§I Euclidean space

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i}\right)^2 \quad \text{on} \quad \mathbb{R}^n$$

recall $\quad k(t,x,y) = \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\dfrac{|x-y|^2}{4t}\right)$

$$\in C^\infty\left((0,\infty) \times \left(\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}\right)\right)$$

It has the following properties

(i) $k(t,x,y) = k(t,y,x)$

(ii) $\left(\frac{\partial}{\partial t} - \Delta_x\right) k = 0$

(iii) $\displaystyle \int_{y \in \mathbb{R}^n} k(t,x,y) f(y)\, dy \xrightarrow{t \to 0^+} f(x)$

$\tilde{f}(t,x) :=$

(iv) $\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{f} = 0 \\ \tilde{f}(0,x) = f(x) \end{cases}$

instantaneously smoothing

Namely, k is the fundamental solution to the heat equation subject to the initial condition $(t=0)$

§II formal expression

Now, work on (M,g)

$\Delta \varphi_i = -\lambda_i \varphi_i \qquad \lambda_i \geq 0 \qquad \{\varphi_i\}$: basis for $L^2(M)$

$$f = \sum_{i \geq 0} a_i \varphi_i$$

Solve $\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{f} = 0 \\ \tilde{f}(0,x) = f(x) \end{cases}$

Similarly, write $\tilde{f}(t,x) = \sum_{i\geq 0} c_i(t)\, \varphi_i(x)$

$$c_i(0) = a_i$$

$$\sum_{i\geq 0} (c_i' + \lambda_i c_i)\, \varphi_i = 0$$

$$\Rightarrow \quad c_i(t) = a_i e^{-\lambda_i t}$$

Therefore,

$$\tilde{f}(t,x) = \sum_{i\geq 0} \underset{\underset{(f,\varphi_i)}{\parallel}}{\underline{a_i}}\, e^{-\lambda_i t} \varphi_i(x)$$

$$= \sum_{i\geq 0} \underline{(f, \varphi_i)}\, e^{-\lambda_i t} \varphi_i(x)$$

$$= \int_{y\in M} f(y) \Big(\sum_{i\geq 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \Big)\, d\mu$$

$$\Rightarrow \quad e(t,x,y) = \sum_{i\geq 0} e^{-\lambda_i t} \varphi_i(x)\, \varphi_i(y)$$

symmetric in x & y.

is the heat kernel

rmk One can show that

$$e(t,x,y) \in C^\infty((0,\infty)\times M \times M)$$

based on the discussion last time

§III. parametrix construction

0° $\Delta(f_1 f_2) = (\Delta f_1) f_2 + 2\langle df_1, df_2\rangle + f_1(\Delta f_2)$

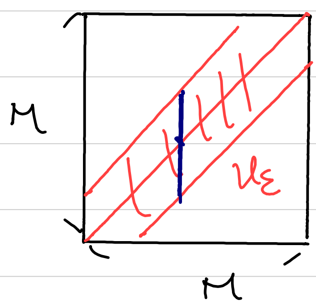
pointwise formula. direct computation

1° Naturally, consider

$$G(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathrm{dist}^2(x,y)}{4t}\right) \in C^\infty(\mathbb{R}^+ \times \mathcal{U}_\varepsilon)$$

Fix $0 < \varepsilon <$ injectivity radius of M

$$\mathcal{U}_\varepsilon = \{(x,y) \in M \times M : \mathrm{dist}(x,y) < \varepsilon\}$$



In general,

$$\left(\frac{\partial}{\partial t} - \Delta_y\right) G \neq 0$$

2° For any $x \in M$, use the geodesic polar coordinate at x

$$(s^1 \cdots, s^n) \in \mathbb{R}^n \cong T_xM$$

$$\longmapsto \exp_x(s^i e_i) \in M$$

$T_xM \ni 0$

$\updownarrow$

x

$\longleftarrow$ the point y

Then use the "polar" coordinate for $\mathbb{R}^n$

$$[0,\infty) \times S^{n-1} \longrightarrow \mathbb{R}^n$$

$$(r, \theta_2, \cdots, \theta_n)$$

For $n=2$, $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$

By Gauss lemma,

$$g = dr^2 + r^2 \sum_{2 \le i,j \le n} \tilde{g}_{ij}(r,\theta)\, d\theta_i\, d\theta_j$$

- no $dr\, d\theta_i$ terms
- $\lim_{r \to 0} \sum_{2 \le i,j \le n} \tilde{g}_{ij}(r,\theta)\, d\theta_i\, d\theta_j \longrightarrow g_{std}$ $\leftarrow S^{n-1}$

- The series expansion of $\tilde{g}$ is given by curvature & its covariant derivatives.
- For $n=2$

$$dr^2 + r^2\left(1 - \tfrac{K}{6}r^2 + \cdots\right)^2 d\theta^2$$

Gaussian curvature at x

$$\Rightarrow \det g = r^{2n-2} \det \tilde{g}_{ij} =: D^2$$

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{1\le i,j\le n} \left(\frac{\partial}{\partial \xi^i} \sqrt{\det g}\, g^{ij} \frac{\partial}{\partial \xi^j} \right)$$

For $f = f(r)$, only $j=1 \Rightarrow i=1$

$$\Rightarrow \Delta f = \frac{1}{r^{n-1}D} \frac{\partial}{\partial r}\left(r^{n-1} D \frac{\partial f}{\partial r} \right)$$

$$= \frac{\partial^2 f}{\partial r^2} + \left(\frac{n-1}{r} + \frac{D'}{D} \right) \frac{\partial f}{\partial r}$$

$\partial_r D$

(Also, $df = \frac{\partial f}{\partial r} dr$)

3° (educated) guess:

Fix $k \in \mathbb{N}$, consider

$$S = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{dist^2(x,y)}{4t}\right) \cdot \left(u_0(x,y) + u_1(x,y)\, t + \cdots + u_k(x,y)\, t^k \right)$$

$$= \underbrace{(4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right)}_{G} \sum_{j\ge 0} u_j(r,\theta)\, t^j$$

Note that $\frac{\partial G}{\partial t} = \left(-\frac{n}{2t} + \frac{r^2}{4t^2}\right) G$

$$\frac{\partial G}{\partial r} = -\frac{r}{2t} G$$

$$\Delta G = \left(-\frac{1}{2t} + \frac{r^2}{4t^2} - \frac{n-1}{2t} - \frac{r}{2t}\frac{D'}{D}\right) G$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \Delta\right) G = \frac{r}{2t}\frac{D'}{D} G$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(G \cdot (u_0 + u_1 t + \cdots + u_k t^k)\right)$$

$$= \frac{r}{2t}\frac{D'}{D} G \cdot (u_0 + u_1 t + \cdots + u_k t^k)$$
$$+ G (u_1 + 2u_2 t + \cdots + k u_k t^{k-1})$$
$$- G (\Delta u_0 + \Delta u_1 t + \cdots + \Delta u_k t^k)$$
$$\underbrace{- 2 \langle \underline{dG}, du_0 + du_1 t + \cdots + du_k t^k \rangle}_{+\frac{r}{t} G\left(\frac{\partial u_0}{\partial r} + \frac{\partial u_1}{\partial r} t + \cdots + \frac{\partial u_k}{\partial r} t^k\right)}$$

$dG = -\frac{r}{2t} G\, dr$

$t^{-1} G$: $\quad r\frac{\partial u_0}{\partial r} + \frac{r}{2}\frac{D'}{D} u_0 = 0$

G: $\quad r\frac{\partial u_1}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + 1\right) u_1 - \Delta u_0 = 0$

$t^{i-1} G$: $\quad r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right) u_i - \Delta u_{i-1} = 0$

$t^k G$: $\quad -\Delta u_k$ (left-over)

$u_0 = D^{-\frac{1}{2}}$ $\qquad$ note that $u_0(0) = 1$

$$r\frac{\partial u_i}{\partial r} + \left(\frac{r}{2}\frac{D'}{D} + i\right) u_i - \Delta u_{i-1} = 0$$

The integral facter is $r^i D^{\frac{1}{2}}$

$$v_i = r^i D^{\frac{1}{2}} u_i$$

$$\Rightarrow \frac{\partial v_i}{\partial r} = - r^{i-1} D^{\frac{1}{2}} \Delta u_{i-1}$$

$$\Rightarrow u_i(r,\theta) = -r^{-i} D^{-\frac{1}{2}}(r,\theta) \cdot \int_0^r \rho^{i-1} D^{\frac{1}{2}}(\rho,\theta) (\Delta u_{i-1})(\rho,\theta)\, d\rho$$

Finally, $\left(\frac{\partial}{\partial t} - \Delta_y\right) S_k$

$$= (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{dist^2(x,y)}{4t}\right) t^k \Delta_y u_k$$

Choose a cut $\chi(r) = \begin{cases} 1 & r \le \varepsilon/2 \\ 0 & r \ge \varepsilon \end{cases}$

$$\Rightarrow \chi(dist(x,y)) S_k \in C^\infty((0,\infty) \times M \times M)$$

<u>Lem</u> (i) $\left(\frac{\partial}{\partial t} - \Delta\right)(\chi S_k) \in C^\ell([0,\infty) \times M \times M)$

if $\ell < k - \frac{n}{2}$

(ii) $\lim_{t \to 0} \int_{y \in M} (\chi S_k)(t,x,y) f(y)\, d\mu = f(x)$

pf: direct computation and calculus

key for (ii) $u_0(x,x) = 1$ ※

§IV. True heat kernel ?

: V S

Denote by