GEOMETRY: HOMEWORK 10

DUE NOVEMBER 28

(1) Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

  (a) $U = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$;

  (b) $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$;

  (c) $W = (y+2)x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

(2) Determine whether the following distribution is involutive.

  (a) $\text{span}\{\frac{\partial}{\partial z}, \cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\}$ on $\mathbb{R}^3$.

  (b) $\text{span}\{\frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\}$ on $\mathbb{R}^3\backslash\{z\text{-axis}\}$.

(3) Give an example of vector fields U, V, W on $\mathbb{R}^2$ such that

  - on the x-axis, $U = \frac{\partial}{\partial x} = V$;
  - $L_U W \neq L_V W$ at the origin.

  This exercise says that to compute the Lie derivative $L_V W = [V, W]$, it is not sufficient just to know V at a point, or even along an integral curve. The Lie derivative $L_V W$ depends on the behavior of V on an open neighborhood.

(4) (a) Denote by B the open unit ball in $\mathbb{R}^n$. For any two distinct points $\mathbf{x}_0, \mathbf{x}_1 \in B$, show that there is a diffeomorphism $\psi : B \to B$ which sends $\mathbf{x}_0$ to $\mathbf{x}_1$.

  Hint: Let σ be the line segment connecting $\mathbf{x}_0, \mathbf{x}_1$. Try to construct a vector field which is equal to $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$ on a neighborhood of σ, and $\text{supp}\,\mathbf{v} \subset B$.

  (b) Let M be a compact, connected manifold. Let $\text{Aut}(M)$ be the group[1] of self-diffeomorphisms of M. Prove that $\text{Aut}(M)$ acts *transitively* on M. Namely, for any $p, q \in M$, there exists some $\varphi \in \text{Aut}(M)$ such that $\varphi(p) = q$.

(5) For any three tangent vector fields, U, V, W, check by calculation in coordinate that they obey the *Jacobi identity*:

$$[U,[V,W]] + [W,[U,V]] + [V,[W,U]] = 0\ . \qquad (\spadesuit)$$

[1]The binary operation is the composition.

(6) Suppose that there is a diffeomorphism $\psi : M \to N$. Check that

$$\psi_*([U,V]) = [\psi_* U, \psi_* V] \quad \text{for any } U, V \in \Gamma(M;TM)\ . \tag{$\diamondsuit$}$$

Here, ψ_* is the differential of ψ.

Hint: In local coordinate, ψ is given by $\mathbf{y} = \mathbf{y}(\mathbf{x})$, where $\mathbf{x}$ and $\mathbf{y}$ are local coordinates for M and N, respectively.

(7) Let φ_t be the one-parameter family of diffeomorphism generated by U. It follows from ($\diamondsuit$) that

$$(\varphi_{-t})_*([V,W]) = [(\varphi_{-t})_*(V), (\varphi_{-t})_*(W)]\ .$$

Use this to give another proof of the Jacobi identity ($\spadesuit$).