DIFFERENTIAL GEOMETRY II: HOMEWORK 7

DUE APRIL 28

(1) Let $E \to M$ be a vector bundle, and M is compact without boundary. Endow M a Riemannian metric g. Endow E a bundle metric, and a metric connection ∇.

For any $s \in \Gamma(E)$, and any tangent vectors X, Y, let

$$\nabla^2 s(X,Y) = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s$$

where $\nabla_X Y$ is the Levi-Civita connection of (M, g). Define

$$\Box\, s = \operatorname{tr}_g \nabla^2 s \ .$$

(a) Show that

$$\int_M \langle \nabla s_1, \nabla s_2 \rangle \, \mathrm{d}\mu_g = -\int_M \langle \Box\, s_1, s_2 \rangle \, \mathrm{d}\mu_g \ .$$

(b) Show that

$$\Delta |s|^2 = 2\langle \Box\, s, s \rangle + 2|\nabla s|^2 \ .$$

(2) Let (M, g) be a Riemannian manifold, and ∇ be its Levi-Civita connection. For a tensor $\Psi = \psi_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$, its covariant derivative is $\nabla \Psi = \psi_{ij;k}\, \mathrm{d}x^k \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j$. In other words,

$$\begin{aligned} \psi_{ij;k} &= (\nabla_{\partial_k} \Psi)(\partial_i, \partial_j) \\ &= \partial_k \psi_{ij} - \Gamma^{\ell}_{ik} \psi_{\ell j} - \Gamma^{\ell}_{jk} \psi_{i\ell} \ . \end{aligned}$$

Similarly,

$$\begin{aligned} \psi_{ij;k\ell} = \psi_{ij;k;\ell} &= (\nabla_{\partial_\ell}(\nabla \Psi))(\partial_k, \partial_i, \partial_j) \\ &= \partial_\ell \psi_{ij;k} - \Gamma^q_{i\ell} \psi_{qj;k} - \Gamma^q_{j\ell} \psi_{iq;k} - \Gamma^q_{k\ell} \psi_{ij;q} \ . \end{aligned}$$

This notations is also defined in the same way for tensors of other types.

(a) If Ψ is symmetric, $\psi_{ij} = \psi_{ji}$, verify that $\psi_{ij;k} = \psi_{ji;k}$.

(b) Verify that $\Delta \operatorname{tr}(\Psi) = \operatorname{tr}(\Box\, \Psi)$.

(3) Let $M^n \subset \mathbb{R}^{n+1}$ be a 2-sided minimal hypersurface. Let $\vec{\nu}$ be a unit normal field. Consider its second fundamental form:

$$A = h_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

where $\{x^i\}$ is a local coordinate for M. Prove that $\Box\, A = -|A|^2 A$.

Hint: You have to use the Gauss equation and Codazzi equation.

(4) Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^1$ satisfies

$$\sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\frac{\partial_i f}{\sqrt{1+|Df|^2}} \right) = 0 \ .$$

We have shown that Γ_f is a volume minimizer, and hence is stable. Prove that Γ_f is stable, without invoking the volume minimizing property of Γ_f.