DIFFERENTIAL GEOMETRY II: HOMEWORK 3

DUE MARCH 17

(1) Let $E \to M$ be a rank k real vector bundle with a connection ∇. Verify that the horizontal distribution $\mathcal{H} \subset TE$ is well-defined. To be more precise, check differential trivializations lead to the same $\mathcal{H}$.

(2) The Heisenberg is a matrix group diffeomorphic to $\mathbb{R}^3$:

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(3; \mathbb{R}) \right\} .$$

Its tangent space at the identity is

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} : u, v, w \in \mathbb{R} \right\} .$$

A direct computation finds the matrix exponential

$$\exp\left(\begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & u & w + \frac{1}{2}uv \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix} .$$

(a) Check that the matrix exponential coincides with the Lie theoretical[1] exponential.

(b) Check by direct computation that the matrix bracket coincides with the Lie theoretical bracket.

(c) Let

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} , \qquad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Denote by $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ their left-invariant extensions. Is $\mathrm{span}\{\tilde{U}, \tilde{V}\}$ involutive?

(d) Check that $\mathrm{span}\{\tilde{U}, \tilde{W}\}$ is involutive. Find its integration (subgroup) through the identity matrix.

(e) Construct three linearly independent left-invariant 1-forms on H. Equivalently, find the entries of $g^{-1}\mathrm{d}g$.

[1] It means the one comes from the left-invariant vector field construction.

(3) Consider the matrix group

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \ : \ x > 0 \text{ and } y \in \mathbb{R} \right\} .$$

Its tangent space at the identity can be identified with

$$\mathfrak{g} = \left\{ \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix} \ : \ u, v \in \mathbb{R} \right\} .$$

(a) Construct two linearly independent left-invariant 1-forms on G, and calculate their pull-back under the exponential map.

(b) Construct a left-invariant and a right-invariant area form on G, and show that the only bi-invariant 2-form on G is zero.

Remark. A compact Lie group admits a bi-invariant Riemannian metric. In general, a non-compact Lie group may not admit a bi-invariant volume form.

(4) Consider $\mathrm{SL}(2;\mathbb{R}) = \{\mathfrak{m} \in \mathrm{GL}(2;\mathbb{R}) \mid \det \mathfrak{m} = 1\}$. Denote the identity matrix by $\mathbf{I}$. Its tangent space at $\mathbf{I}$ can be identified with 2×2, traceless matrices. Denote it by

$$\mathfrak{sl}(2;\mathbb{R}) = \left\{ \mathfrak{a} \in \mathbb{M}(2;\mathbb{R}) \mid \mathrm{tr}(\mathfrak{a}) = 0 \right\} .$$

Consider the exponential map

$$\begin{array}{rccl} \exp : & \mathfrak{sl}(2;\mathbb{R}) & \to & \mathrm{SL}(2;\mathbb{R}) \\ & \mathfrak{a} & \mapsto & \mathbf{I} + \mathfrak{a} + \frac{1}{2}\mathfrak{a}^2 + \cdots + \frac{1}{k!}\mathfrak{a}^k + \cdots \end{array} .$$

(a) Prove that for any $\mathfrak{a} \in \mathfrak{sl}(2;\mathbb{R})$, the eigenvalues of $\exp(\mathfrak{a})$ lie either in the unit circle, or in the positive real line.

It follows that $-\mathbf{I} \in \mathrm{SL}(2;\mathbb{R})$ does not belong to the image of the exponential map, and the exponential map is *not surjective*.

(b) For any $\mathfrak{a} \in \mathfrak{sl}(2;\mathbb{R})$, prove that

$$\exp(\mathfrak{a}) = (\cosh\lambda)\,\mathbf{I} + \frac{\sinh\lambda}{\lambda}\mathfrak{a}$$

where $\lambda = (-\det\mathfrak{a})^{\frac{1}{2}}$ is one of the eigenvalues of $\mathfrak{a}$. When $\det\mathfrak{a} = 0$, the above formula reads $\exp(\mathfrak{a}) = \mathbf{I} + \mathfrak{a}$.

(5) (a) Verify that $\mathbb{R}^3$ with the standard cross product constitutes a Lie algebra.

(b) Show that the Lie algebra in (a) is isomorphic to $\mathfrak{o}(3)$, the Lie algebra of the orthogonal group in dimension 3.