DIFFERENTIAL GEOMETRY II: HOMEWORK 6

DUE APRIL 7

(1) Consider the hyperbolic space: $B^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : |x|^2 = \sum_{j=1}^n (x^j)^2 < 1\}$ with the metric

$$\frac{4}{(1-|x|^2)^2} \sum_{j=1}^n (\mathrm{d}x^j)^2 \ .$$

Does it contain non-trivial compact minimal submanifolds? Give your reason.
Hint: Its isometry group acts transitively.

(2) Consider $\mathbb{R}^3 \oplus \mathbb{R}^3 \cong \mathbb{C}^3$. Denote its coordinate by x^1, x^2, x^3 and y^1, y^2, y^3. Let g be the standard metric, and let

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \mathrm{d}x^2 \wedge \mathrm{d}y^2 + \mathrm{d}x^3 \wedge \mathrm{d}y^3 \ .$$

Let J be the endomorphism of $T\mathbb{C}^3$ given by

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^j} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^j}\right) = -\frac{\partial}{\partial x^j}$$

for $j = 1, 2, 3$. Note that $J^2 = -\mathbf{I}$,

$$g(U, V) = \omega(U, JV) = -\omega(JU, V) \ ,$$
$$\omega(U, V) = g(JU, V) = -g(U, JV)$$

for any vectors U, V.

For a collection of vectors $U_1, \ldots, U_k$, denote by $|U_1 \wedge \cdots \wedge U_k|$ the volume of the k-parallelotope spanned by them.

(a) For any U_1, U_2, U_3, show that

$$|U_1 \wedge U_2 \wedge U_3 \wedge J(U_1) \wedge J(U_2) \wedge J(U_3)| \le |U_1 \wedge U_2 \wedge U_3|^2 \ ,$$

and the equality holds if and only if ω vanishes on the 3-space spanned by U_1, U_2, U_3.

(b) Let

$$\alpha = \mathrm{Re}(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3) \quad \text{and} \quad \beta = \mathrm{Im}(\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3)$$

where $\mathrm{d}z^j = \mathrm{d}x^j + i\,\mathrm{d}y^j$. For any U_1, U_2, U_3, show that

$$\begin{aligned} &[\alpha(U_1, U_2, U_3)]^2 + [\beta(U_1, U_2, U_3)]^2 \\ &= |U_1 \wedge U_2 \wedge U_3 \wedge J(U_1) \wedge J(U_2) \wedge J(U_3)| \ . \end{aligned}$$

(c) Show that for any U_1, U_2, U_3

$$|\beta(U_1, U_2, U_3)| \leq |U_1 \wedge U_2 \wedge U_3| \ .$$

Moreover, the equality holds if and only if

- $\alpha(U_1, U_2, U_3) = 0$ and
- ω vanishes on the 3-space spanned by U_1, U_2, U_3.

It is clear that $\mathrm{d}\beta = 0$. With the Stokes theorem, one can show that a 3-dimensional submanifold L satisfying $\beta|_L = \mathrm{dvol}\,|_L$ must be a volume minimizer within its homology class. In particular, it is a minimal submanifold. This exercise says that the condition is equivalent to that $\omega|_L$ and $\alpha|_L$ both vanish.

Remark. $n = 3$ plays no role here. It works in any dimension.