§I. heat kernel estimates

theorem 0 (Berard - Besson - Gallot . Invent. Math 85)

$$r_0(M,g) = \inf\{ \mathrm{Ricci}(u,u) : u \text{ unit tangent vector to } M\}$$

$$d(M,g) = \text{diameter of } (M,g) = \sup\{ d(x,y) : x,y \in M\}$$

If $r_0^2 \cdot d \geq (n-1)\,\varepsilon\,\alpha^2$ for some $\alpha > 0$, $\varepsilon \in \{1,0,-1\}$

Then, $\mathrm{Vol}(M)\, k(t,x,x) \leq Z_{S^n}(t/R^2)$ for some $R = R(n,\varepsilon,\alpha)$

unit sphere

rmk (i) $g \rightsquigarrow Rg \qquad \lambda \rightsquigarrow \frac{1}{R^2}\lambda$

$$S^n(1) : \lambda_0, \lambda_1, \ldots$$
$$S^n(R) : \frac{\lambda_0}{R^2}, \frac{\lambda_1}{R^2} \ldots$$

$$\Rightarrow Z_{S^n(R)}(t) = \sum_{i \geq 0} e^{-\frac{\lambda_0}{R^2}t} = Z_{S^n(1)}\left(\frac{t}{R^2}\right)$$

(ii) From the discussion last time. it remains to show $h_{S^n(R)}(\beta)$ is an Isoperimetric estimator of (M,g)

defn $\mathcal{M}_{n,k,D} = \{ (M^n, g) : r_0(M,g) \geq (n-1)k,\ d(M,g) \leq D \}$

$k \in \mathbb{R},\ D > 0$

<u>theorem 1</u> Fix (n, k, D). $\forall (M, g) \in \mathcal{M}_{n,k,D}$

i) $\lambda_j \geq c\, j^{2/n}$

ii) $N(\lambda) = \#\{ j : \lambda_j \leq \lambda \} \leq 1 + c\lambda^{n/2}$

iii) $\forall x \in M$, $\alpha \geq 0$

$$\sum_{j\geq 1} \lambda_j^{\alpha} \exp(-t\lambda_j)\, \varphi_j^2(x) \leq \frac{c}{Vol(M)}\, t^{-\frac{n+2\alpha}{2}}$$

Here, $c = c(n, k, D, \alpha)$

pf: <u>step 1</u> If $k \geq 0$, $r_0 d^2 \geq 0$

If $k \leq 0$, $r_0 d^2 \geq (n-1) k D^2$

$\Rightarrow$ Theorem 0 applies

$$Z_M(t) = \int_{x\in M} k_M(t, x, x) \leq Vol(M) \sup_{x\in M} \{ k(t, x, x) \}$$

$$\leq Z_{S^n(1)}(t/R^2) = Z_{S^n(R)}(t)$$

$$Z_{S^n(R)}(t) = 1 + \sum_{j\geq 1} e^{-\lambda_j t}$$

For $t < 1$, by the parametrix construction.

$$Z_{S^n(R)}(t) \lesssim t^{-n/2}$$

For $t > 1$, we already know $\lambda_j \gtrsim j^{(s+\frac{n}{2})^{-1}}$

of $S^n(R)$

$\gtrsim j^{\frac{1}{n}}$

$$Z_{S^n(R)}(t) - 1 = \sum_{j\geq 1} e^{-\lambda_j t} \lesssim \sum_{j\geq 1} e^{-j^{\frac{1}{n}} t}$$

of $S^n(R)$

$$\lesssim \int_1^\infty e^{-s^{\frac{1}{n}} t} ds \lesssim \int_1^\infty e^{-s^{\frac{1}{n}} t} s^{\frac{1}{n}-1} ds$$

$$\approx \int_1^\infty e^{-s^{\frac{1}{n}} t} ds^{\frac{1}{n}}$$

$$= e^{-t}$$

Hence. $Z_{S^n(R)}(t) - 1 \lesssim t^{-n/2} \quad \forall t$

<u>step 2</u> $\quad Z_M(t) - 1 = \sum_{j\geq 1} e^{-\lambda_j t}$

$\Rightarrow \quad \bar{j} \leq N(\lambda_{\bar{j}}) - 1$

due to multiplicity

If $\lambda_i \leq \lambda_{\bar{j}} \Rightarrow \lambda_i / \lambda_{\bar{j}} \leq 1 \Rightarrow -\lambda_i / \lambda_{\bar{j}} \geq -1$

($\bar{j}$: fixed)

$$\Rightarrow e^{-\lambda_i / \lambda_{\bar{j}}} \geq e^{-1}$$

$$\Rightarrow e \cdot e^{-\lambda_i \frac{1}{\lambda_{\bar{j}}}} \geq 1$$

$$\leq e \sum_{0<\lambda_i\leq\lambda_{\bar{j}}} e^{-\lambda_i \frac{1}{\lambda_{\bar{j}}}} \leq e \cdot \left(Z_M\left(1/\lambda_{\bar{j}}\right) - 1 \right)$$

$$\leq e \cdot \left(Z_{S^n(R)}\left(\tfrac{1}{\lambda_{\bar{j}}}\right) - 1 \right)$$

$$\lesssim \lambda_{\bar{j}}^{n/2}$$

$$\Rightarrow \text{i)} \ \& \ \text{ii)}$$

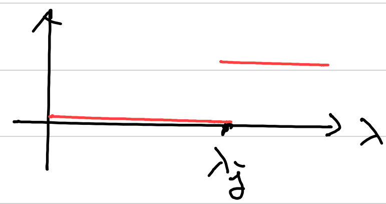
<u>step 3</u> $\quad \forall x \in M$

weight $\qquad$ delta distribution

Consider $d\mu_x = \sum_{j\geq 1} \varphi_j^2(x) \, \delta_{\lambda_j}$

a measure on $\mathbb{R}_{>0}$

$$\int_0^\lambda \delta_{\lambda_j} = \begin{cases} 1 & \lambda \geq \lambda_j \\ 0 & \lambda < \lambda_j \end{cases}$$



$$\Rightarrow \int_0^\lambda d\mu_x = \sum_{0<\lambda_j\leq\lambda} \varphi_j^2(x) =: \mu_x([0,\lambda])$$

$$\sum_{j\geq 1} \lambda_j^\alpha e^{-t\lambda_j} \varphi_j^2(x) = \int_0^\infty \underbrace{\lambda^\alpha e^{-t\lambda}}_{\downarrow d} d\mu_x(\lambda)$$

$$\alpha\lambda^{\alpha-1} e^{-t\lambda} - t\lambda^\alpha e^{-t\lambda}$$

$$= \int_0^\infty \left(t\lambda^\alpha e^{-t\lambda} - \alpha\lambda^{\alpha-1} e^{-t\lambda}\right) \mu_x([0,\lambda])\, d\lambda$$

$$\leq \int_0^\infty (\lambda t + \alpha)\lambda^{\alpha-1} e^{-t\lambda} \underbrace{\left(\sum_{0<\lambda_j\leq\lambda} \varphi_j^2(x)\right)} d\lambda$$

$$\leq e \cdot \sum_{0<\lambda_j\leq\lambda} e^{-\lambda_j \frac{1}{\lambda}} \varphi_j^2(x)$$

$$\leq e \cdot \sup_{x\in M}\left(k_M\left(\tfrac{1}{\lambda}, x, x\right) - \frac{1}{\mathrm{vol}(M)}\right)$$

$$\leq \frac{e}{\mathrm{vol}(M)}\left(Z_{S^n(R)}\left(\tfrac{1}{\lambda}\right) - 1\right) \lesssim \lambda^n$$

$$\lesssim \int_0^\infty (\lambda t + \alpha)\lambda^{\alpha-1} e^{-t\lambda} \lambda^n\, d\lambda \lesssim t^{-(\alpha+\frac{n}{2})}$$

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§II. embedding (M, g) into ℓ^2

(M, g) $\quad 0 = \lambda_0 < \lambda_1 \le \lambda_2 < \cdots < \lambda_j < \cdots$

$\quad (\mathrm{Vol}(M))^{-1/2} = \varphi_0 \quad \varphi_1 \quad \cdots \quad \varphi_j$

<u>defn</u> Fix an orthonormal eigenbasis. let

$$\Psi_t : M \to \ell^2$$

$$x \mapsto \underbrace{\sqrt{2}\,(4\pi)^{n/2}\, t^{\frac{n+4}{2}}}\ \{e^{-\lambda_j t/2}\varphi_j(x)\}_{j\ge 1}$$

will see the reason later

<u>rmk</u>
- φ_0 is dropped
- $\ell^2 \ni \{a_i\}_{i\ge1}$ with $\sum_{i\ge1} |a_i|^2 < \infty$

$$\sum_{j\ge1} |e^{-\lambda_j t/2}\varphi_j(x)|^2 = \sum_{j\ge1} e^{-\lambda_j t}\varphi_j(x)\varphi_j(x)$$

$$= k(t, x, x) - 1$$

- $h^1 \ni \{a_i\}_{i\ge1}$ with $\sum_{i\ge1} |a_i|^2 (1 + i^{\frac{2}{n}}) < \infty$

$\quad \mathrel{\wr\wr}$

$\quad H_1(M) \qquad$ Rellich lemma: $h^1 \xrightarrow{\text{cpt}} \ell^2$

<u>theorem 2</u> i) $\forall t > 0$. $\Psi_t : M \to \ell^2$ is an embedding

ii) $\Psi_t^*(g_0) = g + \frac{t}{3}\left(\frac{1}{2} R(g)\cdot g - \mathrm{Ricci}(g)\right) + O(t^2)$ as $t \to 0^+$

standard metric on ℓ^2 (an ∞-dim'l vector space) pf: <u>step 1</u> $\Psi_t(x) = \{ e^{-\lambda_j t} \varphi_j(x) \}_{j\ge 1}$

$$\| \Psi_{t'}(x') - \Psi_t(x) \|_{\ell^2}^2 = \sum_{j\ge1} \left| e^{-\lambda_j \frac{t'}{2}} \varphi_j(x') - e^{-\lambda_j \frac{t}{2}} \varphi_j(x) \right|^2$$

$$= \sum_{j\ge1} \left(e^{-\lambda_j t'} \varphi_j(x') \varphi_j(x') - 2 e^{-\lambda_j \frac{t+t'}{2}} \varphi_j(x) \varphi_j(x') + e^{-\lambda_j t} \varphi_j(x) \varphi_j(x) \right)$$

$$= k(t', x', x') + k(t, x, x) - 2 k\left(\tfrac{t+t'}{2}, x, x'\right)$$

$\Rightarrow$ $\Psi_t(x)$ is continuous

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$\Rightarrow$ $\Psi_t(M)$ is compact in ℓ^2

<u>step 2</u> $\forall t>0$, $\Psi_t(X)$ is injective

If NOT, $\exists\, x_0 \neq x_1 \in M$

with $\varphi_j(x_0) = \varphi_j(x_1)$ $\forall j$

But this cannot happen:

Choose $f \in C^\infty(M)$ with $\int f = 0$

$f(x_0) = 0$, $f(x_1) = 1$

$\Rightarrow$ $f(x) = \sum_{j\ge1} a_i \varphi_j(x)$

Since $\varphi_j(x_0) = \varphi_j(x_1)$ $\Rightarrow$ $f(x_0) = f(x_1)$ $\rightarrow\leftarrow$

Hence $\Psi_t : M \to \ell^2$ is homeomorphic to its image

If $(d\Psi_t)|_x(V) = 0$

for some $x \in M$, $V \in T_xM \setminus \{0\}$

$\Rightarrow d\varphi_j|_x(V) = 0$: component of $d\Phi_t$

$\Rightarrow df|_x(V) = 0 \quad \forall f \in C^\infty(M)$

$\Rightarrow V = 0 \quad \rightarrow\leftarrow$

Hence, Ψ_t is an embedding