GEOMETRY: HOMEWORK 9

DUE NOVEMBER 21

(1) Prove that any n-dimesnional manifold M can be immersed in $\mathbb{R}^{2n}$.

(2) Suppose that M^n be a submanifold in $\mathbb{R}^N$. Let

$$TM = \{(p, v) \in \mathbb{R}^N \times \mathbb{R}^N \,|\, p \in M \ , \ v \in \text{image}(DX^{-1}|_{X(p)}) \\ \text{for some coordinate chart } (\mathcal{U}, X) \text{ around } p\} \ ,$$

and

$$S(M) = \{(p, v) \in TM \subset \mathbb{R}^N \times \mathbb{R}^N \,|\, |v| = 1\} \ .$$

Check that TM is a $2n$-dimensional submanifold of $\mathbb{R}^N \times \mathbb{R}^N$, and $S(M)$ is a $(2n-1)$-dimensional submanifold of $\mathbb{R}^N \times \mathbb{R}^N$.

Remark: The argument shall imply that TM is diffeomorphic to the abstractly defined tangent bundle of M.

(3) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, yz, zx, xy) \ .$$

Note that on $\mathbb{S}^2$, $F(x, y, z) = F(-x, -y, -z)$. It follows that F descends to a map $f : \mathbb{RP}^2 \to \mathbb{R}^4$. Show that f is an embedding.

Hint: We constructed a coordinate cover of $\mathbb{RP}^2$, which consists of three open sets.

(4) Consider the "height function" x^n on the standard sphere $\mathbb{S}^{n-1} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid \sum_{j=1}^n (x^j)^2 = 1\}$. Check that it is a Morse function with only two critical points.