

Fall 2022 (111-1)

控制系統
Control Systems

Unit 3F
Stability

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NTU-EE

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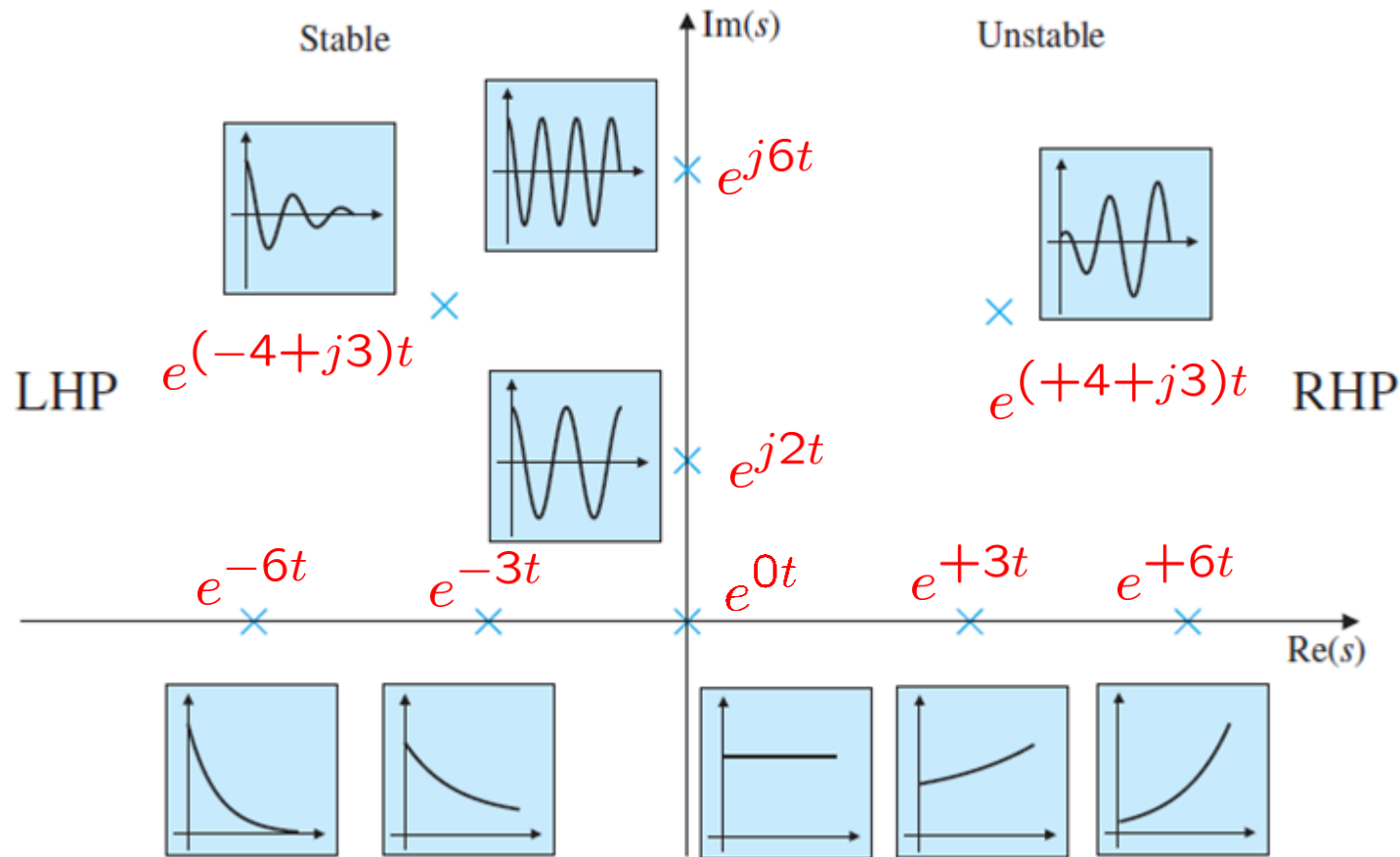
■ Stability:

- An LTI system is said to be **stable**
- if all the **roots** of the transfer function **denominator** polynomial
- have **negative real parts**
(that is, they are all in the **left-hand** s-plane)
- and is **unstable** otherwise.

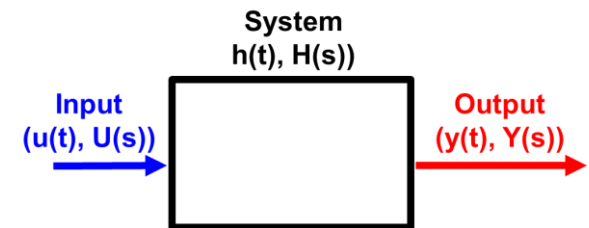
■ Stable System:

- A system is **stable**
- if its initial conditions **decay to zero**
- and is **unstable** if they **diverge**.

- Time functions associated with points in the s-plane
(LHP, left half-plane; RHP, right half-plane)



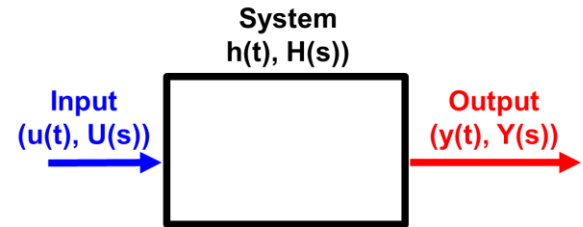
- **Bounded Input-Bounded Output Stability (BIBO Stable)**
 - A system is said to have **BIBO stability**
 - if every **bounded input** results in a **bounded output** (regardless of what goes on inside the system).



- If the system has **input** $u(t)$, **output** $y(t)$, and impulse response $h(t)$, then

$$y(t) = \int_{-\infty}^{\infty} h(t) u(t - \tau) d\tau$$

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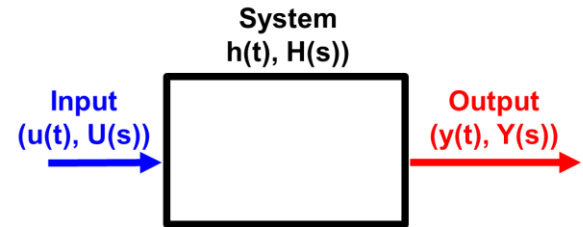


- If **input $u(t)$ is bounded** $|u(\cdot)| \leq M < \infty$
- And **output $y(t)$ is bounded** by

$$|y| = \left| \int_{-\infty}^{\infty} h u d\tau \right| \leq \int_{-\infty}^{\infty} |h| |u| d\tau \leq M \int_{-\infty}^{\infty} |h| d\tau$$

- That is, **output $y(t)$ is bounded** if $\int_{-\infty}^{\infty} |h| d\tau$ is **bounded**.

$$y(t) = \int_{-\infty}^{\infty} h(t) u(t - \tau) d\tau$$

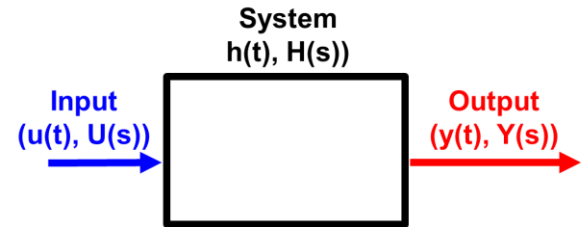


- On the other hand, suppose the integral is **not bounded** and the input is **bounded**,
- e.g., $u(t-\tau) = +1$ if $h(\tau) > 0$
and $u(t-\tau) = -1$ if $h(\tau) < 0$

$$|y| = \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

- The **output y(t)** is **not bounded**.

$$y(t) = \int_{-\infty}^{\infty} h(t) u(t - \tau) d\tau$$



Mathematical Definition of BIBO Stability

- The system with impulse response $h(t)$ is **BIBO stable**
- if and only if

the integral

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

- Example 3.31: BIBO Stability for a Capacitor

$$\Rightarrow h(\tau) = \mathbf{1}(t)$$



- Capacitor driven by current source

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \int_{-\infty}^{\infty} d\tau \rightarrow \infty$$

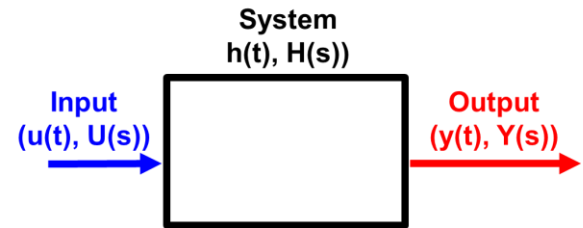
- Consider the LTI whose transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{b(s)}{a(s)}$$

$$= K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}, \quad m \leq n$$

$$\Rightarrow y(t) = \sum_{i=1}^n K_i e^{p_i t}$$

- p_i are the roots of $a(s)$, denominator polynomial
- K_i depend on the initial condition and zero locations



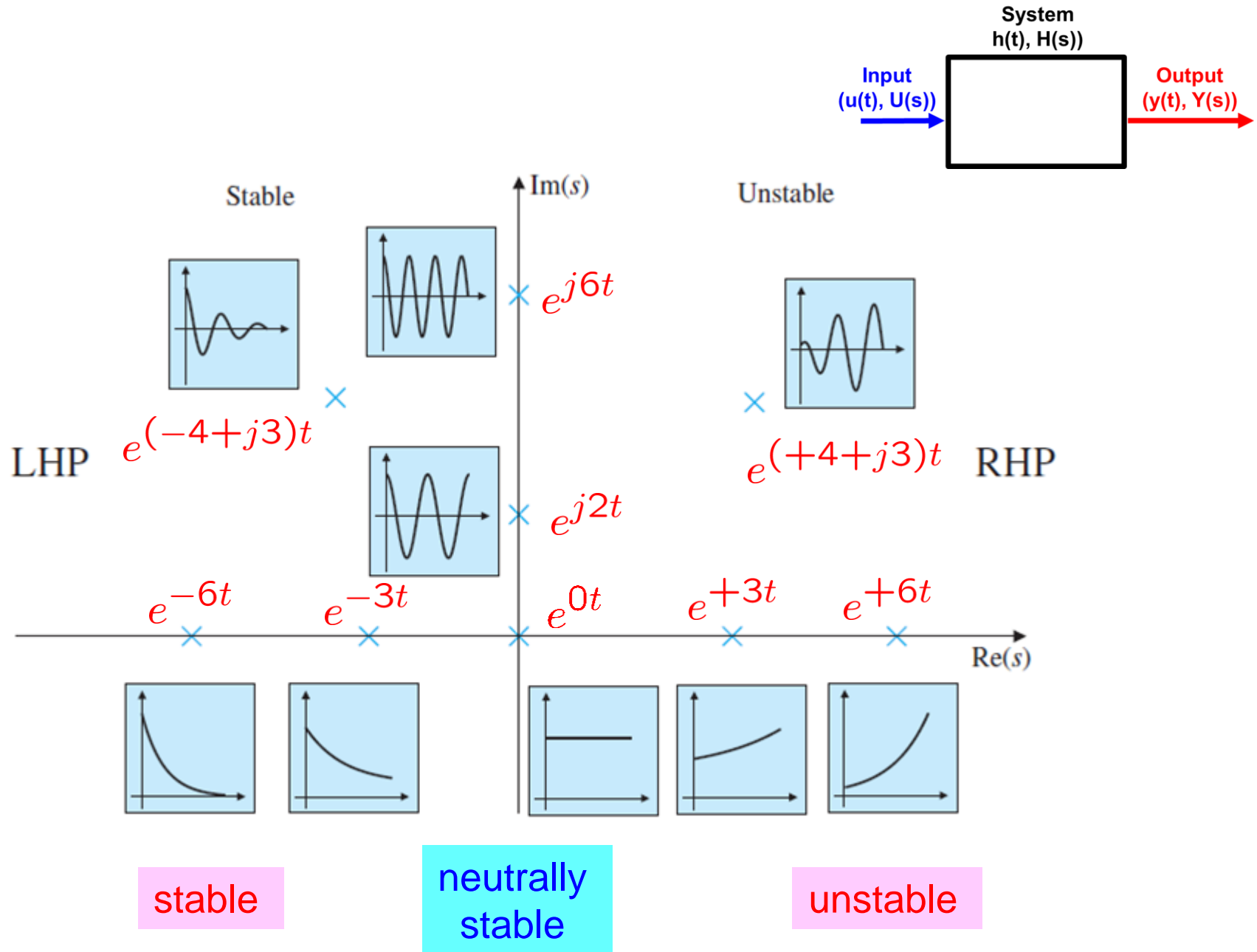
- The system is **stable**
- if and only if (necessary and sufficient condition)
- every term **goes to zero** as time goes to infinity

$$\Rightarrow e^{p_i t} \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad \text{for all } p_i$$

- This will happen
- if all the **poles** of the system are **strictly in the LHP**:

$$\Rightarrow \operatorname{Re}\{ p_i \} < 0$$

- This is called **internal stability**



- Consider the **characteristic equation** of an nth-order system:

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

- A **necessary** (but **not sufficient**) condition for **stability** of the system is that
 - all of the **roots** have **negative real parts**
 - which in turn requires that
 - all the $\{ a_i \}$ be **positive**.
- Equivalent test** were independently proposed by **Routh in 1874** and **Hurwitz in 1895**.

Routh's Stability Criterion

- Routh showed that
- a system is **stable** if and only if
- all the elements in the **first column** of the **Routh array**
- are **positive**.

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

▪ Routh array:

Row n	s^n	:	1	a_2	a_4	\dots		
Row $n - 1$	s^{n-1}	:	a_1	a_3	a_5	\dots	$b_1 = - \frac{\det \begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1}$	
Row $n - 2$	s^{n-2}	:	$b_1 \quad b_2 \quad b_3 \quad \dots$					$b_2 = - \frac{\det \begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$
Row $n - 3$	s^{n-3}	:	c_1	c_2	c_3	\dots	$b_3 = - \frac{\det \begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1}$	

- Routh showed that
 - a system is **stable** if and only if
 - all the elements in the **first column** of the Routh array
 - are **positive**.

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

Routh array:

Row n	s^n	:	1	a_2	a_4	\dots
Row $n - 1$	s^{n-1}	:	a_1	a_3	a_5	\dots
Row $n - 2$	s^{n-2}	:	b_1	b_2	b_3	\dots
Row $n - 3$	s^{n-3}	:	c_1	c_2	c_3	\dots

$$c_1 = - \frac{\det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1}$$

$$c_2 = - \frac{\det \begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

$$c_3 = - \frac{\det \begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1}$$

- Routh showed that
- a system is **stable** if and only if
- all the elements in the **first column** of the **Routh array**
- are **positive**.

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

- Routh array:

Row n s^n : 1 a_2 a_4 \dots

Row $n - 1$ s^{n-1} : a_1 a_3 a_5 \dots

Row $n - 2$ s^{n-2} : b_1 b_2 b_3 \dots

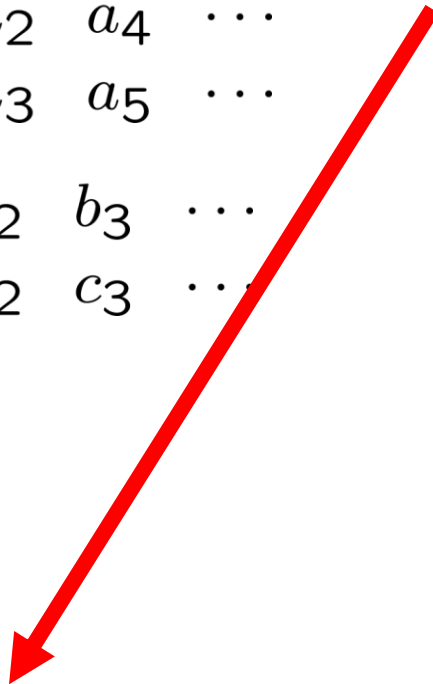
Row $n - 3$ s^{n-3} : c_1 c_2 c_3 \dots

Row : : :

Row 2 s^2 : * *

Row 1 s : *

Row 0 s^0 : *



Example 3.32: Routh's Test

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{l} s^6 : \\ s^5 : \end{array} \begin{array}{cccc} 1 & 3 & 1 & 4 \\ 4 & 2 & 4 & 0 \end{array}$$

$$s^4 : \quad -\frac{1 \cdot 2 - 4 \cdot 3}{4} = \frac{5}{2} \quad -\frac{4 \cdot 1 - 4 \cdot 1}{4} = 0 \quad -\frac{1 \cdot 0 - 4 \cdot 4}{4} = 4$$

$$s^3 : \quad -\frac{4 \cdot 0 - 5/2 \cdot 2}{5/2} = 2 \quad -\frac{4 \cdot 4 - 5/2 \cdot 4}{5/2} = -\frac{12}{5}$$

$$s^2 : \quad -\frac{5/2 \cdot (-12/5) - 2 \cdot 0}{2} = 3 \quad -\frac{5/2 \cdot 0 - 2 \cdot 4}{2} = 4$$

$$s : \quad -\frac{5/2 \cdot (-12/5) - 2 \cdot 0}{2} = -\frac{76}{15} \quad 0$$

$$s^0 : \quad -\frac{0 - (-76/15) \cdot 4}{(-76/15)} = 4$$

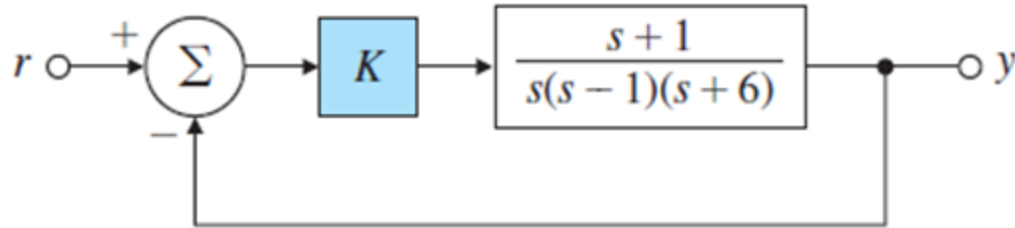
```
roots([1 4 3 2 1 4 4])
```

ans =

```
-3.2644 + 0.0000i
 0.6797 + 0.7488i
 0.6797 - 0.7488i
-0.6046 + 0.9935i
-0.6046 - 0.9935i
-0.8858 + 0.0000i
```


Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability



- The characteristic equation for the system:

$$1 + K \frac{s+1}{s(s-1)(s+6)} = 0 \quad s^3 + 5s^2 + (K-6)s + K = 0$$

$$\begin{array}{l} s^3 : \\ s^2 : \end{array} \quad \begin{array}{|c|} \hline 1 \\ \hline 5 \\ \hline \end{array} \quad \begin{array}{l} K-6 \\ K \end{array}$$

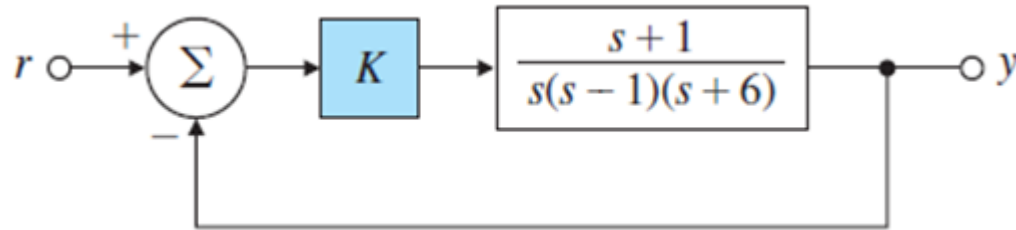
$$\begin{array}{l} s \\ s^0 \end{array} \quad \begin{array}{|c|} \hline (4K-30)/5 \\ \hline K \\ \hline \end{array} \quad \Rightarrow \frac{(4K-30)}{5} > 0 \Rightarrow K > 7.5$$

$$\Rightarrow K > 0$$

Routh's Stability Criterion

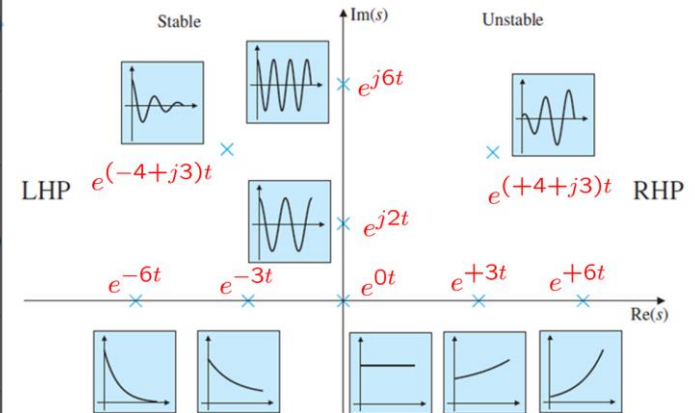
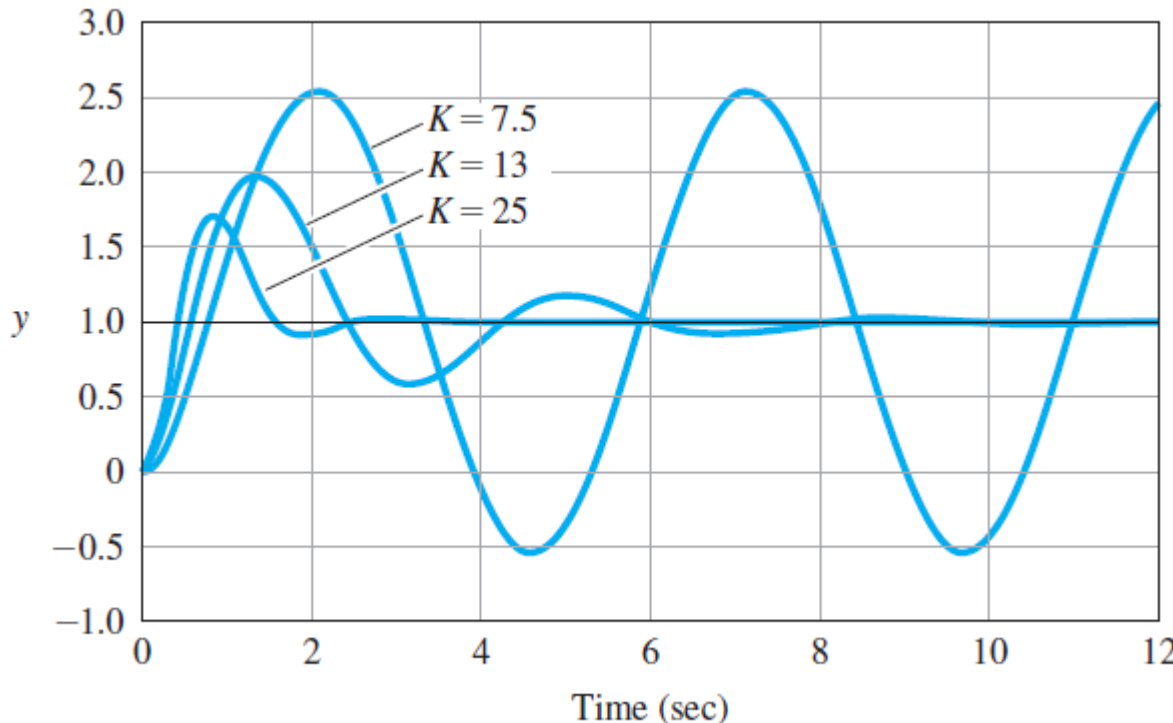
Example 3.33: Stability versus Parameter Range

- A feedback system for testing stability



```

set_K = [ 25  13  7.5  5 ];
for i = 1:4
k = set_K(i);
sysCL = k*(s+1)/(s^3+5*s^2+(k-6)*s+k)
y = step( sysCL, t);
end
    
```

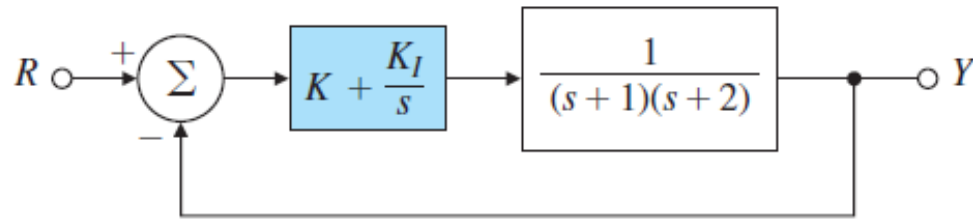


- Transient responses for the system

Routh's Stability Criterion

Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



- The characteristic equation for the system:

$$1 + \left(K + \frac{K_I}{s}\right) \frac{1}{(s+1)(s+2)} = 0$$

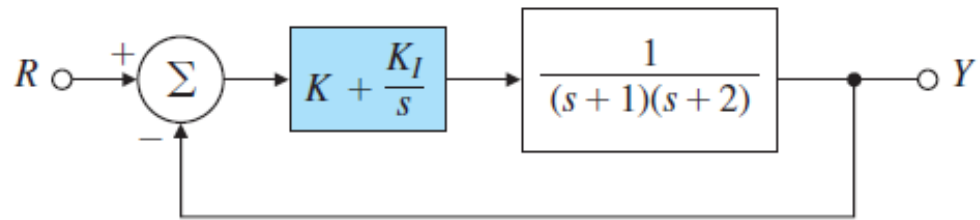
$$s^3 + 3s^2 + (2+K)s + K_I = 0$$

$$\begin{array}{l} s^3 : \\ s^2 : \end{array} \begin{array}{l} 1 \quad 2+K \\ 3 \quad K_I \end{array}$$

$$s : \quad (6 + 3K - K_I)/3 \quad \Rightarrow \quad K > \frac{1}{3}K_I - 2$$

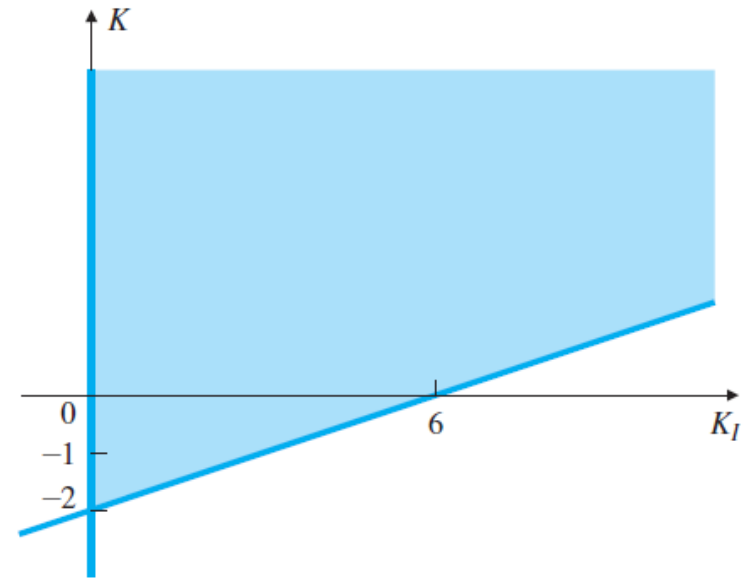
$$s^0 : \quad K_I \quad \Rightarrow \quad K_I > 0$$

- Example 3.34: Stability versus Two Parameter Ranges
- System with proportional-integral (PI) control



$$\Rightarrow K > \frac{1}{3}K_I - 2$$

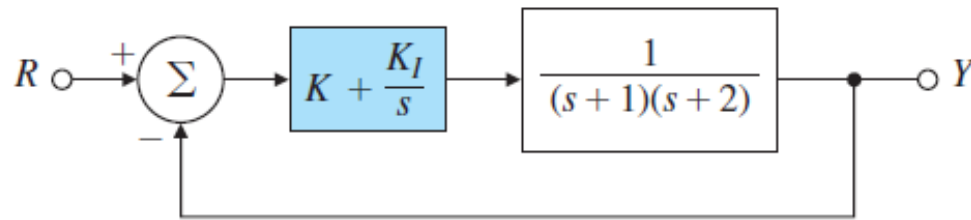
$$\Rightarrow K_I > 0$$



- Allowable region for stability

Example 3.34: Stability versus Two Parameter Ranges

- System with proportional-integral (PI) control



$$\Rightarrow K > \frac{1}{3}K_I - 2$$

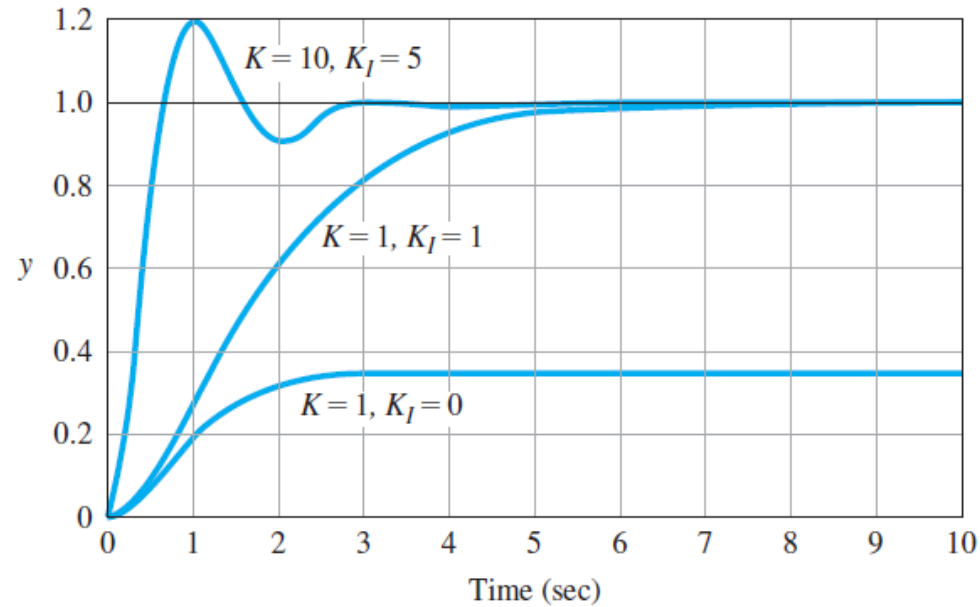
$$\Rightarrow K_I > 0$$

```
set_KI = [ 0 1 5];
set_K = [ 1 10];
```

```
for i = 1:3
for j = 1:2
KI = set_KI(i);
K = set_K(j);
```

```
sysCL = (K*s+KI)/(s^3+3*s^2+(2+K)*s+KI);
y = step( sysCL, t );
```

```
end
end
```



- Transient response for the system