§ I eigenvalue of a domain

$\Omega \subset \mathbb{R}^n$. smooth, bounded domain
(Ω: open. $\partial\Omega$: smooth)

Let $\lambda_1(\Omega)$ be the 1st Dirichlet eigenvalue of Δ

$$= \inf \left\{ \int_\Omega |\nabla u|^2 \Big/ \int_\Omega |u|^2 \ : \ \begin{matrix} u|_{\partial\Omega} = 0 \\ u \not\equiv 0 \end{matrix} \right\}$$

- Similarly. L^2_0 has eigenbasis (f_i, λ_i) $i = 1, 2, \cdots$

  $0 < \lambda_1 < \lambda_2 \leq \cdots$

  $\Delta f_i = -\lambda_i f_i$. $\qquad f_i|_{\partial\Omega} = 0$

  $f_1 > 0$ on Ω
- The only constant function is zero

  $\Rightarrow$ 0 is not an eigenvalue

<u>theorem</u> For any Ω as above. let Ω^* be the open ball in $\mathbb{R}^n$ with

$$Vol(\Omega^*) = Vol(\Omega)$$

Then. $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$ $\quad$ Faber – Krahn inequality

sketch of pf: For brevity, write f for f_1

$\forall\, t > 0$. let $\Omega_t = \{ x \in \Omega : f(x) > t \}$

Let Ω_t^* be the ball with

$$Vol(\Omega_t^*) = Vol(\Omega_t)$$

Note that $\Omega^*_{t_1} \subset \Omega^*_{t_2}$ if $t_1 > t_2$
and $\Omega^*_0 = \Omega^*$

Let $f^* : \Omega^* \to \mathbb{R}^+$ be the radial symmetric function with

$$f^*(x) = t \quad \text{for} \quad x \in \partial \Omega^*_t$$

It follows that f is non-increasing in the radius (= distance to the origin)

Intuitively, f^* is the "symmetrization" of f

<u>Recall</u> coarea formula

$$\int_{f \le a} u\, |df| = \int_0^a \Big(\int_{f^{-1}(t)} u \Big)\, dt$$

or

$$\int_{f \le a} u = \int_0^a \Big(\int_{f^{-1}(t)} u\, |df|^{-1} \Big)\, dt \quad \text{—} (*)$$

Now, consider $a = \sup_\Omega f$, $u = |df|^2$

$$\Rightarrow \int_\Omega |df|^2 = \int_0^a \Big(\int_{f^{-1}(t)} |df| \Big)\, dt$$

Also, consider $(*)$ with $u = 1$, and take $\frac{d}{da}$ $\Rightarrow$ $\frac{d}{dt} \mathrm{Vol}(\Omega_t) = \int_{f^{-1}(t)} |df|^{-1}$

$$\int_{f^{-1}(t)} |df|^{-1} \int_{f^{-1}(t)} |df| \geq \int_{f^{-1}(t)} 1 = Vol(\underset{\partial\Omega_t}{\underset{\shortparallel}{f^{-1}(t)}})$$

Similarly. $\int_{\Omega} |df^*|^2 = \int_0^a \left(\int_{(f^*)^{-1}(t)} |df^*| \right) dt$

Since $|df^*| = \text{const}$ on $(f^*)^{-1}(t)$

$$\int_{(f^*)^{-1}(t)} |df^*|^{-1} \int_{(f^*)^{-1}(t)} |df^*| = \int_{(f^*)^{-1}(t)} 1 = Vol(\partial\Omega_t^*)$$

<u>recall</u> classical Isoperimetric inequality

$$\frac{Vol(\partial\Omega_t)^n}{Vol(\Omega_t)^{n-1}} \geq \frac{Vol(\partial\Omega_t^*)^n}{Vol(\Omega_t^*)^{n-1}}$$

and $Vol(\Omega_t) = Vol(\Omega_t^*)$ by construction

$$\Rightarrow \int_{f^{-1}(t)} |df| \geq \frac{Vol(\partial\Omega_t)}{\int_{f^{-1}(t)} |df|^{-1}} \geq \frac{Vol(\partial\Omega_t^*)}{\int_{f^{-1}(t)} |df|^{-1}}$$

$$= \underbrace{\left(\int_{(f^*)^{-1}(t)} |df^*|^{-1} \Big/ \int_{f^{-1}(t)} |df|^{-1} \right)}_{\shortparallel} \int_{(f^*)^{-1}(t)} |df^*|$$

$$\frac{d}{dt} Vol(\Omega_t^*) \Big/ \frac{d}{dt} Vol(\Omega_t) = 1$$

Hence $\int_\Omega |df|^2 \geq \int_{\Omega^*} |df^*|^2$

$$\int_\Omega f^2 = \int_0^a \Big(\underbrace{\int_{f^{-1}(t)} f^2 |df|^{-1}}_{\substack{\| \\ t^2 \int_{f^{-1}(t)} |df|^{-1} = t^2 \int_{(f^*)^{-1}(t)} |df^*|^{-1} \text{ as before}}} \Big)\, dt \;\underset{\Uparrow}{=}\; \int_{\Omega^*} (f^*)^2$$

$$\Rightarrow \quad \lambda_1(\Omega) = \int_\Omega |df|^2 \Big/ \int_\Omega f^2 \geq \int_{\Omega^*} |df^*|^2 \Big/ \int_{\Omega^*} (f^*)^2$$

$$\geq \inf \cdots = \lambda_1(\Omega^*)$$

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§ II On a Riemannian manifold

1° For $\Omega \subset M$, we can still consider the least area of $\partial\Omega$

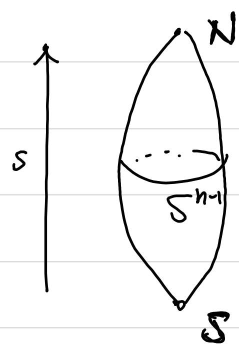
<u>defn</u> The isoperimetric function of (M, g) for $\beta \in [0, 1]$ is

$$h(\beta) = h(M, g; \beta) = \inf \left\{ \frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(M)} : \Omega \subset M,\ \mathrm{Vol}(\Omega) = \beta\, \mathrm{Vol}(M) \right\}$$

An isoperimetric estimator of (M, g) is a function $H : [0, 1] \to \mathbb{R}_{\geq 0}$ such that $h(\beta) \geq H(\beta) \quad \forall \beta \in [0, 1]$

prop.
- $h(\beta) = h(1-\beta)$
- $h(\beta) \sim c_n \, \mathrm{Vol}(M)^{-\frac{1}{n}} \, \beta^{\frac{n-1}{n}}$ for $\beta \sim 0$
- $h(\beta)$ is continuous, has left and righ derivative $\forall \beta$, and is differentiable except on a denumerable set

2° From now on, let us assume (M, g) admits an Isoperimetric estimator $H(\beta)$ and see how to use it



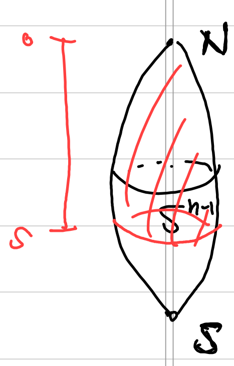
$$M^* = S^{n-1} \times (0, L) \cup \{N, S\}$$

$$g^* = ds^2 + a^2(s)\, g_0$$ ← standard metric on S^{n-1} of radius 1

$$a(0) = 0 = a(L)$$

$$a(s) > 0 \quad \forall s \in (0, L)$$

(if $a'(0) = 1$, $a'(L) = -1 \Rightarrow$ smooth)



Let $V^* = \mathrm{Vol}(M^*, g^*)$

$B_s(N)$

$$A(s) = \mathrm{Vol}\,(\text{ball of radius } s \text{ at } N) \,/\, V^*$$

$$= \frac{\mathrm{Vol}(S^{n-1})}{V^*} \int_0^s a^{n-1}(\rho)\, d\rho \ \in [0, 1]$$

How to choose $a(s)$?
It is natural to require that

$$A'(s) = \frac{\mathrm{Vol}(\partial B_s(N))}{V^*} = H(A(s)) \quad \forall s$$

$$\Rightarrow \frac{1}{H(A(s))} A'(s) = 1$$

$$\Rightarrow \int_0^{A(s)} \frac{d\beta}{H(\beta)} = s$$: this determines $A(s)$

and thus $\int_0^1 \frac{d\beta}{H(\beta)} = L$

3° example (S^2, g_0)

$g = dr^2 + \sin^2 r\, d\theta^2$

$$\mathrm{Vol}(B_r(N)) = \int_0^r \int_0^{2\pi} \sin s\, ds\, d\theta$$

$$= 2\pi (1 - \cos r)$$

$$\mathrm{Vol}(\partial B_r(N)) = 2\pi \sin r$$

$$\mathrm{Vol}(S^2) = 4\pi$$

$\Rightarrow$ If $\beta = \frac{2\pi(1-\cos r)}{4\pi} = \frac{1-\cos r}{2}$

$$h(\beta) = \frac{2\pi \sin r}{4\pi} = \frac{1}{2} \sin r = \sqrt{\beta(1-\beta)}$$

$\cos r = 1 - 2\beta \quad \sin r = \sqrt{1-(1-2\beta)^2} = \sqrt{2\beta(2-2\beta)}$

exercise

Now, suppose that (M^2, g) has isoperimetric estimator $H(\beta) = \sqrt{\beta(1-\beta)}$

$$\Rightarrow \int_0^{A(s)} \frac{d\beta}{\sqrt{\beta(1-\beta)}} = s \Rightarrow A(s) = \sin^2 \frac{s}{2}, \quad L = \pi$$

$$A(s) = \frac{2\pi}{V^*} \int_0^s a(\rho)\, d\rho$$

$$\Rightarrow a(s) = \frac{V^*}{4\pi} \sin s$$

$$g^* = ds^2 + \left(\frac{V^*}{4\pi} \sin s\right)^2 d\theta^2$$

Note that V^* is a formal parameter

If we take $V^* = 4\pi$

$\Rightarrow (M^*, g^*) = (S^2, g_0)$