GEOMETRY: HOMEWORK 7

DUE NOVEMBER 7

(1) **IFT and submanifold in the Euclidean space** Suppose that $q_0 \in \mathbb{R}^m$ is a regular value of a smooth map $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$. That is to say, $DF|_p$ is surjective for any $p \in F^{-1}(q_0)$. Endow $M = F^{-1}(q_0)$ the subspace topology (from $\mathbb{R}^{n+m}$). Prove that M is a smooth manifold of dimension n.

Hint: You do not have to do the part of para-compactness, or second countability. This exercise is nothing more than rephrasing the implicit function theorem.

Another point is that smoothness is a local property. In fact, we did a similar argument for regular surfaces; see p.2 on the lecture note of the second week.

(2) **sphere, orthogonal group, unitary group** For the following sets, apply (1) to show that they are smooth manifolds, and determine their dimension.

(a) $\mathbb{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1\}$.

(b) $\mathrm{O}(n) = \{A \in \mathrm{M}(n \times n; \mathbb{R}) : A^T A = \mathbf{I}\}$.

(c) $\mathrm{U}(n) = \{A \in \mathrm{M}(n \times n; \mathbb{C}) : A^* A = \mathbf{I}\}$.

Hint: Instead of identifying $\mathrm{M}(n \times n; \mathbb{R})$ with $\mathbb{R}^{n^2}$ (and $\mathrm{M}(n \times n; \mathbb{C})$ with $\mathbb{R}^{2n^2}$), it is more convenient to work with $\mathrm{M}(n \times n; \mathbb{R})$ directly. Both left and right multiplication by a matrix are linear maps on $\mathrm{M}(n \times n; \mathbb{R})$.

(3) **Stiefel manifold** Fix $n, k \in \mathbb{N}$ with $k \leq n$. The Stiefel manifold, $\mathrm{V}_k(\mathbb{R}^n)$, is the set of ordered k-tuples of orthonormal vectors in $\mathbb{R}^n$:

$$\mathrm{V}_k(\mathbb{R}^n) = \{(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) : \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \text{ is orthonormal (in } \mathbb{R}^n)\} \ .$$

It is not hard to see that $\mathrm{V}_1(\mathbb{R}^n) = \mathbb{S}^{n-1}$ and $\mathrm{V}_n(\mathbb{R}^n) = \mathrm{O}(n)$.

(a) Show that $\mathrm{V}_k(\mathbb{R}^n)$ is a smooth manifold, and determine its dimension. You can put $\vec{v}_j$'s into an $n \times k$ matrix. Being orthonormal can be translated into a condition on the matrix.

(b) Consider the map

$$\begin{array}{rccl} \pi : & \mathrm{V}_k(\mathbb{R}^n) & \to & \mathrm{Gr}(k, n) \\ & (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) & \mapsto & \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \ . \end{array}$$

For any $P \in \mathrm{Gr}(k, n)$, what do you know about $\pi^{-1}(P)$? (Just explain it set-theoretically.)

(4) Flag manifold Fix $n \in \mathbb{N}$. Choose a sequence of integers,

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = n \ .$$

Denote $(\alpha_0, \alpha_1, \cdots, \alpha_k)$ by $\mathcal{A}$. A flag of type $\mathcal{A}$ in $\mathbb{R}^n$ is a sequence of vector subspaces with

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{R}^n \quad \text{and} \quad \dim V_j = \alpha_j \ .$$

For $\ell = (0, 1, 2, \ldots, n)$, the flag is called a complete flag. Namely, $\dim V_j - \dim V_{j-1} = 1$ for all j. Otherwise it is called a partial flag.

The set of all flags of the same type is called the flag manifold:

$$\mathrm{F}(\mathcal{A}, n) = \{ \text{flags of type } \mathcal{A} \text{ in } \mathbb{R}^n \} \ .$$

The name suggests that the above set can be shown to be a smooth manifold. Note that for $\mathcal{A} = (0, i, n)$, $\mathrm{F}(\mathcal{A}, n) = \mathrm{Gr}(i, n)$.

Now, focus on the case $n = 4$ and $\mathcal{A} = (0, 1, 3, 4)$. The flag manifold $\mathrm{F}((0, 1, 3, 4), 4)$ is 5-dimensional. Denote by $\{\vec{e}_j\}_{j=1}^4$ the standard basis of $\mathbb{R}^4$.

(a) Construct an injective map from $\mathbb{R}^5$ to $\mathrm{F}(\mathcal{A}, 4)$ by the following hint.
Let $0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{R}^4$ be a flag with $\dim V_1 = 1$ and $\dim V_2 = 3$.
- Suppose that the orthogonal projection of V_2 to $\mathrm{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is surjective. Regard flags with this property the "nearby" flags. Given any basis of such V_2, it can be extended to a basis of $V_3 = \mathbb{R}^4$ by adding $\vec{e}_4$.
- It follows that the orthogonal projection of V_1 onto one of the first three coordinate axes must be surjective. Suppose that it is $\mathrm{span}\{\vec{e}_1\}$. Regard flags with this property the "nearby" flags.
- Flags satisfying the above two properties can be parametrized by $\mathbb{R}^5$. Work out an injective map from $\mathbb{R}^5$ to $\mathrm{F}(\mathcal{A}, 4)$.

(b) Construct another map by choosing different coordinate planes. Determine the overlap region, and calculate the transition map. (You are only asked to work out two coordinate charts, not all.)