DIFFERENTIAL GEOMETRY II: HOMEWORK 1

DUE MARCH 3

(1) Consider $\mathbb{CP}^1 = \{\text{complex lines in } \mathbb{C}^2\}$. It is the one-point compactification of $\mathbb{C}$, and is diffeomorphic to $\mathbb{S}^2$.

   (a) Define analogously the tautological (complex) line bundle E over $\mathbb{CP}^1$.

   (b) Recall that $\mathbb{CP}^1 = \dfrac{\mathbb{C} \cup \mathbb{C}}{z \sim w = z^{-1}}$. In terms of this coordinate cover, work out the transition function of the tautological bundle.

(2) Consider the unit sphere in $\mathbb{R}^{n+1}$, $S^n \subset \mathbb{R}^{n+1}$. Show that $TS^n \oplus \underline{\mathbb{R}}$ is isomorphic to $\underline{\mathbb{R}}^{n+1}$.

   The notation $\oplus$ means the direct sum of vector bundles. Specifically, suppose that E has transition $g_{\mathcal{UV}}$. Then, $E \oplus \underline{\mathbb{R}}$ has transition

$$\begin{bmatrix} g_{\mathcal{UV}} & 0 \\ 0 & 1 \end{bmatrix} .$$

(3) The Grassmannian $\mathrm{Gr}(k, n)$, the space of all k-planes in $\mathbb{R}^n$, is a smooth manifold of dimension $k(n-k)$.

   (a) Define analogously the tautological vector bundle E over the Grassmannian $\mathrm{Gr}(k, n)$.
   (b) What is the dimension of E, as a smooth manifold? Give your reason.
   (c) For $\mathrm{Gr}(2, 4)$, work out its local trivializations over *two* coordinate charts, and find the transition functions $g_{\mathcal{UV}}$.

(4) Suppose that $E \xrightarrow{\pi} M$ is a real vector bundle of rank k. Let

$$E \times_M E = \{(e_1, e_2) \in E \times E \mid \pi(e_1) = \pi(e_2)\} .$$

   Namely, it associates $E_p \times E_p$ for every $p \in M$. Locally, $E|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^k$, $E|_{\mathcal{U}} \times E|_{\mathcal{U}} = (\mathcal{U} \times \mathbb{R}^k) \times (\mathcal{U} \times \mathbb{R}^k)$, and $(E \times E)|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^k \times \mathbb{R}^k$.

   A *bundle metric* is a smooth map $\mathfrak{g} : E \times_M E \to \mathbb{R}$ which defines a inner product on E_p for every p.

   (a) Show that any (real) vector bundle always admits a bundle metric. (But it is never unique.)

(b) Prove that for any (real) vector bundle, the transition functions can be required to be orthogonal matrices, i.e.

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{O}(k) \subset \mathrm{GL}(k; \mathbb{R}) \ .$$

(c) Show that any real vector bundle is isomorphic (abstractly) to its dual bundle.