Quiz 2 Problem 1

(a) Find the differential equation and nature response (15%)

1. The capacitor current is given by:

$$i_C(t) = C \frac{dv_{\text{out}}}{dt} = -\beta \, i(t).$$

2. The input voltage is expressed as:

$$v_{\text{in}}(t) = L \frac{di}{dt} + i(t) \, R.$$

3. Using the capacitor current relation, we have:

$$C \frac{dv_{\text{out}}}{dt} = -\beta \, i(t) \quad \Longrightarrow \quad i(t) = -\frac{C}{\beta} \frac{dv_{\text{out}}}{dt}.$$

4. Substitute the expression for $i(t)$ into the equation for $v_{\text{in}}(t)$ (5%):

$$v_{\text{in}}(t) = -\frac{LC}{\beta} \frac{d^2 v_{\text{out}}}{dt^2} - \frac{RC}{\beta} \frac{dv_{\text{out}}}{dt}.$$

   With the given numerical values

$$L = 1\,\text{H}, \quad R = 5\,\Omega, \quad C = 0.5\,\text{F}, \quad \beta = 10,$$

   this becomes:

$$v_{\text{in}}(t) = -0.05 \frac{d^2 v_{\text{out}}}{dt^2} - 0.25 \frac{dv_{\text{out}}}{dt}.$$

5. **Natural (Homogeneous) Response:** Setting $v_{\text{in}}(t) = 0$ yields

$$\frac{d^2 v_{\text{out}}}{dt^2} + 5 \frac{dv_{\text{out}}}{dt} = 0.$$

   The corresponding characteristic equation

$$s(s+5) = 0$$

   has roots $s = 0$ and $s = -5$, so the general natural response is

$$v_{\text{out}}^{(n)}(t) = K_1 + K_2 \, e^{-5t},$$

   where K_1 and K_2 are constants determined by the initial conditions. (10%)

(b) Forced Response (15%)

For the input

$$v_{\text{in}}(t) = -\big(10 + 9\,e^{-2t}\big),$$

we substitute into the differential equation derived earlier:

$$v_{\text{in}}(t) = -\big(10 + 9\,e^{-2t}\big) = \; -\frac{LC}{\beta}\frac{d^2 v_{\text{out}}}{dt^2} - \frac{RC}{\beta}\frac{dv_{\text{out}}}{dt}$$

Since

$$v_{\text{in}}(t) = -\big(10 + 9\,e^{-2t}\big),$$

we look for a particular solution of the form (5%)

$$v_{\text{out}}^{(f)}(t) = A\,t + B\,e^{-2t}.$$

Deriving the Particular Solution

- Compute the first derivative:

$$\frac{d}{dt}v_{\text{out}}^{(f)}(t) = A - 2B\,e^{-2t}.$$

- Compute the second derivative:

$$\frac{d^2}{dt^2}v_{\text{out}}^{(f)}(t) = 4B\,e^{-2t}.$$

- Substitute these into the differential equation, we have (5% each):

$$A = 40, B = -30$$

Thus, the forced response is:

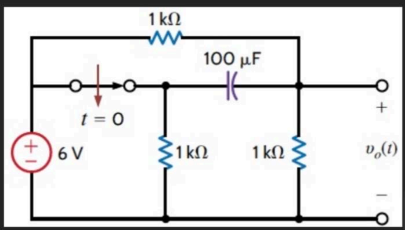
$$v_{\text{out}}^{(f)}(t) = 40\,t + -30\,e^{-2t}.$$

2. (30%) Consider the cirucit shown in Fig. 2. At t = 0, the switch is opened (disconnected).

(a) (10%) Find $v_0(0^+)$.

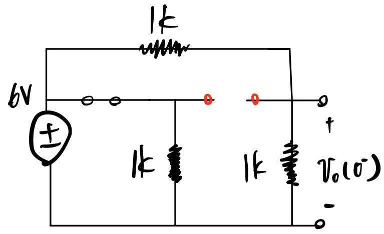
(b) (5%) Find $v_0(\infty)$.

(c) (15%) Find $v_0(t)$.



(Fig. 2)

(a) $t = 0^-$



$v_c(0^-) = 3V$

$V_o(0^-) = 3V$

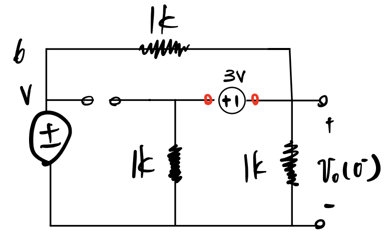
配分 (a)

(1) 寫出 $v_o(0^-)$ 畫出 $v_o(0^-)$ 电路 4分

(2) 解出 $v_o(0^+)$ 6分

(3) 直接解出 $v_o(0^+)$ 10分 ↗ 有錯給分形式

$t = 0^+$

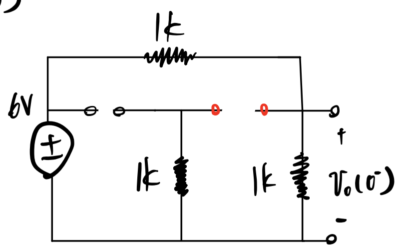


$$\frac{V_1 - 6}{1k} + \frac{V_1 + 3}{1k} + \frac{V_1}{1k} = 0$$

$\Rightarrow V_1 = 1V$

$\Rightarrow V_o(0^+) = 1V$

(b)



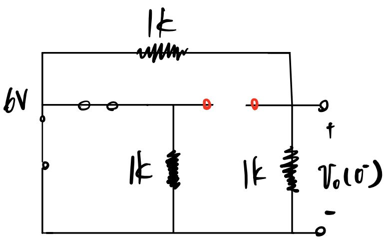
$v_o(\infty) = 3V$

配分 (b)

(1) 畫电路 3分, 解出 $v_o(\infty)$ 2分

(2) 直接解出 $v_o(\infty)$ 5分 ↗ 有錯給分形式

(c)



$R_{eq} = (1k \parallel 1k) + 1k$

$= 1.5k$

$\tau = RC = 1.5k \times 100\mu = 0.15$

$$v_o(\infty) + (V_o(0^+) - v_o(\infty))\, e^{-t/\tau} \Rightarrow v_o(t) = 3 - 2e^{-20t/3} \text{ V}$$

配分 (c)

(1) 解出 R_{eq} 4分

(2) 解出 τ 4分

(3) 畫出电路 2分, 解出 $v_o(t)$ 5分

(4) 直接解出 $v_o(t)$ 15分 ↗ 有錯給分形式

3.

(a) (3) $\boldsymbol{i_R(t) = \frac{v_s(t)-v_C(t)}{R}}$ or $\boldsymbol{v_C(t) = v_s(t) - Ri_R(t)}$

(3) $\boldsymbol{i_R(t) = i_L(t) + i_C(t)}$ or $\boldsymbol{\frac{di_R(t)}{dt} = \frac{di_L(t)}{dt} + \frac{di_C(t)}{dt}}$

(3) $\boldsymbol{v_L(t) = L\frac{di_L(t)}{dt}}$ or $\boldsymbol{\frac{di_L(t)}{dt} = \frac{v_L(t)}{L}}$

(3) $\boldsymbol{i_C(t) = C\frac{dv_C(t)}{dt}}$ or $\boldsymbol{\frac{di_C(t)}{dt} = \frac{d^2v_C(t)}{dt^2}}$

$$\frac{di_R(t)}{dt} = \frac{v_L(t)}{L} + C\frac{d^2v_C(t)}{dt^2}$$

$$\frac{dv_C(t)}{dt} = \frac{d}{dt}[v_s(t) - Ri_R(t)]$$

$$\frac{d^2v_C(t)}{dt^2} = \frac{d^2v_s(t)}{dt^2} - R\frac{d^2i_R(t)}{dt^2}$$

$$\frac{di_R(t)}{dt} = \frac{v_s(t) - Ri_R(t)}{L} + C\left[\frac{d^2v_s(t)}{dt^2} - R\frac{d^2i_R(t)}{dt^2}\right]$$

(3) $\boldsymbol{RC\frac{d^2i_R(t)}{dt^2} + \frac{di_R(t)}{dt} + \frac{R}{L}i_R(t) = \frac{1}{L}v_s(t) + C\frac{d^2v_s(t)}{dt^2}}$

(b) $RC\frac{d^2i_R(t)}{dt^2} + \frac{di_R(t)}{dt} + \frac{R}{L}i_R(t) = \frac{1}{L}v_s(t) + C\frac{d^2v_s(t)}{dt^2}$

(10) $\boldsymbol{t = 0^+}$
$$\begin{cases} i_L(0^+) = i_L(0^-) = -2(A) \\ v_C(0^+) = v_C(0^-) = 0(V) \\ i_R(0^+) = \frac{v_s(0^+)-v_C(0^+)}{R} = 6(A) \\ i_C(0^+) = i_R(0^+) - i_L(0^+) = 8(A) \\ \frac{d^2i_R(t)}{dt^2} + 10\frac{di_R(t)}{dt} + 25i_R(t) = 150(A) \end{cases}$$

Homogeneous solution

$$\frac{d^2 i_R(t)}{dt^2} + 10\frac{di_R(t)}{dt} + 25i_R(t) = 0$$

$$r^2 + 10r + 25 = 0 \ \rightarrow \ r = -5(double\ root)$$

(5) $\boldsymbol{i_{R,h}(t) = (A + Bt)e^{-5t}}$

Particular solution

(5) $\boldsymbol{0 + 0 + 25i_{R,p}(t) = 150 \ \rightarrow \ i_{R,p}(t) = 6}$

General solution

$$i_R(0) = A + 6 = 6 \ \rightarrow \ A = 0$$

$$\frac{di_R(0)}{dt} = B = -10i_C(0) = -80$$

(5) $\boldsymbol{i_R(t) = -80te^{-5t} + 6\ (when\ t > 0)}$