DIFFERENTIAL GEOMETRY II: HOMEWORK 2

DUE MARCH 10

(1) Polar Decomposition for $\mathrm{GL}(n;\mathbb{C})$.
   (a) Show that any $G \in \mathrm{GL}(n;\mathbb{C})$ has a unique decomposition as $G = UP$ where U is unitary, and P is hermitian, positive-definite.
   (b) The space of all $n \times n$ hermitian matrices can be identified with $\mathbb{R}^{n^2}$. Positive-definite ones form an open subset in it. Prove that the space of all $n \times n$ hermitian, positive-definite matrices is contractible.
   (Check the definition of "contractible" from wikipedia or any topology textbook. A "contractible" space is usually regarded as having "trivial topology".)

(2) Given a *complex* vector bundle $E \xrightarrow{\pi} M$, one can construct its dual bundle $E^* \to M$, and its conjugate bundle $\bar{E} \to M$. Show that the dual of the conjugate of E is isomorphic to itself, $(\bar{E})^* \cong E$.

(3) A local trivialization, $E|_{\mathcal{U}} \cong \mathcal{U} \times \mathbb{R}^k$, is equivalent to local trivializing sections, $\{\mathfrak{s}_\mu\}_{\mu=1}^k$. Each section $\mathfrak{s}_\mu$ corresponds to the standard basis $\mathfrak{e}_\mu$ for $\mathbb{R}^k$. Given a connection ∇, $\nabla\mathfrak{s}_\nu$ can be expressed as a linear combination of $\{\mathfrak{s}_\mu\}_{\mu=1}^k$, with the coefficients being 1-forms on $\mathcal{U}$. Namely,

$$\nabla\mathfrak{s}_\nu = \sum_{\mu=1}^k \omega_\nu^\mu \otimes \mathfrak{s}_\mu \quad \text{where } \omega_\nu^\mu \in \Omega^1(\mathcal{U}) \ .$$

The expression is a section of $T^*M \otimes E$ over $\mathcal{U}$. Sometimes $\otimes$ is omitted.

Any local section can be expressed as $\sum_{\mu=1}^k \alpha^\mu\, \mathfrak{s}_\mu$ where $\alpha^\mu \in \mathcal{C}^\infty(\mathcal{U})$. Due to the properties of a connection,

$$\begin{aligned}\nabla(\sum_{\mu=1}^k \alpha^\mu\, \mathfrak{s}_\mu) &= \sum_{\mu=1}^k (\mathrm{d}\alpha^\mu)\mathfrak{s}_\mu + \sum_{\mu=1}^k \alpha^\mu \nabla\mathfrak{s}_\mu \\ &= \sum_{\mu=1}^k (\mathrm{d}\alpha^\mu)\mathfrak{s}_\mu + \sum_{\mu,\nu}^k \alpha^\nu \omega_\nu^\mu \mathfrak{s}_\mu = \sum_{\mu=1}^k (\sum_{\nu=1}^k \mathrm{d}\alpha^\mu + \omega_\nu^\mu\, \alpha^\nu)\mathfrak{s}_\mu \ .\end{aligned}$$

That is to say, ∇ in terms of the trivialization is $\mathrm{d} + [\omega_\nu^\mu]$ acting on $\mathbb{R}^k$-valued functions.
   (a) Endow E a bundle metric. A connection ∇ is called a *metric connection* if

$$\mathrm{d}\langle s, \tilde{s}\rangle = \langle \nabla s, \tilde{s}\rangle + \langle s, \nabla\tilde{s}\rangle$$

for any two $s, \tilde{s} \in \Gamma(E)$[1]. Prove that a metric connection always exists.

(b) Does the metric connection unique? Give your reason.

(c) Suppose that E is a real vector bundle with a bundle metric and a metric connection. In terms of an *orthonormal*, local trivializing sections, what can you say about the matrix-valued 1-form $[\omega^{\mu}_{\nu}]$?

[1]The notation $\Gamma(E)$ is the space of all smooth sections.