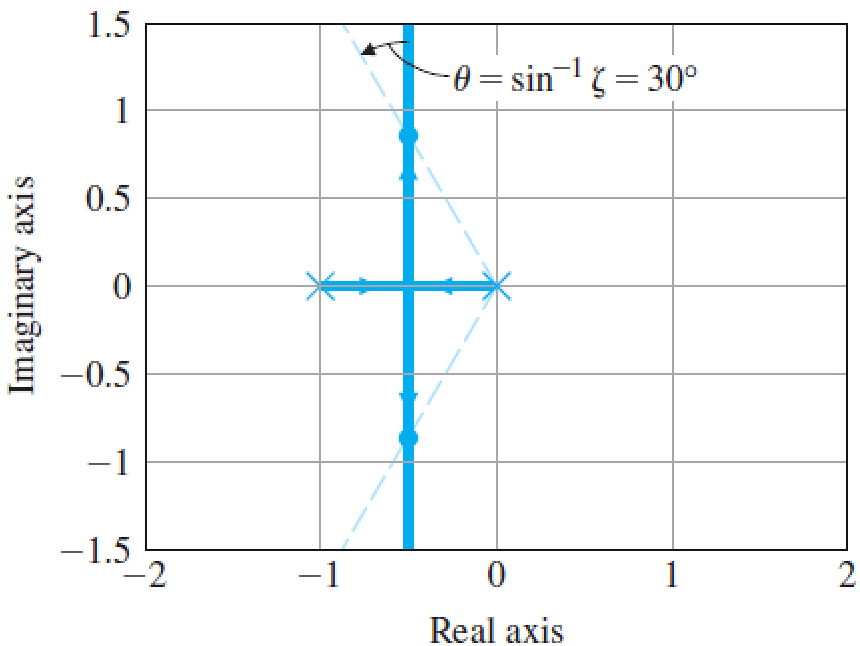
$$\Rightarrow \left(s + \frac{1}{2}\right)^2 + K_2 = 0$$

$$K_2 = 0 \quad \Rightarrow s_{1,2} = -\frac{1}{2}$$

$$2\,\phi_{dep} = -180^o - 360^o(l-1)$$

$$\phi_{dep} = \pm 90^o$$

$$\phi_{arr} = 0^o, 180^o$$



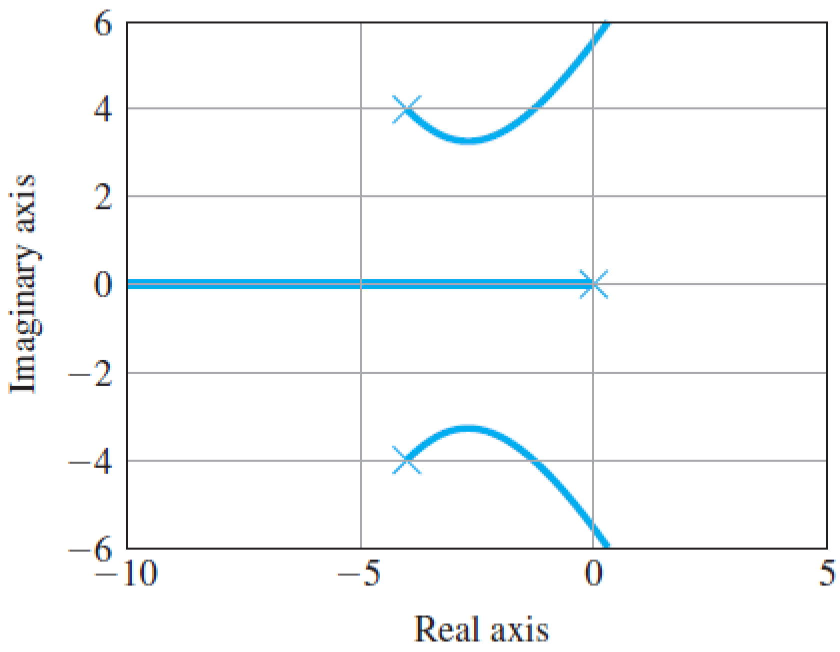
- The third-order example:

$$L(s) = \frac{1}{s\,(\,s^2 + 8s + 32\,)}$$

```
s = tf( 's' )

sysL = ( 1 )/(s*( s^2+8*s+32 ));
sysL = 1/(s*( (s+4)^2+16 ) );

 rlocus( sysL );
```



Summary of the Rules for Determining a Root Locus

- Rule 1:
- The n branches of the locus start at the poles of L(s) and
- m of these branches end on the zeros of L(s).

- Rule 2:
- The loci are on the real axis to the left of an odd number of poles and zeros.

- Rule 3:
- For large s and K, n-m branches of the loci are asymptotic to lines at angles ϕ radiating out from the point s = α on the real axis, where

$$\phi_l = \frac{180^o + 360^o(l - 1)}{n - m} \qquad \alpha = \frac{\sum p_i - \sum z_i}{n - m}$$

$$l = 1, 2, \cdots, n - m$$

Summary of the Rules for Determining a Root Locus

- Rule 4: $$\sum \psi_i - \sum \phi_i = 180^o + 360^o (l - 1)$$
- The angle of departure of a branch of the locus from repeated poles with multiplicity q is given by

$$q \, \phi_{l,dep} = \sum \psi_i - \sum_{i \neq l,dep} \phi_i - 180^o - 360^o (l-1)$$

$$l = 1, 2, \cdots, q$$

- The angle of arrival of a branch at a zero with multiplicity q is given by

$$q \, \psi_{l,arr} = \sum \phi_i - \sum_{i \neq l,arr} \psi_i + 180^o + 360^o (l-1)$$

- Rule 5:
- The locus can have multiple roots at points on the locus and the branches will approach a point of q roots at angles separated by $$\frac{180^o - 360^o (l-1)}{q}$$
- And will depart at angles with same separation.

Selecting the Parameter Value

- The positive root locus
  is a plot of all possible locations
  for roots to the equation $1 + K L(s) = 0$
  for some real positive value of K.

- The purpose of design
  is to select a particular value of K
  that will meet the specifications
  for static and dynamic response.

Selecting the Parameter Value

$$L(s) = \frac{1}{s\,[\,(s+4)^2 + 16\,]}$$

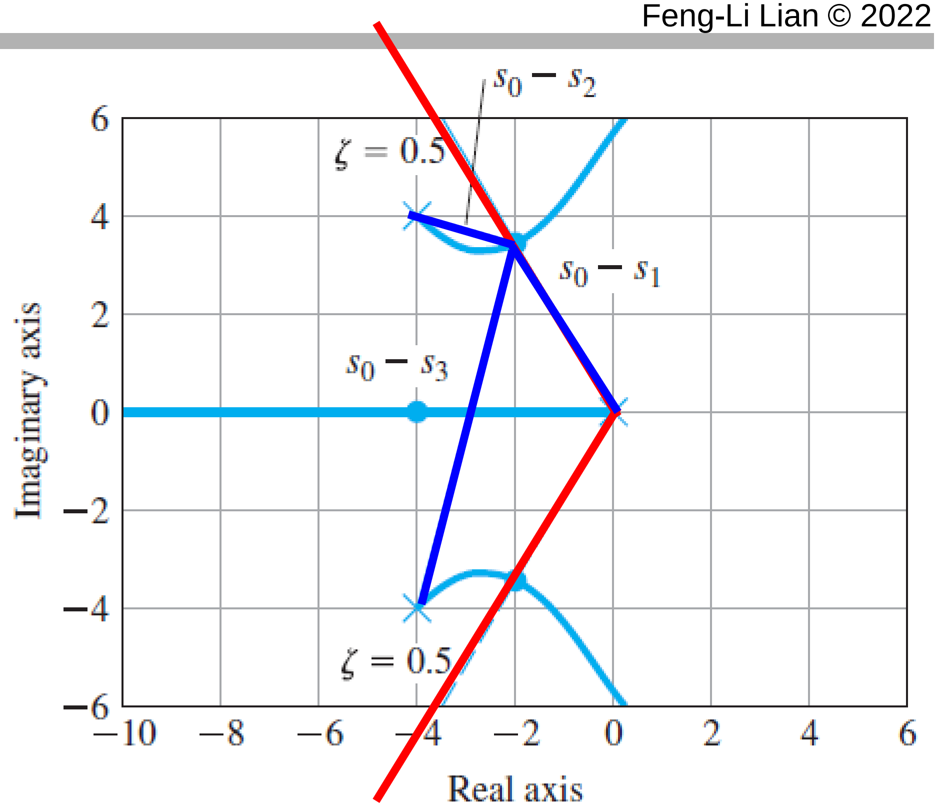
$$L(s_0) = \frac{1}{s_0\,(s_0 - s_2)\,(s_0 - s_3)}$$

$$K = \frac{1}{|L(s_0)|}$$

$$= |s_0|\,|s_0 - s_2|\,|s_0 - s_3|$$

$$\approx 4.0 \times 2.1 \times 7.7$$

$$\approx 65$$




```
s = tf( 's' )

sysL = (1)/( s*( s^2+8*s+32 ) );
sysL = (1)/( s*( (s+4)^2+16 ) );

rlocus( sysL );

[ K, p ] = rlocfind( sysL );
```

$$\Rightarrow 1 + K\,L(s) = 0$$

$$\Rightarrow L(s) = -\frac{1}{K}$$

- Compute the error constant of the control system
- For example,
  the steady-state error in tracking a ramp input
  is giving by the velocity constant:

$$K_v = \lim_{s \to 0} s\, L(s)$$

$$K_v = \lim_{s \to 0} s\, K\, L(s)$$

$$= \lim_{s \to 0} s\, K \frac{1}{s\,[\,(s + 4)^2 + 16\,)\,]}$$

$$= \frac{K}{32}$$

$$= \frac{65}{32} \approx 2 \text{ sec}^{-1}$$