GEOMETRY: HOMEWORK 12

DUE DECEMBER 19

(1) Calculate the exterior derivative of the following differential forms.

(a) $\mathrm{d}z + x\,\mathrm{d}y - y\,\mathrm{d}z$ on $\mathbb{R}^3$.

(b) $\dfrac{x\,\mathrm{d}y - y\,\mathrm{d}x}{1 + x^2 + y^2}$ on $\mathbb{R}^2$.

(c) $\dfrac{1}{|\mathbf{x}|^n}\displaystyle\sum_{j=1}^{n}(-1)^{j-1}x^j\,\mathrm{d}x^1\wedge\cdots\wedge\widehat{\mathrm{d}x^j}\wedge\cdots\wedge\mathrm{d}x^n$ on $\mathbb{R}^n\backslash\{0\}$, where $\widehat{\cdot}$ means that the term is not there.

(2) Check that

$$\mathrm{d}(\omega\wedge\eta) = (\mathrm{d}\omega)\wedge\eta + (-1)^k\omega\wedge(\mathrm{d}\eta)$$

for any $\omega\in\Omega^k(M)$ and $\eta\in\Omega^\ell(M)$.

(3) Consider the 2-form on $\mathbb{R}^{2n}$ defined by

$$\omega = \mathrm{d}x^1\wedge\mathrm{d}x^2 + \mathrm{d}x^3\wedge\mathrm{d}x^4 + \cdots + \mathrm{d}x^{2n-1}\wedge\mathrm{d}x^{2n} \ .$$

Compute $\omega^n = \omega\wedge\omega\wedge\cdots\wedge\omega$.

(4) For any 1-form α and vector fields U, V, prove that

$$\mathrm{d}\alpha(U,V) = U(\alpha(V)) - V(\alpha(U)) - \alpha([U,V]) \ .$$

Hint: This formula is local in nature. Due to $\mathbb{R}$-linearity, it suffices to show it for $\alpha = f(\mathbf{x})\,\mathrm{d}x^1$.

(5) Consider the following two parametrization for $\mathbb{S}^2$:

$$F_+(x^1,x^2) = \frac{1}{1+(x^1)^2+(x^2)^2}\big(2x^1, 2x^2, 1-(x^1)^2-(x^2)^2\big) \ ,$$
$$F_-(y^1,y^2) = \frac{1}{1+(y^1)^2+(y^2)^2}\big(2y^1, -2y^2, -1+(y^1)^2+(y^2)^2\big) \ .$$

For the following differentials forms, find their expression in terms of the $\mathbf{y}$-coordinate.

(a) $\dfrac{4}{(1+(x^1)^2+(x^2)^2)^2}\,\mathrm{d}x^1\wedge\mathrm{d}x^2$.

(b) $\dfrac{x^1\,\mathrm{d}x^2 - x^2\,\mathrm{d}x^1}{(1+(x^1)^2+(x^2)^2)^2}$.

(6) On a vector space V of dimension $n > 0$, an *orientation* is a non-zero element in $\Lambda^n(V^*)$. Two orientation is said to be equivalent if they differ by a scaling of positive factor. Since $\Lambda^n(V^*) \cong \mathbb{R}$, there are indeed two different equivalent classes of orientations. With a choice of basis $\{\frac{\partial}{\partial x^j}\}_{j=1}^n$ for V, the (class of) orientation is given by $\pm \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

A manifold M is said to be orientable if it admits a *nowhere vanishing* n-form. For simplicity, assume M is connected. Show that M is orientable if and only if it admits a coordinate cover such that the determinant of Jacobian of the coordinate transitions is always positive (on where it is defined).

Note: Part (a) of (5) defines an orietation on $\mathbb{S}^2$.

Hint: For a diffeomorphism between open subsets of $\mathbb{R}^n$, $\mathbf{y} = \mathbf{y}(\mathbf{x})$, a straightforward calculation[1] shows that

$$\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n = \det\left[\frac{\partial(y^1,\cdots,y^n)}{\partial(x^1,\cdots,x^n)}\right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \ .$$

For the direction of construction nowhere vanishing n-form, you may need the partition of unity.

[1]Check it by yourself.