On $(M, g)$, denote by $\Delta$ the Laplace-Beltrami operator on real-valued functions

In coordinate, $\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x^j}\right)$
$= -(\delta d + \cancel{d\delta}^{\to 0})$

§I. uniqueness of the solution to heat

**lem** Given $f \in C^\infty(M)$, let $\tilde{f} \in C^0([0,T) \times M) \times C^\infty((0,T) \times M)$ such that
$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right)\tilde{f} = 0 \\ \tilde{f}(0, x) = f(x) \end{cases}$$

Then $\tilde{f}$ is unique

pf: It suffices to show that $\tilde{f}$ must vanish if $f$ is zero

$$\frac{d}{dt} \int_M |\tilde{f}|^2 \, d\mu = \int 2\tilde{f}\, \partial_t \tilde{f} \, d\mu$$
$$= \int 2\tilde{f}\, \Delta\tilde{f} \, d\mu$$
$$= -2\int |\nabla \tilde{f}| \, d\mu$$

$\Rightarrow \int |\tilde{f}|^2 d\mu$ is non-increasing in $t$

Since $f$ is zero at $t=0$,
$$\int |\tilde{f}|^2 d\mu \equiv 0 \quad \forall t \Rightarrow \tilde{f} \equiv 0$$
✲

## §II. coarse estimate on eigenvalues

1° <u>recall</u> from exercise in ch. 6 of Warner

$\exists\, (\lambda_i, \varphi_i) \in \mathbb{R}_{\geq 0} \times C^\infty(M)$ for $i = 0, 1, \cdots$

such that • $\Delta \varphi_i = -\lambda_i \varphi_i$

• $0 = \lambda_0 < \lambda_1 \leq \cdots \longrightarrow \infty$

(no finite accumulation value)

• $\{\varphi_i\}_{i \geq 0}$ is a basis for $L^2(M)$

$\varphi_0 = (\mathrm{Vol}(M))^{1/2}$

2° The growth rate of $\lambda_i$ in $i$?

Fix $k \in \mathbb{N}$

Let $E_k = \mathrm{span}\{\varphi_0, \cdots \varphi_k\}$

<u>goal</u> study $\max |f|$ for $f \in E_k$

i) $H_s = W^{2,s}$

For $s \in \mathbb{N}$. $\|f\|_{H_s}^2 \approx \int |f|^2 + \cdots + |\nabla^{(k)} f|^2$

Elliptic estimate. $\|f\|_{H_2} \lesssim \|(1-\Delta) f\|_{L^2}$

$\Rightarrow \|f\|_{H_{2m}} \lesssim \|(1-\Delta)^m f\|_{L^2}$

ii) By Sobolev. fix $\delta > 0$

$H_{\delta + n/2} \hookrightarrow C^0$

For $f \in E_k$

$\|f\|_{L^\infty} \lesssim \|f\|_H$

$\lesssim (1+\lambda_k)^{\delta + \frac{n}{2}} \|f\|_{L^2}$

iv) For any $y \in M$, consider