GEOMETRY: HOMEWORK 6

DUE OCTOBER 24

(1) Draw a tangent vector field on $\mathbb{S}^2$, which vanishes only at one point.

(2) Let Ω be a bounded, open set in $\mathbb{R}^2$, whose boundary consists of finite collection of continuous, piecewise smooth curves. Suppose that there is a nowhere zero, smooth vector field $(P(x,y), Q(x,y))$ defined on some open set containing $\bar{\Omega}$. Show that

$$\int_{\partial\Omega} \frac{1}{P^2+Q^2}\left((P\,Q_x - Q\,P_x)x' + (P\,Q_y - Q\,P_y)y'\right)\,\mathrm{d}t = 0$$

where $(x(t), y(t))$ is a parametrization for $\partial\Omega$ with counterclockwise orientation (with respect to Ω).

Remark: Suppose that the origin is an isolated zero of (P,Q). One show that the above integral along a small circle around the origin is the index. In other words, one can use this 1D integral to calculate the index of a plane vector field.

(3) Is there a vector field (P,Q) defined on some neighborhood of the origin in the xy-plane, which only vanishes at the origin, and whose index at the origin is zero?

If your answer is YES, give an example of such a vector field. If your answer is NO, prove that such a vector field does not exist.

(4) A regular surface in $\mathbb{R}^3$ is said to be *minimal* if its mean curvature vanishes everywhere. Show that there is no *compact* regular surface which is minimal.

Hint: If $H(p) = 0$, what can be said about $K(p)$?