GEOMETRY: HOMEWORK 8

DUE NOVEMBER 14

(1) (a) All the $n \times n$ invertible matrices, $\mathrm{GL}(n;\mathbb{R})$, is an open subset of $\mathbb{R}^{n^2}$, and thus a manifold of dimension n^2. Denote by $\mathfrak{x}_{ij}$ the coordinate. At a point $(\mathfrak{m}_{ij}) \in \mathrm{GL}(n;\mathbb{R})$, show that

$$D(\det[\mathfrak{x}_{ij}])|_{\mathfrak{m}_{ij}} = \det([\mathfrak{m}_{ij}]) \sum_{k,\ell=1}^{n} (\mathfrak{m}^{-1})_{k\ell} \; D\mathfrak{x}_{\ell k}|_{\mathfrak{m}_{ij}} \ .$$

(b) Prove that $\mathrm{SL}(n;\mathbb{R}) = \{X \in \mathrm{GL}(n;\mathbb{R}) \mid \det X = 1\}$ is a smooth manifold, and determine its dimension.

(2) Fix $n \in \mathbb{N}$ and $a > 0$, let $B(0;a) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| < a\}$. Prove that

$$\mathbf{x} \mapsto \frac{a\,\mathbf{x}}{\sqrt{a^2 - |\mathbf{x}|^2}}$$

is a diffeomorphism from $B(0;a)$ to $\mathbb{R}^n$.

(3) Suppose that $M^m \subset N^n$ is a submanifold. For any $p \in M$, prove that there exists a coordinate chart $(\mathcal{U}, X = (x^1, \ldots, x^n))$ around p of N such that $M \cap \mathcal{U}$ is defined by the equation $x^j = 0$ for $j = m+1, \ldots, n$.

Hint: This is a local property. That is to say, it suffices to consider the case that N is an open subset set of $\mathbb{R}^n$. Denote by $\mathbf{y}$ the coordinate. Similar to the case of regular surfaces, M can be written as a graph. We may assume that M is given by $y^j = g_j(y^1, y^2, \ldots, y^m)$ for $j = m+1, \ldots, n$.

(4) Given a smooth map $F : M^m \to N^n$, $q \in N$ is called a *singular value* if there exists some $p \in F^{-1}(\{q\})$ such that $\mathrm{rank}(DF|_p) < n$. Otherwise it is called a *regular value*. By applying (1) of Homework 7, $F^{-1}(\{q\})$ is a submanifold of dimension $m - n$ if q is a regular value.

Sard theorem asserts that the set of singular values has measure zero. It follows that there are lots of regular values.

Let q be a regular value, denote $F^{-1}(\{q\})$ by S. Prove that $T_pS = \ker DF|_p$ for any $p \in S$. Try to argue it by using the derivations.

(5) A smooth map $F : M \to N$ is called a *submersion* if $DF|_p : T_pM \to T_{F(p)}N$ is surjective for any $p \in M$. That is to say, every point in N is a regular value, and thus $F^{-1}(\{q\})$ is a smooth manifold of dimension $m - n$ for every $q \in N$.

Prove that for any $p \in M$, there exist coordinate charts around p and $f(p)$ such that $F(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$. More precisely, there exist a coordinate chart $(\mathcal{U}, X = (x^1, \ldots, x^m))$ around p and a coordinate chart $(\mathcal{V}, Y = (y^1, \ldots, y^n))$ around $f(p)$ such that the j-th component of $(Y \circ F \circ X^{-1})(x^1, \ldots, x^n)$ is exactly x^j for $j = 1, \ldots, n$.

Hint: Start with any coordinate charts $(\mathcal{U}, (u^1, \ldots, u^m))$ around p, and $(\mathcal{V}, (v^1, \ldots, v^n))$ around $F(p)$. The map in coordinate is $v^j = v^j(\mathbf{u})$. The differential DF is represented by the Jacobian $\frac{\partial v}{\partial u}$, which is an $n \times m$ matrix. By assumption, the matrix is surjetive on where it is defined. After re-numbering the coordiantes, we may assume that the first $n \times n$-block is invertible at (the points corresponding to) p. By shrinking $\mathcal{U}$, we may assume $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$. Consider the map

$$\begin{array}{rcl} \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 & \to & \mathcal{V} \times \mathcal{U}_2 \\ ((u^1, \ldots, u^n), (u^{n+1}, \ldots, u^m)) & \mapsto & ((v^1(\mathbf{u}), \ldots, v^n(\mathbf{u})), ((u^{n+1}, \ldots, u^m))) \ . \end{array}$$