

Fall 2022 (111-1)

控制系統  
Control Systems

Unit 20  
Dynamic Models

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NTU-EE

Sep 2022 – Dec 2022

- For the plant to be analyzed and controlled
  - Dynamic Models
  - Mathematical Models
- Methodology
  - Based on Physics and By Differential Equations
  - From Experimental Data (System Identification)
- Key Ingredients:
  - Physics, Chemistry, Biology, Sociology, Economics, etc.
  - Differential Equations (Equations of Motion, Dynamic Equations)
  - Laplace Transforms, Fourier Transforms
  - Transfer Function (From Input to Output)

## ■ Mechanical Systems

- U2A: Translational Motion
- U2B: Rotational Motion
- U2C: Combined Rotation and Translation
- U2D: Distributed Parameter Systems

## ■ Electrical Circuits

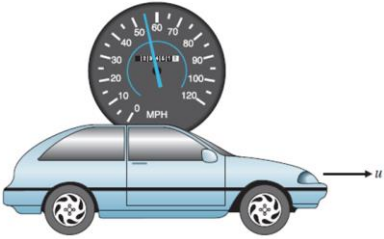
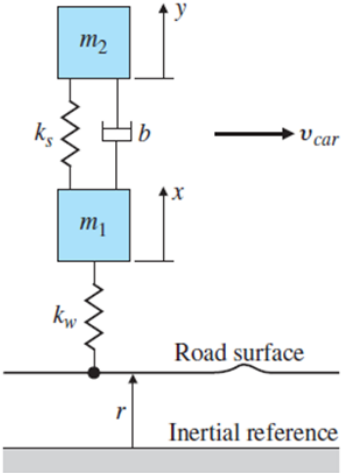
- U2E: Kirchhoff's Current Law (KCL)
- U2E: Kirchhoff's Voltage Law (KVL)
- U2E: Operational Amplifier

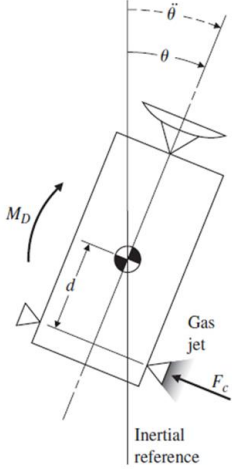
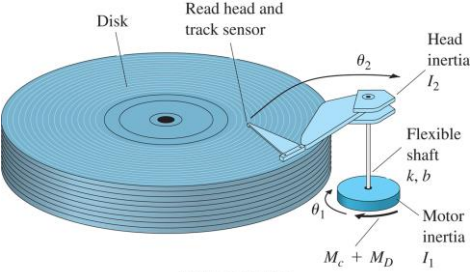
## ■ Electromechanical Systems

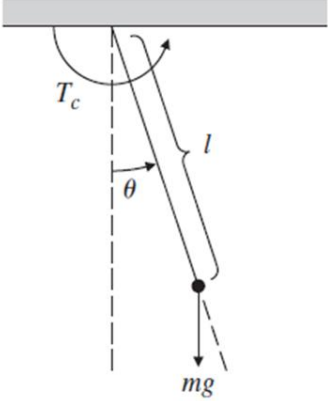
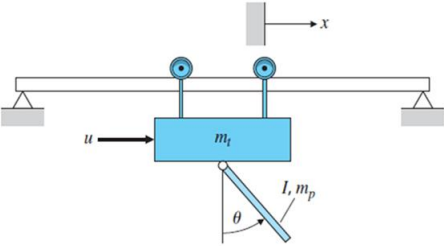
- U2F: Loudspeakers
- U2F: Motors
- U2F: Gears

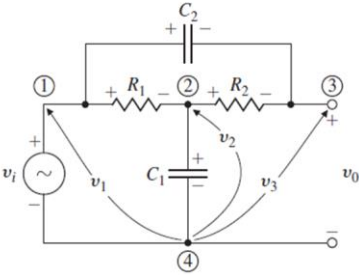
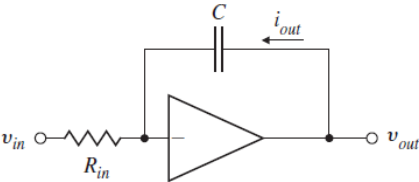
## ■ Heat and Fluid-Flow Models

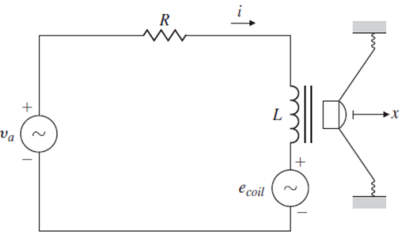
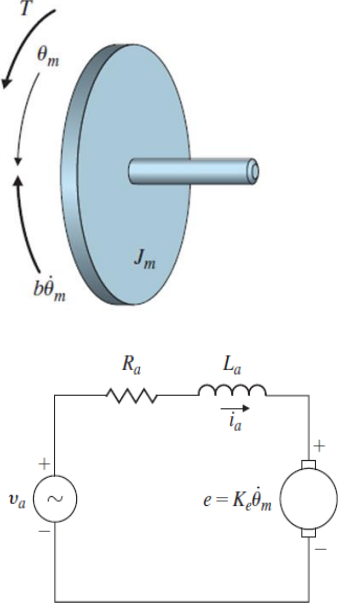
- U2G: Heat Flow
- U2G: Incompressible Fluid Flow

System	Differential Equation	Transfer Function
<p>Ex. 2.1: Cruise Control</p> 	$\dot{v} + \frac{b}{m} v = \frac{u}{m}$	$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}$
<p>Ex. 2.2: Two-Mass System</p> 	$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1}(x) = \frac{k_w}{m_1}(r)$ $\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0$	$\frac{Y(s)}{R(s)},$ $Y(s) = \frac{k_w b}{m_1 m_2} \left( s + \frac{k_s}{b} \right)$ $R(s) = s^4 + \left( \frac{b}{m_1} + \frac{b}{m_2} \right) s^3 + \left( \frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1} \right) s^2 + \frac{k_w b}{m_1 m_2} s + \frac{k_w k_s}{m_1 m_2}$

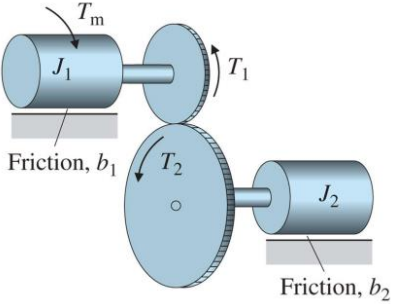
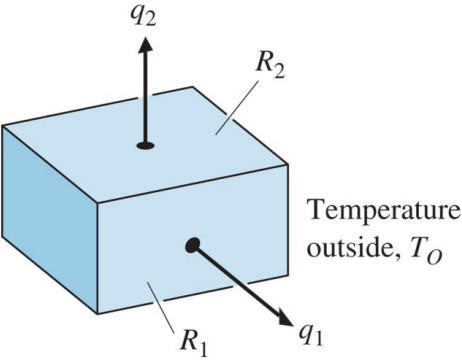
System	Differential Equation	Transfer Function
<p>Ex. 2.3: <b>Satellite Control</b></p> 	$F_c \cdot d + M_D = I \cdot \ddot{\theta}$	$\frac{\Theta(s)}{U(s)} = \frac{1}{I} \cdot \frac{1}{s^2}$ $(F_c \cdot d + M_D = u)$
<p>Ex. <b>Disk Read/Write Head</b></p> 	$I_1 \ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = M_c + M_D$ $I_2 \ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0$	$\frac{\Theta_2(s)}{M_c(s)} = \frac{k}{I_1 I_2 s^2 (s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$ $\frac{\Theta_1(s)}{M_c(s)} = \frac{I_2 s^2 + k}{I_1 I_2 s^2 (s^2 + \frac{k}{I_1} + \frac{k}{I_2})}$

System	Differential Equation	Transfer Function
<p>Ex. 2.6: <b>Pendulum</b></p> 	$\ddot{\theta} + \frac{g}{l}\theta = \frac{T_c}{ml^2}$	$\frac{\Theta(s)}{T_c(s)} = \frac{1}{s^2 + \frac{g}{l}}$
<p>Ex. 2.8: <b>Hanging Crane</b></p> 	$(I + m_p l^2)\ddot{\theta} + m_p g l \theta = -m_p l \ddot{x}$ $(m_t + m_p)\ddot{x} + b\dot{x} + m_p l \ddot{\theta} = u$	$\frac{\Theta(s)}{U(s)},$ $\Theta(s) = -m_p l$ $U(s) = s^2((I + m_p l^2)(m_t + m_p) - m_p^2 l^2) + m_p g l (m_t + m_p)$

System	Differential Equation	Transfer Function
<p>Ex. 2.9: <b>Bridged Tee Circuit</b></p> 	$-\frac{v_1 - v_2}{R_1} + \frac{v_2 - v_3}{R_2} + C_1 \frac{dv_2}{dt} = 0$ $\frac{v_3 - v_2}{R_2} + C_2 \frac{d(v_3 - v_1)}{dt} = 0$	$\frac{V_2(s)}{V_1(s)} = \frac{\dots}{C_1 s + \dots}$ $\frac{V_3(s)}{V_1(s)} = \frac{\dots}{C_2 s + \dots}$
<p>Ex. 2.12: <b>Integrator</b></p> 	$v_{out} = -\frac{1}{R_{in}C} \int_0^t v_{in}(\tau) d\tau + v_{out}(0)$	$V_{out}(s) = -\frac{1}{s R_{in}C} V_{in}(s)$

System	Differential Equation	Transfer Function
<p>Ex. 2.13 &amp; 14: <b>Loudspeaker</b></p> 	$L \frac{di}{dt} + Ri = v_a - e_{coil}$ $= v_i$ $M\ddot{x} + b\dot{x} = Bli$	$\frac{I(s)}{V_i(s)} = \frac{1}{Ls + R}$ $\frac{X(s)}{I(s)} = \frac{Bl}{(Ms^2 + bs)}$
<p>Ex. 2.15: <b>Motors</b></p>  <p>(a)</p>	$J_m \ddot{\theta}_m + b \dot{\theta}_m = T = K_t \cdot i_a$ $L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \cdot \dot{\theta}_m$	$\frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s[(J_m s + b)(L_a s + R_a) + K_t K_e]}$



System	Differential Equation	Transfer Function
<p>Ex. <b>Gears</b></p> 	$J_1 \ddot{\theta}_1 + b_1 \dot{\theta}_1 = T_m - T_1$ $J_2 \ddot{\theta}_2 + b_2 \dot{\theta}_2 = T_2$	$\frac{\Theta_2(s)}{T_m(s)} = \frac{n}{J_{eq}s^2 + b_{eq}s}$
<p>Ex. <b>Heat flow</b></p> 	$\dot{T}_I = \frac{1}{C_I} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) (T_O - T_I)$	$\frac{T_I(s)}{T_O(s)} = \frac{\dots}{s + \dots}$

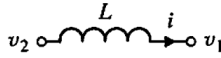
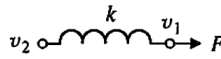

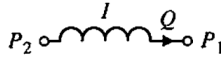
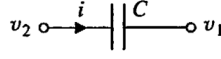
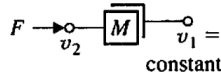
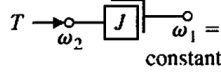
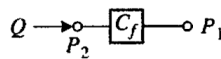
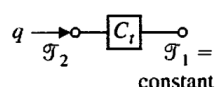
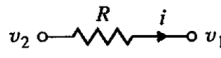
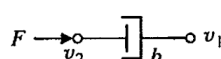
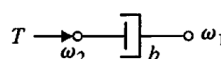
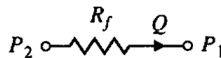
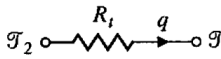
## Key Equations for Dynamic Models

System	Important Laws or Relationships	Associated Equations	Equation Number(s)
Mechanical	Translational motion (Newton's law)	$F = ma$	(2.1)
	Rotational motion	$M = I\alpha$	(2.14)
Electrical	Operational amplifier		(2.46), (2.47)
Electromechanical	Law of motors	$F = Bli$	(2.53)
	Law of generators	$e = Blv$	(2.56)
	Torque developed in a rotor	$T = K_t i_a$	(2.60)
Back emf	Voltage generated as a result of rotation of a rotor	$e = K_e \dot{\theta}_m$	(2.61)
Gears	Effective inertia	$J_{eq} = J_2 + J_1 n^2$	(2.80)
Heat flow	Heat-energy flow	$q = 1/R(T_1 - T_2)$	(2.81)
	Temperature as a function of heat-energy flow	$\dot{T} = \frac{1}{C} q$	(2.82)
	Specific heat	$C = mc_v$	(2.83)
Fluid flow	Continuity relation (conservation of matter)	$\dot{m} = w_{in} - w_{out}$	(2.88)
	Force of a fluid acting on a piston	$f = pA$	(2.90)
	Effect of resistance to fluid flow	$w = 1/R(p_1 - p_2)^{1/\alpha}$	(2.91)

# Summary of Unit 2: Key equations

Dorf & Bishop, 2011  
Modern Control Systems,

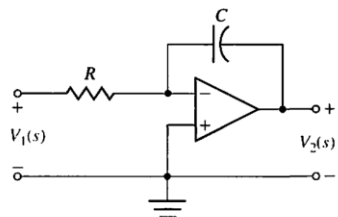
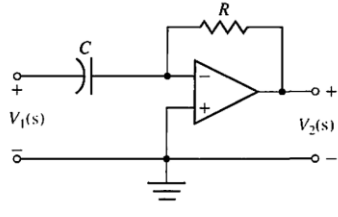
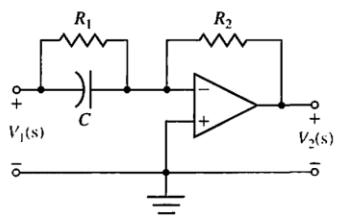
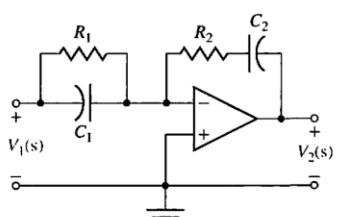
**Table 2.2 Summary of Governing Differential Equations for Ideal Elements**

Type of Element	Physical Element	Governing Equation	Energy $E$ or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L \frac{di}{dt}$	$E = \frac{1}{2} Li^2$	
	Translational spring	$v_{21} = \frac{1}{k} \frac{dF}{dt}$	$E = \frac{1}{2} \frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k} \frac{dT}{dt}$	$E = \frac{1}{2} \frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I \frac{dQ}{dt}$	$E = \frac{1}{2} IQ^2$	
Capacitive storage	Electrical capacitance	$i = C \frac{dv_{21}}{dt}$	$E = \frac{1}{2} Cv_{21}^2$	
	Translational mass	$F = M \frac{dv_2}{dt}$	$E = \frac{1}{2} Mv_2^2$	
	Rotational mass	$T = J \frac{d\omega_2}{dt}$	$E = \frac{1}{2} J\omega_2^2$	
	Fluid capacitance	$Q = C_f \frac{dP_{21}}{dt}$	$E = \frac{1}{2} C_f P_{21}^2$	
	Thermal capacitance	$q = C_t \frac{d\mathcal{T}_2}{dt}$	$E = C_t \mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R} v_{21}$	$\mathcal{P} = \frac{1}{R} v_{21}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}^2$	
	Fluid resistance	$Q = \frac{1}{R_f} P_{21}$	$\mathcal{P} = \frac{1}{R_f} P_{21}^2$	
	Thermal resistance	$q = \frac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t} \mathcal{T}_{21}$	

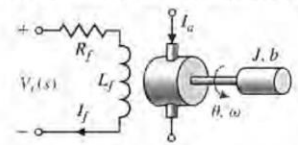
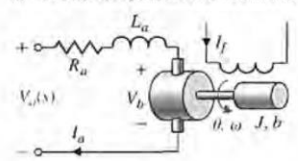
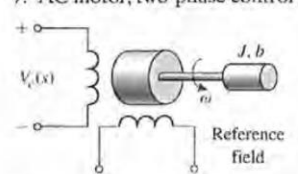
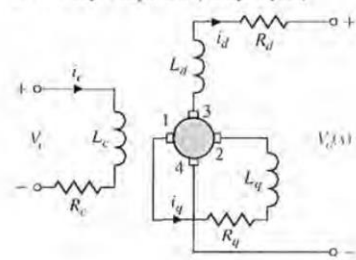
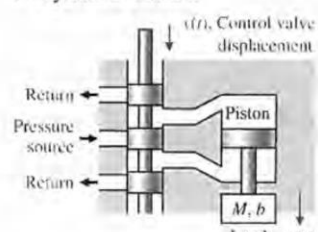
# Summary of Unit 2: Key equations

Dorf & Bishop, 2011  
Modern Control Systems,

**Table 2.5 Transfer Functions of Dynamic Elements and Networks**

Element or System	$G(s)$
1. Integrating circuit, filter	
	$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$
2. Differentiating circuit	
	$\frac{V_2(s)}{V_1(s)} = -RCs$
3. Differentiating circuit	
	$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$
4. Integrating filter	
	$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$

**Table 2.5 Continued**

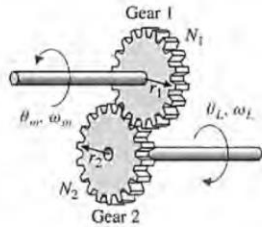
Element or System	$G(s)$
5. DC motor, field-controlled, rotational actuator	
	$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$
6. DC motor, armature-controlled, rotational actuator	
	$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_t K_\phi]}$
7. AC motor, two-phase control field, rotational actuator	
	$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$ $\tau = J/(b - m)$ $m = \text{slope of linearized torque-speed curve (normally negative)}$
8. Rotary Amplifier (Amplidyne)	
	$\frac{V_o(s)}{V_c(s)} = \frac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$ $\tau_c = L_c/R_c, \quad \tau_q = L_q/R_q$ for the unloaded case, $i_d \approx 0$ , $\tau_c \approx \tau_q$ , $0.05 \text{ s} < \tau_c < 0.5 \text{ s}$ $V_q, V_{34} = V_d$
9. Hydraulic actuator	
	$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}$ $K = \frac{Ak_x}{k_p}, \quad B = \left(b + \frac{A^2}{k_p}\right)$ $k_x = \left. \frac{\partial g}{\partial x} \right _{x_0}, \quad k_p = \left. \frac{\partial g}{\partial P} \right _{P_0}$ $g = g(x, P) = \text{flow}$ $A = \text{area of piston}$

**Table 2.5 Continued**

Element or System

G(s)

10. Gear train, rotational transformer

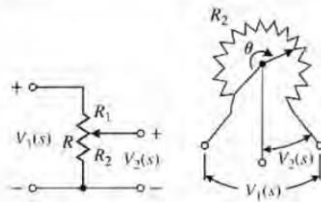


$$\text{Gear ratio} = n = \frac{N_1}{N_2}$$

$$N_2 \theta_L = N_1 \theta_m, \quad \theta_L = n \theta_m$$

$$\omega_L = n \omega_m$$

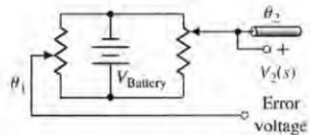
11. Potentiometer, voltage control



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{\max}}$$

12. Potentiometer, error detector bridge

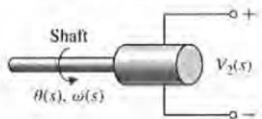


$$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$$

$$V_2(s) = k_s \theta_{\text{error}}(s)$$

$$k_s = \frac{V_{\text{Battery}}}{\theta_{\max}}$$

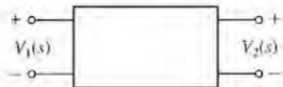
13. Tachometer, velocity sensor



$$V_2(s) = K_t \omega(s) = K_t s \theta(s)$$

$$K_t = \text{constant}$$

14. DC amplifier



$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s\tau + 1}$$

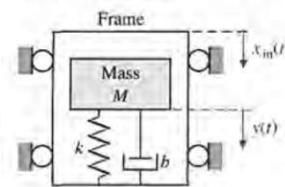
$R_o$  = output resistance  
 $C_o$  = output capacitance  
 $\tau = R_o C_o, \tau \ll 1s$   
and is often negligible for controller amplifier

**Table 2.5 Continued**

Element or System

G(s)

15. Accelerometer, acceleration sensor



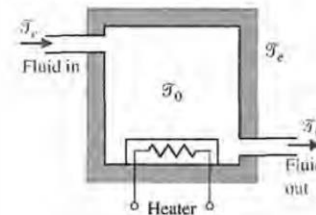
$$x_o(t) = y(t) - x_{in}(t),$$

$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where  $\omega < \omega_n$ ,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \approx \frac{\omega^2}{k/M}$$

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$  = temperature difference due to thermal process  
 $C_t$  = thermal capacitance  
 $Q$  = fluid flow rate = constant  
 $S$  = specific heat of water  
 $R_t$  = thermal resistance of insulation  
 $q(s)$  = transform of rate of heat flow of heating element

17. Rack and pinion



$$x = r\theta$$

converts radial motion to linear motion