

§ 0

heat equation

(M, g) : compact, connected, oriented, $\partial M = \emptyset$

$\Rightarrow \exists k(t, x, y)$: fundamental solution
to the heat equation

- $(\frac{\partial}{\partial t} - \Delta_x) k = 0$
- $k(t, x, y) = k(t, y, x)$
- $\int_{y \in M} k(t, x, y) f(y) d\mu_y \rightarrow f(x)$ as $t \rightarrow 0$
- $k(t, x, y) = \sum_{i \geq 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)$

Solve $\begin{cases} (\frac{\partial}{\partial t} - \Delta) u = v(t, x) \\ u(0, x) = u_0(x) \end{cases}$ ← given

recall $(\frac{\partial}{\partial t} + \lambda) u(t) = v(t)$

$$\Rightarrow \frac{\partial}{\partial t} (e^{t\lambda} u) = e^{t\lambda} v(t)$$

$$\Rightarrow e^{t\lambda} u(t) - u(0) = \int_0^t e^{s\lambda} v(s) ds$$

$$\Rightarrow u(t) = e^{-t\lambda} u(0) + \int_0^t e^{(t-s)\lambda} v(s) ds$$

$$u(t, x) = \int_{y \in M} k(t, x, y) u_0(y) d\mu_y$$

$$+ \int_0^t \int_{y \in M} k(t-s, x, y) v(s, y) d\mu_y ds$$

It can be verified by a direct computation for smooth $u_0(x)$ and $V(t, x)$

defn $Z(t) = \sum_{\lambda \geq 0} e^{-\lambda t}$ is called the partition function of (M, g)
(or $Z(t; M, g)$)

Note that $Z(t) = \int_{x \in M} k(t, x, x) d\mu_x$

rmk • $Z(t)$ is equivalent to the data $\{\lambda_i\}_{i \geq 0}$

$$\bullet k(t, x, x) = \frac{1}{(4\pi t)^{n/2}} \mathbb{1} \cdot (\underbrace{u_0(x, x)}_{\mathbb{1}} + O(t)) \quad \text{as } t \rightarrow 0$$

$$\Rightarrow Z(t) \sim \frac{1}{(4\pi t)^{n/2}} \text{Vol}(M, g)$$

§ I. isoperimetric estimator and heat kernel

Suppose that (M, g) admits an isoperimetric estimator $H(\beta)$

with $H(\beta) \sim c\beta^\alpha$ for $\beta \sim 0$

($\alpha \in [\frac{n-1}{n}, 1)$)

(and $\sim c(1-\beta)^\alpha$ for $\beta \sim 1$)

$$M^* = S^{n-1} \times (0, 1) \cup N \cup S$$

$$\int_0^1 \frac{1}{\beta} d\beta = \infty$$

(with a choice of $V^* = \text{Vol}(M^*, g^*)$)

theorem Denote by $k_*(t, N, N)$ the heat kernel (assume its existence). Then,

$$\begin{aligned} Z(t) &= \text{Vol}(M, g) \sup_x k(t, x, x) \\ &\leq \text{Vol}(M^*, g^*) k_*(t, N, N) \end{aligned}$$

sketch Solve $\begin{cases} (\frac{\partial}{\partial t} - \Delta)u = 0 \\ u(0, x) = u_0(x) \end{cases}$

and the "symmetrized one" on (M^*, g^*)

As $u_0(x) \rightarrow \delta_{x_0}(x)$ the Dirac measure
 \Rightarrow obtain the comparison at some $x_0 \in M$

$$\left(\int k(t, x, y) u_0(y) dy \rightarrow k(t, x, x_0) \right)$$

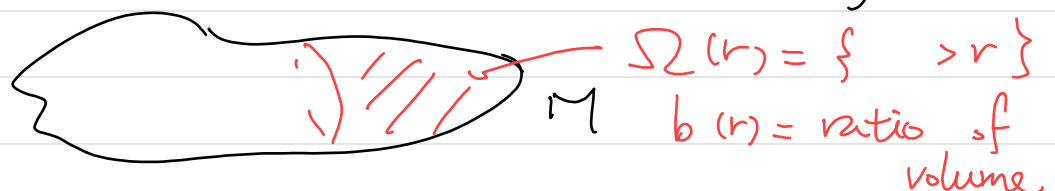
step 1 : $M \rightarrow \mathbb{R}_{>0}$ smooth

goal construct ID comparison datum depending on $\beta \in [0, 1]$

Range $c \in [0, c = \sup]$

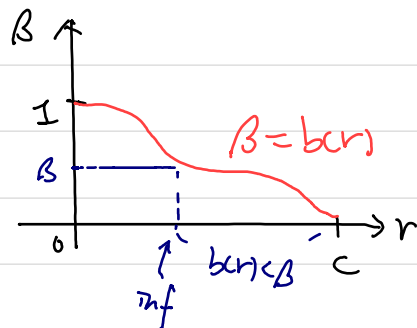
Let $D(r) = \{x \in M : u(x) > r\}$

$$b(r) = \frac{\text{Vol}(D(r))}{\text{Vol}(M)} \quad \downarrow \text{in } r$$



$\beta \in [0, 1] \rightarrow [0, c]$
 ideally $\beta \in [0, 1] \Rightarrow \beta = b(r)$ for some $r \in [0, c]$

Define $b(\beta) =$ that r
 $\Rightarrow b(\beta) = \inf \{ r : b(r) < \beta \}$



$\Rightarrow (b(r)) = r$ (a.e.)

$\beta =$ { volume ratio of sub-level set of

\rightarrow value of

Now, let $\Omega(\beta) = D(\beta) = \{ \dots \}$

Consider $F(\beta_0) = \int_{\Omega(\beta_0)} 1(x) d\mu_x \Rightarrow \Omega(b(r)) = D(r)$

$$= \int_0^c \left(\int_{\partial \Omega(b(r))} 1 \Gamma^1 \right) dr$$

$$= \int_0^c \left(r \int_{\partial \Omega(b(r))} 1 \Gamma^1 \right) dr$$

$$\text{Vol}(M) b(r) = \text{Vol}(D(r))$$

$$= \int_r^c \left(\int_{\partial D(p)} 1 \Gamma^1 \right) dp$$

$$\Rightarrow \text{Vol}(M) b'(r) = - \int_{\partial D(r)} \Gamma^1$$

$$F(\beta_0) = \int^C (-r \text{Vol}(M) b'(r)) dr$$

Use $\beta = b(r)$ $d\beta = b'(r) dr$
 $r \in [r, C] \iff \beta \in [0, \beta_0]$
 and $r =$

$$F(\beta_0) = \text{Vol}(M) \int_0^{\beta_0} d\beta$$

Relate $\int_{\Omega(\beta)} \Delta$ to $F''(\beta)$:

- $F'(\beta) = \text{Vol}(M)$

$$(b(r)) = r \Rightarrow (\beta) = \frac{1}{b'(r)} = \frac{1}{b'(r)}$$

$$\Rightarrow F''(\beta) = \text{Vol}(M) / b'(r(\beta))$$

- $\int_{\Omega(\beta)} \Delta d\mu_x = \int_{\partial\Omega(\beta)} \frac{\partial}{\partial \nu}$

$\partial\Omega(\beta)$ is the level set of $\beta \Rightarrow \nu \parallel \nabla f \nearrow$ along inward $\Rightarrow \nu \leftarrow$

$$\Rightarrow \int_{\Omega(\beta)}$$

\Rightarrow

$$\int_{\partial\Omega(B)}$$

Again. $\int_{\partial\Omega(B)} \Delta \cdot \int_{\partial\Omega(B)} k \geq (\text{Vol}(\partial\Omega(B)))^2$

$$\int_{\partial\Omega(B)} \Delta \leq - \frac{(\text{Vol}(\partial\Omega(B)))^2}{\int_{\partial\Omega(B)} \Delta}$$
$$= \frac{(\text{Vol}(\partial\Omega(B)))^2}{\text{Vol}(M) b'(\cdot)}$$
$$= \left(\frac{\text{Vol}(\partial\Omega(B))}{\text{Vol}(M)} \right)^2 \frac{\text{Vol}(M)}{b'(\cdot)}$$

Upshot

$$\int_{\partial\Omega(B)} \Delta \leq$$