GEOMETRY: HOMEWORK 13

DUE DECEMBER 26

(1) Let $\eta = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ is a smooth 1-form on $\mathbb{R}^3$. Suppose that S is a regular surface with boundary. Still denote by η the restriction of η on S. Check that the Stokes theorem

$$\iint_S \mathrm{d}\eta = \int_{\partial S} \eta$$

gives the usual Stokes theorem in vector calculus.

(2) Similar to (1), suppose that Ω is an open subset of $\mathbb{R}^3$, endowed with the orientation $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, with $\partial\Omega$ be a compact regular surface. Let $\eta = P\,\mathrm{d}y \wedge \mathrm{d}z + Q\,\mathrm{d}z \wedge \mathrm{d}x + R\,\mathrm{d}x \wedge \mathrm{d}y$ be a smooth 2-form on $\mathbb{R}^3$. Check that the Stokes theorem

$$\iiint_\Omega \mathrm{d}\eta = \iint_{\partial S} \eta$$

gives the usual Gauss theorem in vector calculus.

(3) [re-visiting the degree] Let M and N be two connected, oriented, closed[1] n-dimensional manifold. It can be shown that $\mathrm{H}^n_{\mathrm{dR}}(M) \cong \mathbb{R} \cong \mathrm{H}^n_{\mathrm{dR}}(N)$, and the isomorphism is given by integration over the manifold. For any smooth map $f : M \to N$, define its degree by

$$\deg(f) = \int_M f^*\omega \quad \text{where } [\omega] \in \mathrm{H}^n_{\mathrm{dR}}(N) \text{ with } \int_N \omega = 1 \ .$$

Note that for any $q \in N$, one can choose ω whose support is contained in a small neighborhood of q. By using the same argument as that in that note `5.degree of map to S`^{2}`.pdf` (see also [BT, pp.40–42]), degree can shown to be an integer. Moreover, if $q \in N$ is a regular value,

$$\deg(f) = \sum_{p \in f^{-1}(q)} \operatorname{sgn}\left(\det(f_*|_p)\right) \ ,$$

where determinant is taken with respect to oriented bases for T_pM and T_qN.

(a) Suppose that $f_0, f_1 : M \to N$ are homotopic to each other, show that $\deg(f_0) = \deg(f_1)$.

[1]compact without boundary

(b) Suppose that there is an n-dimensional manifold with boundary X, which is compact and oriented, and whose boundary is M. Let $F : X \to N$ be a smooth map, and denote by f its restriction on $\partial X = M$. Show that $\deg(f) = 0$.

Remark. For $f_0, f_1 : M \to \mathbb{S}^n$, a theorem of Hopf asserts that the inverse direction of (a) holds true. That is to say, if $\deg(f_0) = \deg(f_1)$, then they are homotopic. See for instance §7 in [J. Milnor, Topology from the Differentiable Viewpoint].

(4) Let M be a manifold without boundary, and let $\Sigma \subset M$ be closed, oriented submanifold of dimension k.

(a) Prove that the integration over Σ gives a well-defined linear functional on $\mathrm{H}^k_{\mathrm{dR}}(M)$.

(b) Use part (a) and part (1.c) in homework 12 to show that $\mathrm{H}^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \backslash \{0\})$ is not trivial.