GEOMETRY: HOMEWORK 11

DUE DECEMBER 5

(1) The Heisenberg is a matrix group diffeomorphic to $\mathbb{R}^3$:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(3;\mathbb{R}) \right\} .$$

Its tangent space at the identity is

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} : u, v, w \in \mathbb{R} \right\} .$$

A direct computation finds the matrix exponential

$$\exp\left(\begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & u & w + \frac{1}{2}uv \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix} .$$

(a) Check that the matrix exponential coincides with the Lie theoretical[1] exponential.

(b) Check that the matrix bracket coincides with the Lie theoretical bracket.

(c) Let

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} , \qquad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Denote by $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ their left-invariant extensions. Is $\mathrm{span}\{\tilde{U}, \tilde{V}\}$ involutive?

Remark: You can compare $\exp(sU)\exp(tV)$ with $\exp(tV)\exp(sU)$.

(d) Check that $\mathrm{span}\{\tilde{U}, \tilde{W}\}$ is involutive. Find its integration (subgroup) through the identity matrix.

(e) What is the Killing form of $\mathfrak{g}$?

(2) Check that $\mathbb{R}^3$ with the cross product constitutes a Lie algebra.

Hint: $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle \, \mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \, \mathbf{w}$.

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[1]It means the one comes from the left-invariant vector field construction.

(3) Consider

$$\mathrm{SO}(3) = \{A \in \mathrm{GL}(3;\mathbb{R}) \,|\, A^T A = \mathbf{I}\ ,\ \det A = 1\}\ .$$

Its Lie algebra is

$$\mathfrak{so}(3) = \{U \in \mathrm{M}(3\times 3;\mathbb{R}) \,|\, U + U^T = 0\}\ .$$

In general, for matrix groups, the matrix exponential coincides with the Lie theoretical exponential, and the matrix bracket coincides with the Lie theoretical bracket. You can use this fact here.

(a) Show that $(\mathfrak{so}(3), [\ ,\ ])$ and $(\mathbb{R}^3, \times)$ are isomorphic as a Lie algebra.

Hint: Here is the commonly used basis for $\mathfrak{so}(3)$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\ ,\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\ ,\ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ .$$

Definition. Let G be a group acting on two sets D and C. A function $f : D \to C$ is called *G-equivariant* if $f(g \cdot p) = g \cdot f(p)$ for any $g \in G$ and $p \in D$.

For instance, let $G = \mathrm{SO}(3)$ acting on $\mathfrak{so}(3)$ by conjugation (also on $\mathfrak{so}(3) \times \mathfrak{so}(3)$). Then, the bracket is SO(3)-equivariant:

$$[g \cdot U, g \cdot V] = (gUg^T)(gVg^T) - (gVg^T)(gUg^T) = g[U,V]g^T = g \cdot [U,V]\ .$$

One can check that this is always true for Adjoint action of G on $\mathfrak{g}$.

(b) Show that the map in (a) is SO(3)-*equivariant.*

Note: SO(3) acts on $\mathbb{R}^3$ by the usual multiplication.